US Particle Accelerator School

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Design of Electron Storage and Damping Rings

Part 3: Nonlinear Dynamics

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Lectures 1 and 2: summary

In Lecture 1, we:

- derived expressions for the damping times of the vertical, horizontal, and longitudinal emittances;
- derived expressions for the equilibrium horizontal and longitudinal emittances in an electron storage ring in terms of the lattice functions and beam energy.

In Lecture 2, we derived expressions for the natural emittance in storage rings with different lattice styles, in terms of the number of cells and the beam energy.

The momentum compaction factor is:

$$\alpha_p = \frac{I_1}{C_0}. \tag{1}$$

The energy loss per turn is:

$$U_0 = \frac{C_\gamma}{2\pi} E_0^4 I_2, \qquad C_\gamma \approx 8.846 \times 10^5 \text{ m/GeV}^3. \tag{2}$$

The natural energy spread and bunch length are given by:

$$\sigma_\delta^2 = C_q \gamma^2 \frac{I_3}{j_z I_2}, \qquad \sigma_z = \frac{\alpha_p c}{\omega_s} \sigma_\delta. \tag{3}$$

The natural emittance is:

$$\varepsilon_0 = C_q \gamma^2 \frac{I_5}{j_x I_2}, \qquad C_q \approx 3.832 \times 10^{-13} \text{ m}. \tag{4}$$

The damping partition numbers are:

$$j_x = 1 - \frac{I_4}{I_2}, \qquad j_z = 2 + \frac{I_4}{I_2}. \tag{5}$$

The synchrotron radiation integrals are:

$$I_1 = \oint \frac{\eta_x}{\rho} \, ds, \tag{6}$$

$$I_2 = \oint \frac{1}{\rho^2} \, ds, \tag{7}$$

$$I_3 = \oint \frac{1}{|\rho|^3} \, ds, \tag{8}$$

$$I_4 = \oint \frac{\eta_x}{\rho} \left(\frac{1}{\rho^2} + 2k_1 \right) ds, \qquad k_1 = \frac{e}{P_0} \frac{\partial B_y}{\partial x}, \tag{9}$$

$$I_5 = \oint \frac{\mathcal{H}_x}{|\rho|^3} \, ds, \qquad \mathcal{H}_x = \gamma_x \eta_x^2 + 2\alpha_x \eta_x \eta_{px} + \beta_x \eta_{px}^2. \tag{10}$$

Lecture 3 objectives

In this lecture, we shall discuss issues associated with nonlinear dynamics in storage rings.

In particular, we shall:

- show that lattices constructed using only dipoles and quadrupoles have significant chromaticity (tune variation with beam energy);
- show how the adverse effects of chromaticity can be avoided by correcting the chromaticity using sextupoles;
- show that the use of sextupoles for correcting chromaticity has a side effect in limiting the dynamic aperture, which in turn limits the beam lifetime.

Nonlinear effects cannot be avoided in storage rings, and are of crucial importance in determining practical limitations on injection efficiency and beam lifetime.

Effect of a focusing error on the betatron tune

Our first goal is to derive an expression showing how the betatron tune in a storage ring changes with particle energy (for fixed magnet strengths).

For simplicity, we shall consider the dynamics in just one transverse degree of freedom.

The transfer matrix for a particle moving through a thin quadrupole is:

$$M = \begin{pmatrix} 1 & 0 \\ -K & 1 \end{pmatrix}, \tag{11}$$

where:

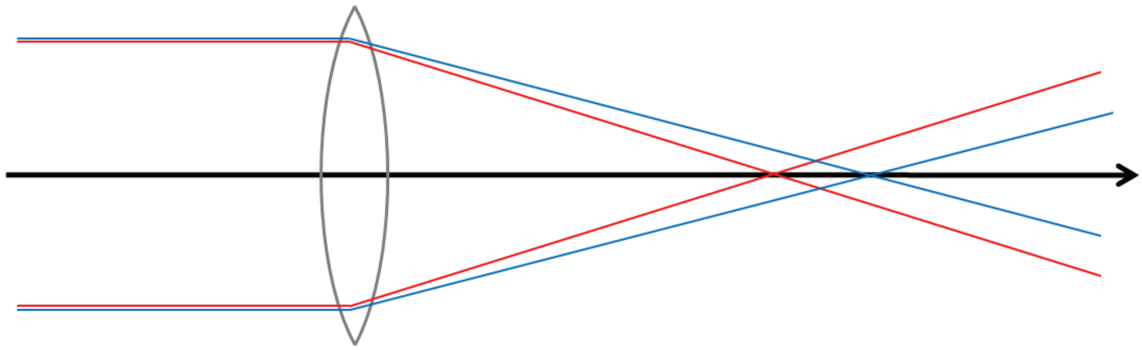
$$K = \frac{q}{P} \int \frac{\partial B_y}{\partial x} \, ds. \tag{12}$$

Note that:

$$K = \frac{q}{P} \int \frac{\partial B_y}{\partial x} \, ds = \frac{1}{f}, \tag{13}$$

where f is the focal length of the magnet.

P is the momentum of the particle. A small increase in P leads to a small increase in focal length (i.e. a reduction in focusing strength).



Under the transformation:

$$P \mapsto (1 + \delta)P, \tag{14}$$

the focusing strength transforms:

$$K \mapsto \frac{K}{1 + \delta} \approx (1 - \delta)K, \tag{15}$$

and the transfer matrix transforms:

$$M \mapsto \begin{pmatrix} 1 & 0 \\ -(1 - \delta)K & 1 \end{pmatrix} \approx \begin{pmatrix} 1 & 0 \\ -K & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ \delta \cdot K & 1 \end{pmatrix}. \tag{16}$$

The effect of the energy deviation can be represented by inserting a thin quadrupole alongside each real quadrupole in the lattice.

This affects the betatron tune, as we shall now show.

Consider the single-turn matrix starting just after a given quadrupole in a storage ring. We write the matrix in the standard form:

$$R = \begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{pmatrix}, \tag{17}$$

where α, β, γ are the Twiss parameters.

μ gives the phase advance around the storage ring, i.e. the rotation angle in phase space when a particle makes one turn of the ring.

Note that the eigenvalues of R are:

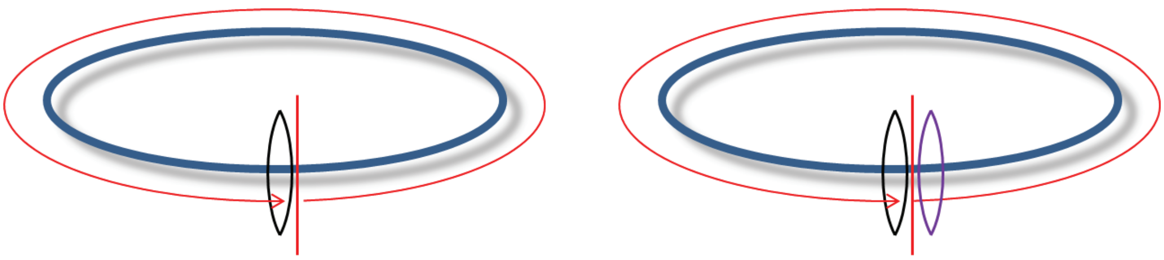
$$\lambda_{\pm} = e^{\pm i\mu}. \tag{18}$$

This gives a way of finding the phase advance from any given single-turn transfer matrix.

The single-turn transfer matrix is just the product of the transfer matrices for all successive elements in the storage ring.

Therefore, in the presence of a single focusing error (of strength dK) at the chosen quadrupole, the single-turn matrix becomes:

$$R' = \begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -dK & 1 \end{pmatrix}. \tag{19}$$



The phase advance in the presence of the focusing error can be found from the eigenvalues of R'.

After some algebra, we find:

$$\mu' = \mu + d\mu, \tag{20}$$

where:

$$d\mu \approx \frac{1}{2}\beta\, dK. \tag{21}$$

Chromaticity

In the case of a focusing error arising from an energy error on a particle moving through the lattice, we would have:

$$dK = -\delta \cdot K. \tag{22}$$

To take account of the effect of focusing errors on *all* the quadrupoles (which would arise from the energy deviation), we simply integrate around the ring:

$$\Delta\mu = -\frac{1}{2}\oint \beta\, \delta \cdot k_1\, ds. \tag{23}$$

where:

$$k_1 = \frac{q}{P_0}\frac{\partial B_y}{\partial x}. \tag{24}$$

P_0 is the reference momentum, and $\Delta\mu$ is the change in phase advance with respect to a particle with the reference momentum (i.e. zero energy deviation).

The *linear chromaticity* ξ is defined as the first-order derivative of the betatron tune ν as a function of the energy deviation δ.

Since $\mu = 2\pi\nu$, we can write for the horizontal chromaticity:

$$\xi_x = -\frac{1}{4\pi}\oint \beta_x k_1\, ds. \qquad (25)$$

Since horizontally focusing quadrupoles are vertically defocusing, and *vice versa*, the vertical chromaticity is:

$$\xi_y = \frac{1}{4\pi}\oint \beta_y k_1\, ds. \qquad (26)$$

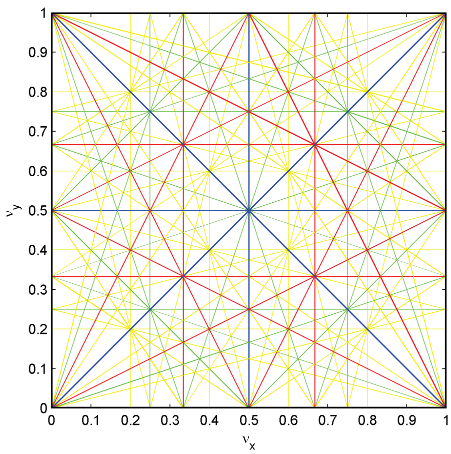
In any lattice, the beta function will tend to reach its largest values in focusing quadrupoles (horizontally, $k_1 > 0$), and its smallest values in defocusing quadrupoles (horizontally, $k_1 < 0$).

Therefore, the *natural chromaticity* (i.e. the linear chromaticity without any correction by sextupoles) will always be negative.

Resonances occur when the tunes satisfy:

$$m\nu_x + n\nu_y = \ell, \qquad (27)$$

for integer values of m, n and ℓ.



The natural chromaticity of a storage ring can easily be large enough that the tunes of particles with even modest energy deviation can hit integer or half-integer resonances. This can lead to rapid loss of particles from the beam.

Also, certain beam instabilities (collective effects) are sensitive to the chromaticity. Operating with a chromaticity that is close to zero, or even slightly positive, can increase the limit on the amount of current that can be stored in the ring before the beam becomes unstable.

Fortunately, there is a (relatively) easy way to control the chromaticity in a storage ring, even with a fixed linear lattice design.

Particles with an energy deviation δ oscillate around a closed orbit that is displaced from the closed orbit for $\delta = 0$ by a distance:

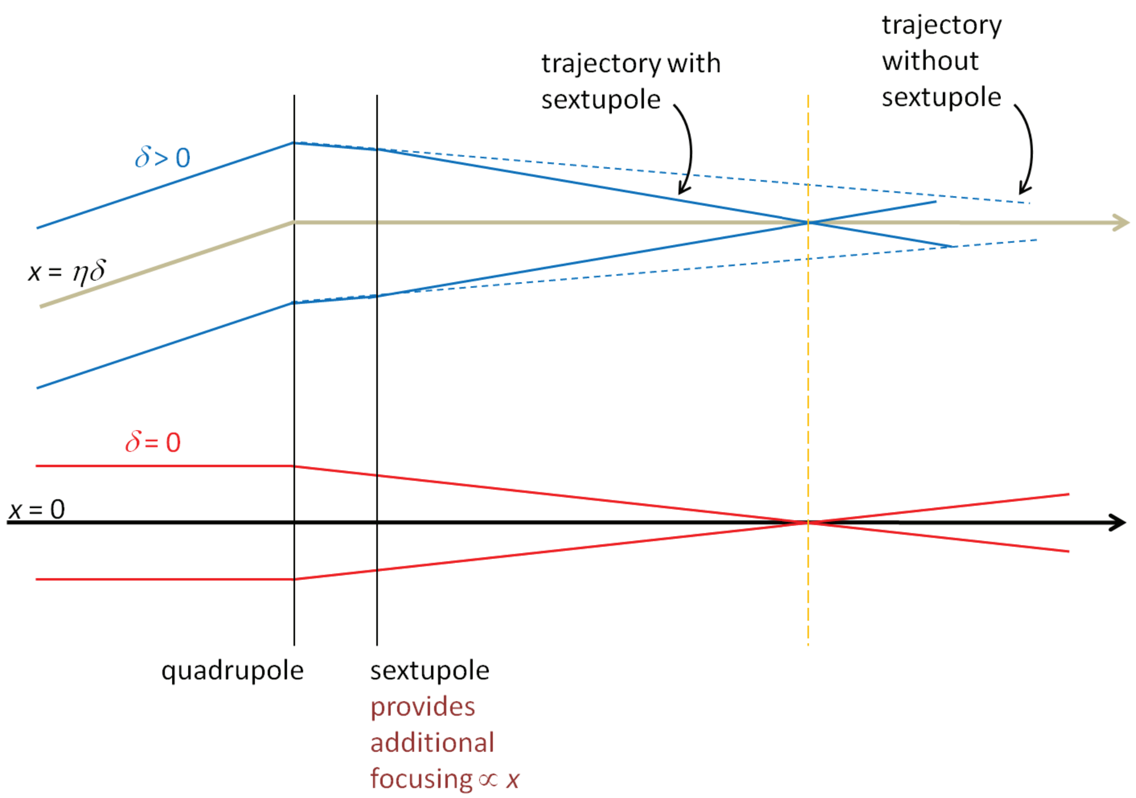
$$x = \eta\delta, \tag{28}$$

where η is the dispersion.

In a sextupole magnet, the focusing strength varies with horizontal position:

$$k_2 = \frac{q}{P_0}\frac{\partial^2 B_y}{\partial x^2}, \qquad \frac{q}{P_0}\frac{\partial B_y}{\partial x} = xk_2. \tag{29}$$

Locating sextupoles where the dispersion is large allows us to provide additional focusing for off-energy particles, to compensate the chromaticity of the quadrupoles.



Combining equations (28) and (29), we find that the linear focusing provided by a sextupole is:

$$k_{1,\text{sext}} = \eta\delta \cdot k_2. \qquad (30)$$

Notice that this has a direct dependence on the energy deviation δ.

We can treat the focusing from sextupoles as a perturbation, in the same way as we did the focusing error from the energy deviation of a particle in a quadrupole.

Then, the total linear chromaticity, including quadrupoles and sextupoles is:

$$\xi = -\frac{1}{4\pi}\oint \beta \left(k_1 - \eta k_2\right) \, ds. \qquad (31)$$

Strictly speaking, equation (31) applies to the horizontal motion:

$$\xi_x = -\frac{1}{4\pi}\oint \beta_x \left(k_1 - \eta_x k_2\right) \, ds. \qquad (32)$$

But we can derive a similar expression for the vertical chromaticity, using the same arguments:

$$\xi_y = \frac{1}{4\pi}\oint \beta_y \left(k_1 - \eta_x k_2\right) \, ds. \qquad (33)$$

Note that β_y is largest in vertically focusing quadrupoles, $k_1 < 0$; so the natural vertical chromaticity in a storage ring is always negative (like the natural horizontal chromaticity).

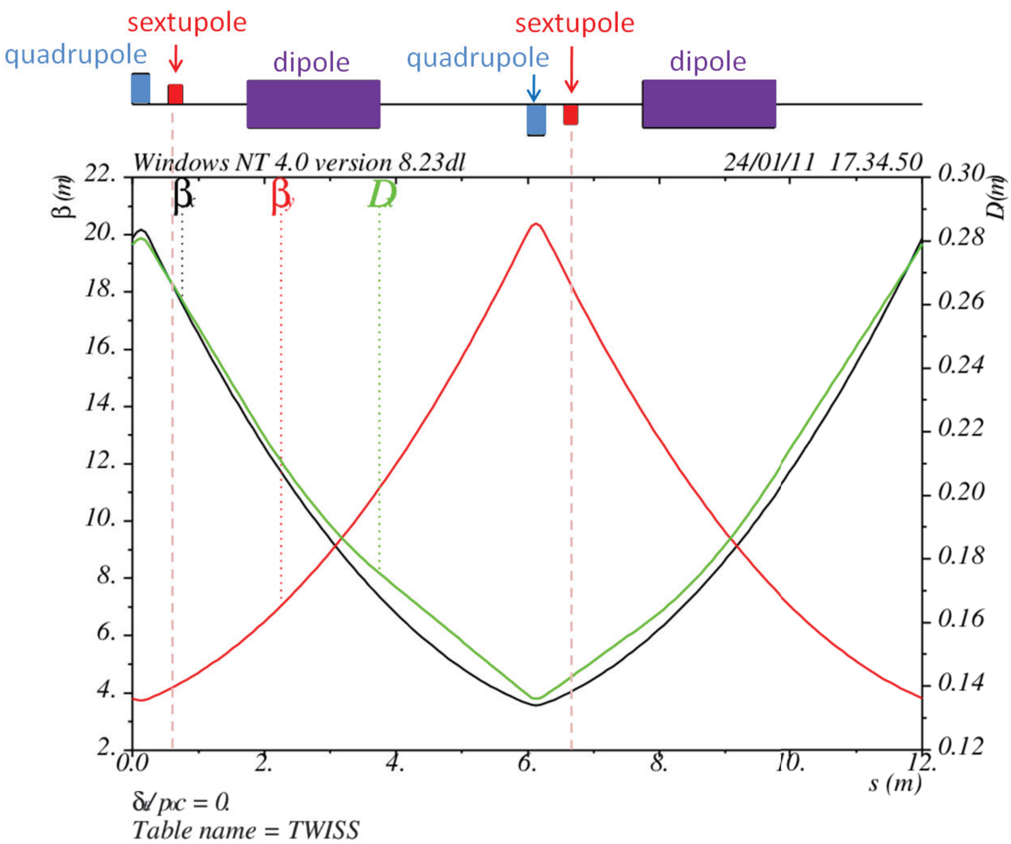
Also note that sextupoles used to cancel the horizontal chromaticity will tend to make the vertical chromaticity *more negative*, and vice versa.

However, the chromatic effect of a sextupole depends on the beta function, as well as on the dispersion and the strength of the sextupole.

By locating sextupoles with $k_2 > 0$ where β_x is large and β_y is small, we can correct the horizontal chromaticity with relatively little impact on the vertical chromaticity.

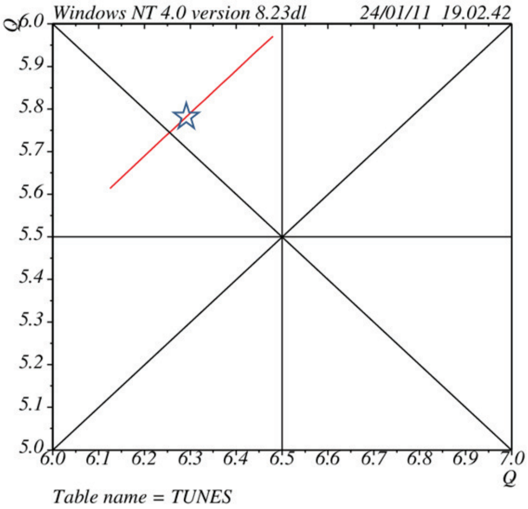
Similarly, by locating sextupoles with $k_2 < 0$ where β_y is large and β_x is small, we can correct the vertical chromaticity with relatively little impact on the horizontal chromaticity.

Example: correcting chromaticity in a FODO lattice

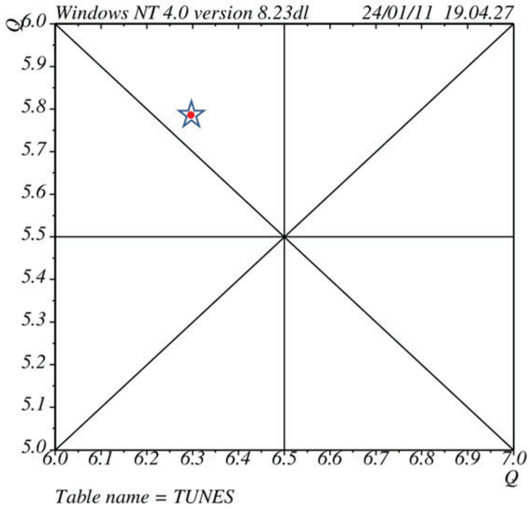


Example: correcting chromaticity in a FODO lattice

Tune variation in a 24-cell FODO lattice, with energy deviation from -2.5% to +2.5%:

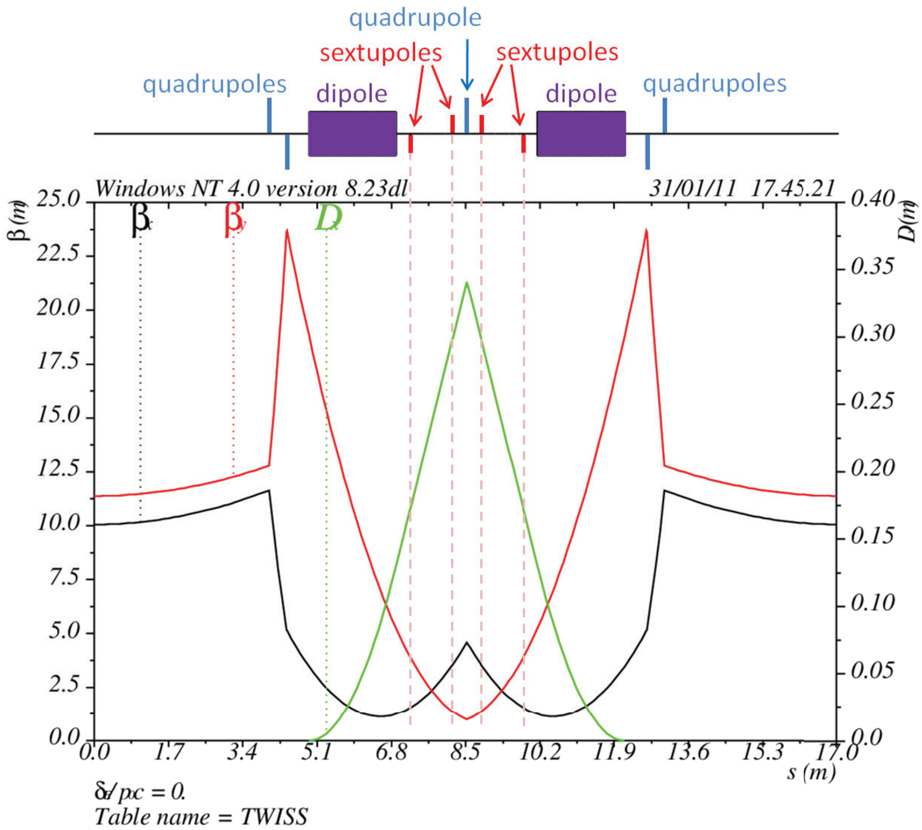


Without sextupoles.



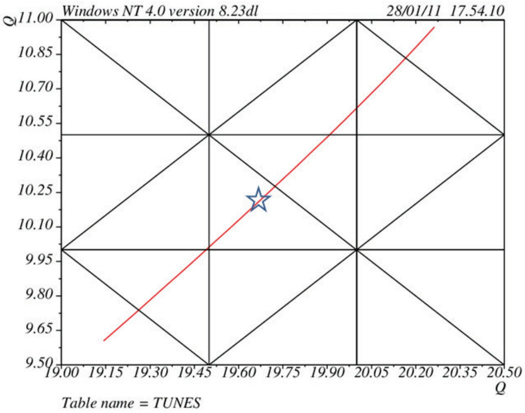
With sextupoles.

Example: correcting chromaticity in a DBA lattice

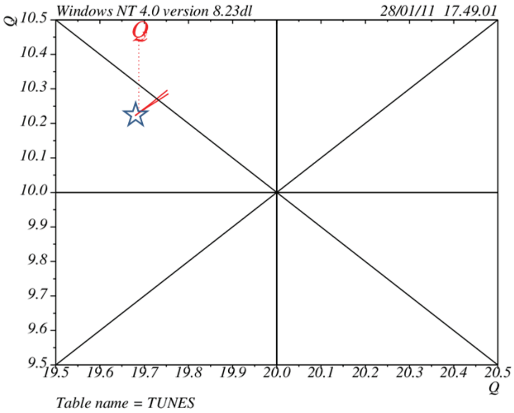


Example: correcting chromaticity in a DBA lattice

Tune variation in a 24-cell DBA lattice, with energy deviation from -2.0% to +2.0%:



Without sextupoles.



With sextupoles.

Notice that even with sextupoles tuned to give zero chromaticity, there are significant changes in tune with energy, because of the higher-order chromaticity.

Adverse effects of sextupoles: dynamic aperture

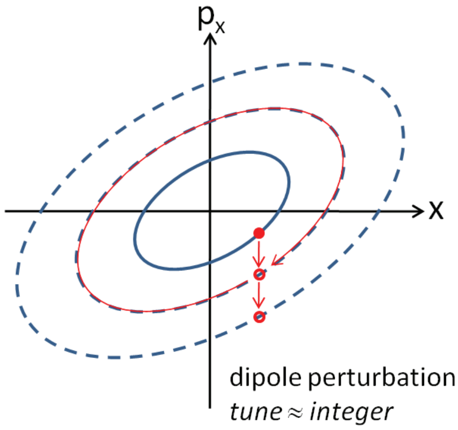
Sextupoles are necessary for correcting the dynamics of particles which do not have exactly the energy for which the lattice is designed.

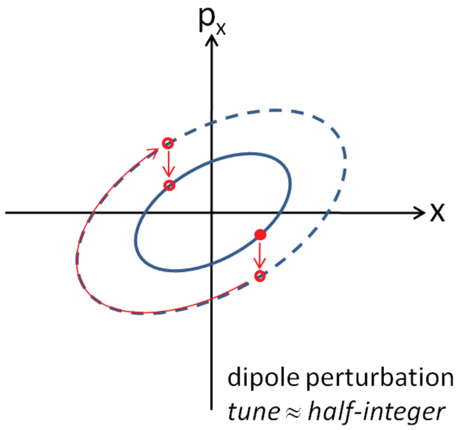
Unfortunately, because the fields in sextupoles are nonlinear, the sextupoles have "side effects" for particles that may have the right energy, but are performing betatron oscillations (i.e. are not following a closed orbit).

To understand the possible impact, let us calculate the change in the betatron action resulting from a series of sextupole "kicks" as a particle performs multiple turns around the ring.

The effects of different kinds of perturbation (dipole, quadrupole, sextupole...) can be understood by considering motion of a particle in phase space.

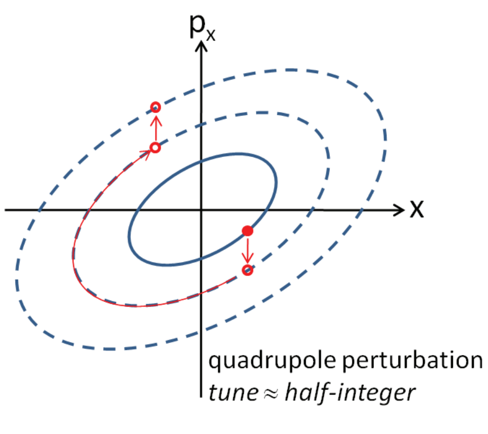
For example, with a dipole perturbation (Δp_x independent of x), we can see that the betatron amplitude of a particle increases rapidly if the tune is near an integer, but only slowly near a half-integer.

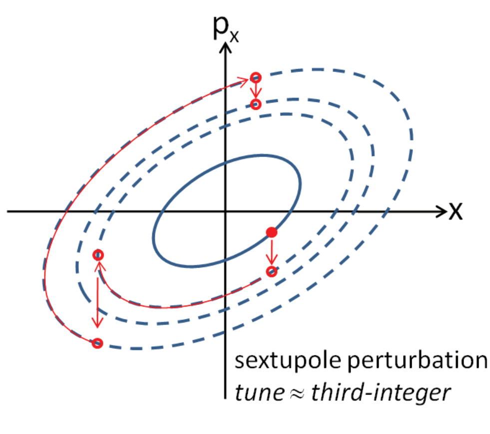




However, if the tune is near a half-integer, a quadrupole perturbation ($\Delta p_x \propto x$) leads to rapid growth in betatron amplitude.

Similarly, a sextupole perturbation ($\Delta p_x \propto x^2$) leads to a rapid growth in betatron amplitude if the tune is close to a third-integer.





We can quantify the effects of sextupoles more precisely in terms of the rate of change of the betatron action.

Under linear symplectic transport, the betatron action J_x (that characterises the amplitude of a betatron oscillation) is constant:

$$2J_x = \gamma_x x^2 + 2\alpha_x x p_x + \beta_x p_x^2. \tag{34}$$

When a particle passes through a sextupole, it receives a transverse kick:

$$\Delta p_x = -\frac{1}{2} k_2 x^2 \, \Delta s. \tag{35}$$

The corresponding change in the action is:

$$\Delta J_x = -\frac{1}{2} k_2 \left(\alpha_x x^3 + \beta_x x^2 p_x \right) \Delta s. \tag{36}$$

For simplicity, let us assume that α_x is small (i.e. the beta function varies slowly around the ring).

Then, using:

$$x = \sqrt{2\beta_x J_x} \cos \phi_x, \tag{37}$$

$$p_x \approx -\sqrt{\frac{2J_x}{\beta_x}} \sin \phi_x, \tag{38}$$

we find:

$$\frac{dJ_x}{ds} \approx \frac{1}{8} k_2 \left(2\beta_x J_x \right)^{\frac{3}{2}} \left(\sin \phi_x + \sin 3\phi_x \right). \tag{39}$$

Since β_x and k_2 are periodic functions of s (with period equal to the circumference of the ring, C), we can write:

$$\beta_x^{3/2} k_2 = \sum_n \tilde{k}_{2,n} e^{-i[\psi(\theta)+n\theta]} \tag{40}$$

where:

$$\theta = 2\pi \frac{s}{C}, \tag{41}$$

and the Fourier amplitudes $\tilde{k}_{2,n}$ are given by:

$$\tilde{k}_{2,n} = \frac{1}{C} \int_0^C \beta_x^{3/2} k_2 \, e^{i[\psi(\theta)+n\theta]} \, d\theta. \tag{42}$$

$\psi(\theta)$ is any function with the same periodicity in s as β_x and k_2. The reason for introducing this function will become clear shortly.

Substituting the Fourier decomposition (40) into equation (39) gives:

$$\frac{dJ_x}{ds} \approx \frac{1}{8} (2J_x)^{\frac{3}{2}} \sum_n \tilde{k}_{2,n} e^{-i[\psi(\theta)+n\theta]} (\sin\phi_x + \sin 3\phi_x). \tag{43}$$

Note that there are two "oscillating" factors in the right hand side: one representing the variation of k_2 (weighted by the beta function), and another representing the phase advance.

If the frequencies of these two oscillations are different, then they combine to give a rapid oscillation, which averages to zero: the average rate of change of the action will be small.

But for particular cases, the frequency of the variation of k_2 can *resonate* with the phase advance: in such cases, if $\tilde{k}_{2,n}$ is large, the action can be quickly driven to very large values.

Inspecting equation (43):

$$\frac{dJ_x}{ds} \approx \frac{1}{8}(2J_x)^{\frac{3}{2}} \sum_n \tilde{k}_{2,n} e^{-i[\psi(\theta)+n\theta]} (\sin\phi_x + \sin 3\phi_x),$$

we see that resonance with the term containing $\sin\phi_x$ occurs when:

$$\psi(\theta) = \mu_x(s) - \nu_x\theta, \tag{44}$$

and n is the integer closest to ν_x.

The Fourier coefficient in this case:

$$\tilde{k}_{2,n} = \frac{1}{C}\int_0^C \beta_x^{3/2} k_2 \, e^{i[\mu_x(s)-\nu_x\theta+n\theta]} \, d\theta, \tag{45}$$

represents the strength of a driving term for an *integer resonance*, since this term has the largest impact when ν_x is an integer.

Again inspecting equation (43):

$$\frac{dJ_x}{ds} \approx \frac{1}{8}(2J_x)^{\frac{3}{2}} \sum_n \tilde{k}_{2,n} e^{-i[\psi(\theta)+n\theta]} (\sin\phi_x + \sin 3\phi_x),$$

we see that resonance with the term containing $\sin 3\phi_x$ occurs when:

$$\psi(\theta) = 3(\mu_x(s) - \nu_x\theta), \tag{46}$$

and n is the integer closest to $3\nu_x$.

The Fourier coefficient in this case:

$$\tilde{k}_{2,n} = \frac{1}{C}\int_0^C \beta_x^{3/2} k_2 \, e^{i[3(\mu_x(s)-\nu_x\theta)+n\theta]} \, d\theta, \tag{47}$$

represents the strength of a driving term for a *third integer resonance*, since this term has the largest impact when $3\nu_x$ is an integer.

Note that there are two conditions for the action of a particle to be driven to large values by the sextupoles in a lattice:

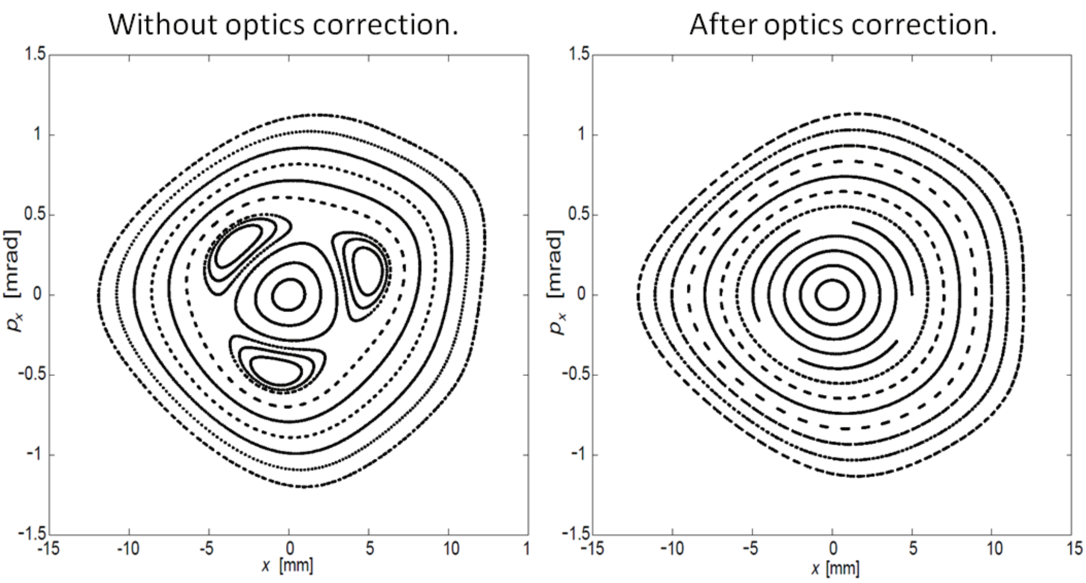
1. The tune of the lattice must be close to an integer, or a third integer.

2. The resonant driving term must be significantly large.

Usually, of course, we wish to *avoid* particles reaching large betatron amplitudes.

If a lattice is not carefully designed to avoid both the conditions above, then it is quite likely that trajectories with even small initial amplitudes rapidly become unstable.

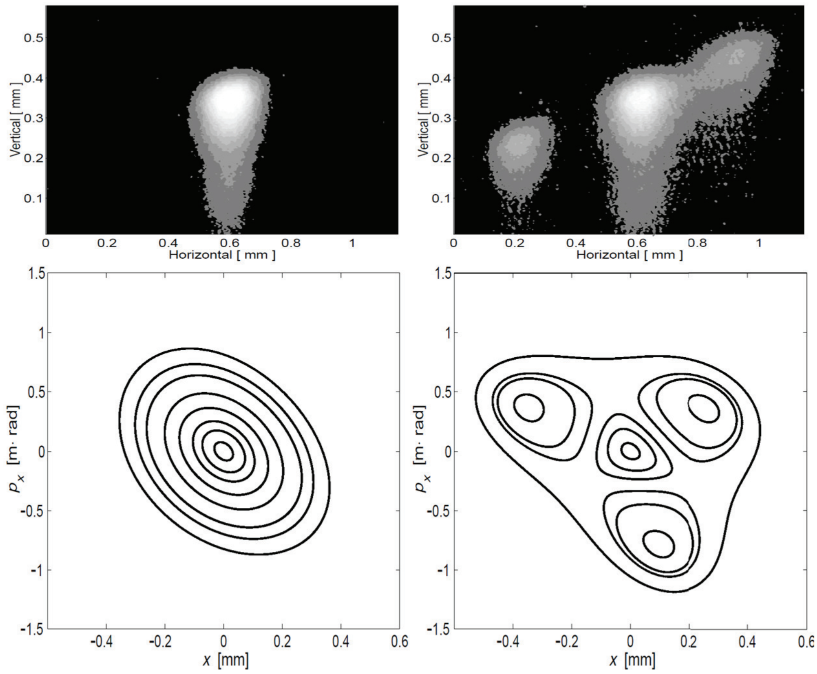
Such trajectories are said to be outside the *dynamic aperture* of the lattice.



Horizontal phase space in the ALS, close to a third-integer resonance, produced by tracking in a model of the lattice.

D. Robin, J. Safranek, W. Decking, "Realizing the benefits of restored periodicity in the Advance Light Source," PRST-AB, 2-044001 (1999).

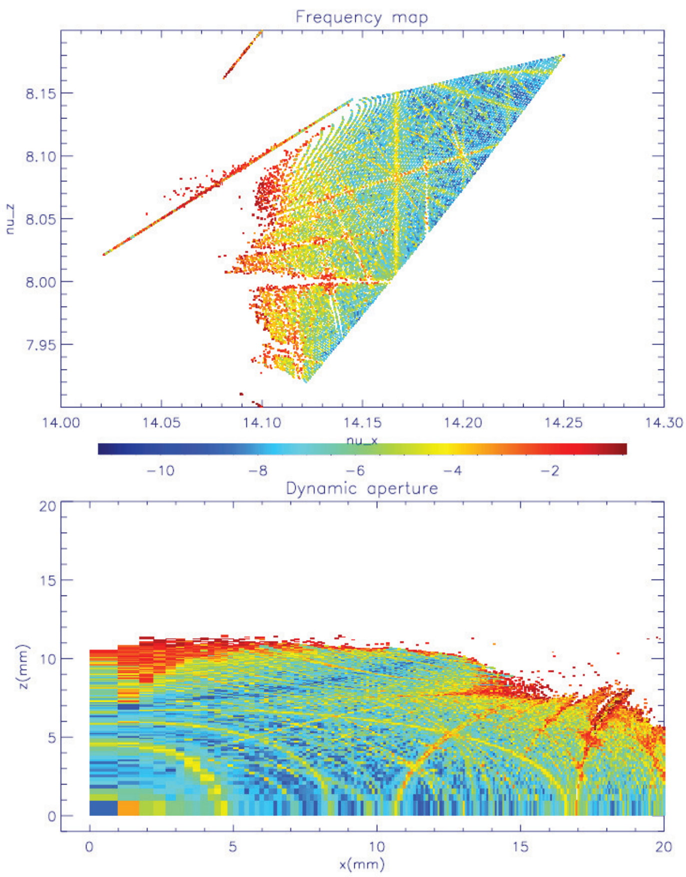
Resonances: effects in phase space



Effect of third-integer resonance on beam distribution. Top: SR light monitor images. Bottom: (simulated) horizontal phase space.

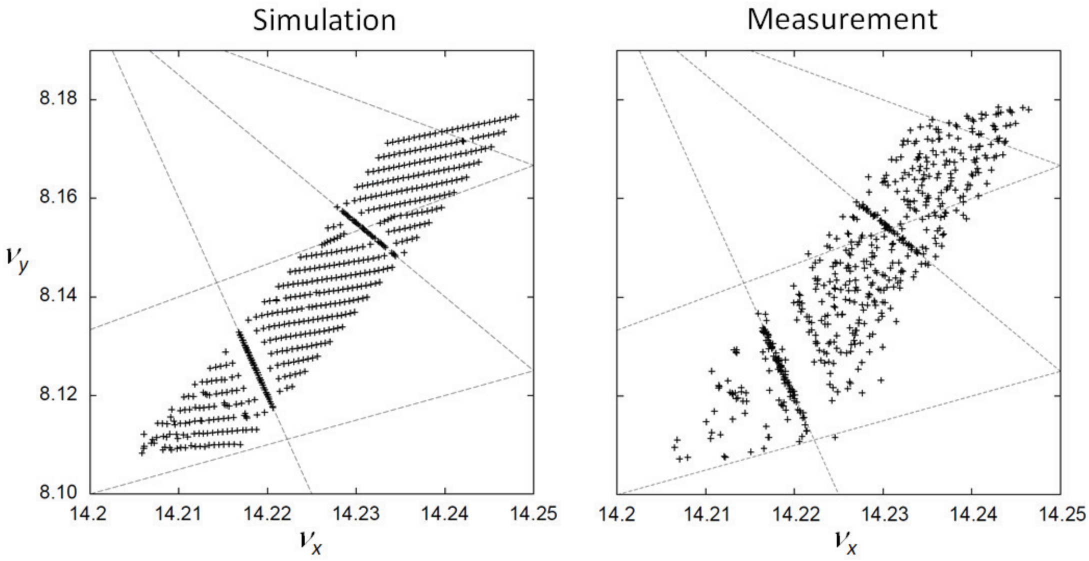
D. Robin, J. Safranek, W. Decking, "Realizing the benefits of restored periodicity in the Advance Light Source," PRST-AB, 2-044001 (1999).

Frequency map analysis of particle dynamics in the ALS



C. Steier, D. Robin, J. Laskar, L. Nadolski, "Lattice model calibration and frequency map measurements at the ALS," Proc. EPAC2000, Vienna, Austria (2000).

D. Robin, C. Steier, J. Laskar, L. Nadolski, "Global dynamics of the Advanced Light Source revealed through experimental frequency map analysis," Phys. Rev. Lett. 85, 3, pp. 558-561 (2000).



C. Steier, D. Robin, J. Laskar, L. Nadolski, "Lattice model calibration and frequency map measurements at the ALS," Proc. EPAC2000, Vienna, Austria (2000).

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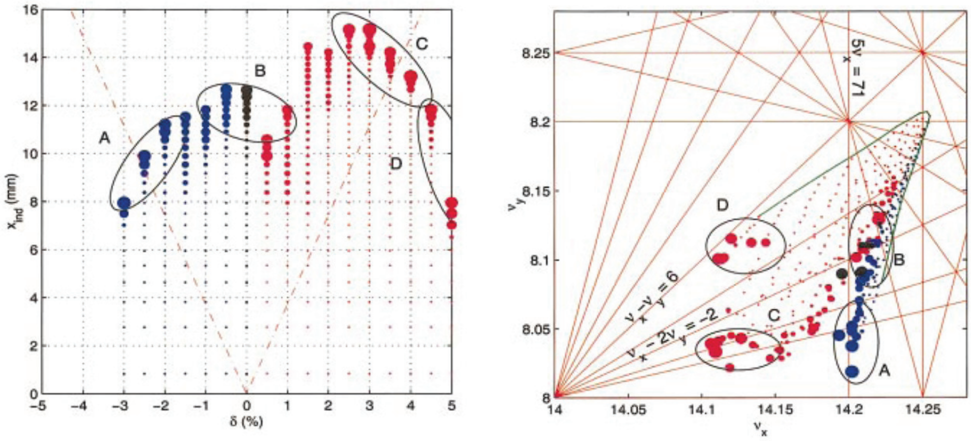
The dynamic aperture depends on the energy deviation.

The range of energy deviation over which the (transverse) dynamic aperture is non-zero is called the *dynamic energy acceptance*.

The energy acceptance is of significant importance in (low emittance) storage rings for light sources, because it plays a major role in determining the beam lifetime, which in turn is one of the major performance metrics.

Usually, the dynamic energy acceptance is determined using tracking simulations; but it can also be explored using experimental techniques.



Left: Particle loss as a function of energy deviation (horizontal axis) and horizontal kick amplitude (vertical axis). Right: corresponding points in tune space.

C. Steier, D. Robin, L. Nadolski, W. Decking, Y. Wu, J. Laskar, "Measuring and optimizing the momentum aperture in a particle accelerator," Phys. Rev. E, 65, 056506 (2002).

If the energy deviation of a particle in a storage ring becomes too large, then the particle will be lost from the beam. The *energy acceptance* is the maximum energy deviation that a particle can have and remain stored within the ring.

The energy acceptance of a storage ring is limited by two effects:

1. RF acceptance: at large energy deviations, the RF voltage becomes insufficient to restore the particle energy.

2. Dynamic acceptance: the dynamic aperture generally shrinks with energy deviation, because of chromatic and nonlinear effects.

In practice, the energy acceptance is the smaller of the RF and dynamic acceptance.

Achieving a good energy acceptance is essential for achieving a good beam lifetime, as we shall discuss shortly.

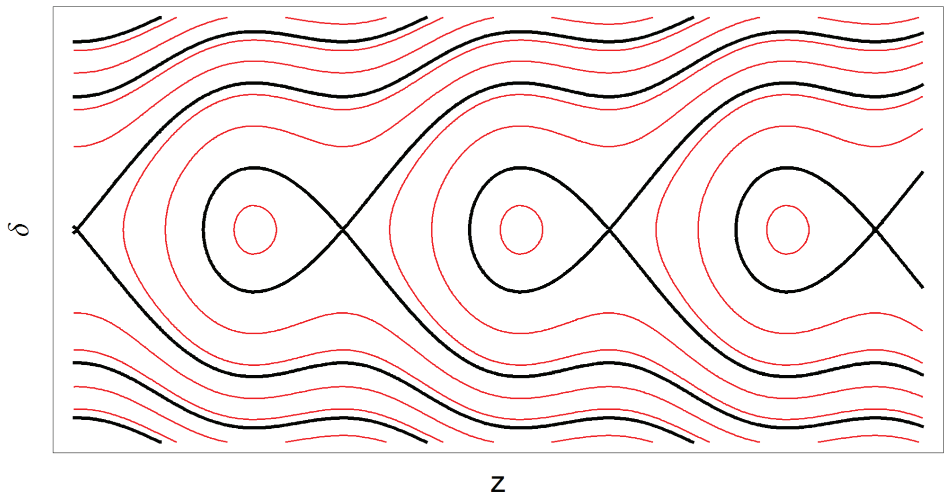
The RF acceptance is determined by parameters including the RF voltage and frequency, momentum compaction factor, and the energy loss per turn.

Recall the equations for longitudinal motion:

$$\frac{dz}{ds} = -\alpha_p \delta, \tag{48}$$

$$\frac{d\delta}{ds} = \frac{qV_{\text{RF}}}{C_0 P_0 c} \left[\sin(\phi_s - kz) - \sin\phi_s \right]. \tag{49}$$

Solving these equations of motion gives a characteristic phase space portrait...



There are distinct regions of stable motion (lines in closed loops) and unstable motion (lines extending to infinity).

Regions of stable motion are bounded by the *separatrices*. Separatrices intersect at unstable fixed points.

The equations of motion can be derived from a Hamiltonian:

$$H = -\frac{\alpha_p}{2}\delta^2 - \frac{qV_{\mathrm{RF}}}{kC_0P_0c}\left[\cos(\phi_s - kz) - kz\sin\phi_s\right]. \qquad (50)$$

On any line in the phase space portrait, the value of H is constant. This provides a way to determine the energy acceptance:

1. Determine the phase space coordinates of an unstable fixed point, from the condition $dz/ds = d\delta/ds = 0$.

2. Substitute these coordinates into the Hamiltonian, to find the value H_s of the Hamiltonian on a separatrix.

3. Determine the energy deviation for which $d\delta/ds = 0$, subject to the constraint that the Hamiltonian takes the value H_s.

The result is that the RF acceptance of a storage ring is given by:

$$|\delta|_{\mathrm{max,RF}} = \frac{2\nu_s}{h\alpha_p}\sqrt{1 - \left(\frac{\pi}{2} - \phi_s\right)\tan\phi_s}, \qquad (51)$$

where ϕ_s is the synchronous phase, given by:

$$\phi_s = \pi - \sin^{-1}\left(\frac{U_0}{eV_{RF}}\right), \qquad (52)$$

and ν_s is the synchrotron tune:

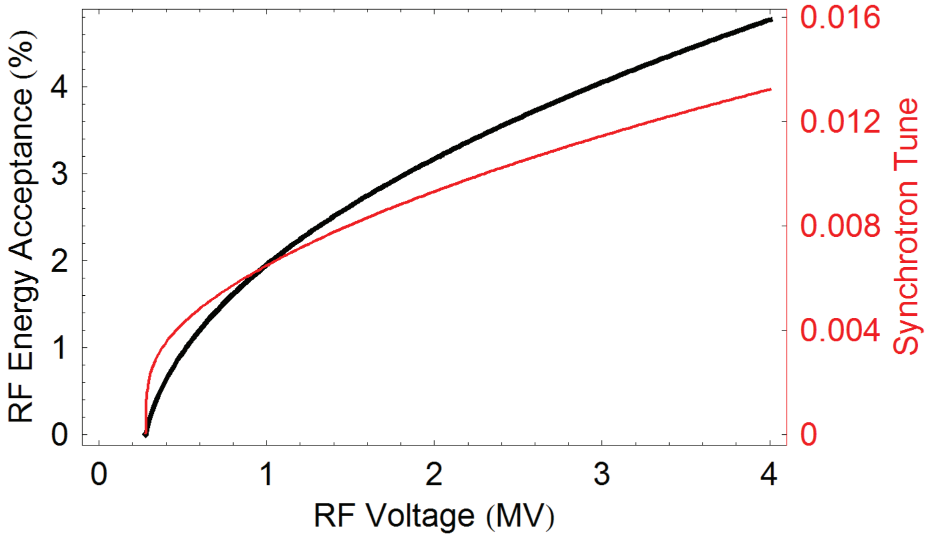
$$\nu_s = \sqrt{-\frac{eV_{RF}}{E_0}\frac{h\alpha_p}{2\pi}\cos\phi_s}. \qquad (53)$$

V_{RF} is the RF voltage; E_0 the beam energy; U_0 the energy loss per turn; h the harmonic number; and α_p is the momentum compaction factor.

RF energy acceptance: example

Based on parameters (similar to) the ALS:

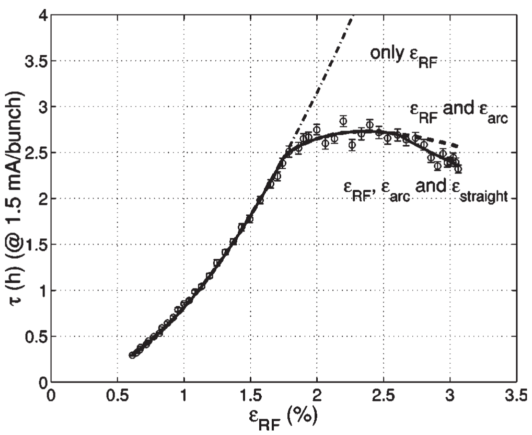
Ring circumference	196.8 m
Beam energy	1.9 GeV
RF frequency	500 MHz
Momentum compaction factor	1.6×10^{-3}
Energy loss per turn	280 keV



RF energy acceptance

The RF system of an electron storage ring is usually specified to provide an energy acceptance of several percent.

Increasing the RF energy acceptance beyond a few percent rarely provides any benefits, because the energy acceptance is then limited by dynamic effects.



Left: Touschek lifetime in the ALS as a function of the RF acceptance.

C. Steier, D. Robin, L. Nadolski, W. Decking, Y. Wu, J. Laskar, "Measuring and optimizing the momentum aperture in a particle accelerator," Phys. Rev. E, 65, 056506 (2002).

The dynamic energy acceptance is best found by particle tracking in an accelerator modelling code.

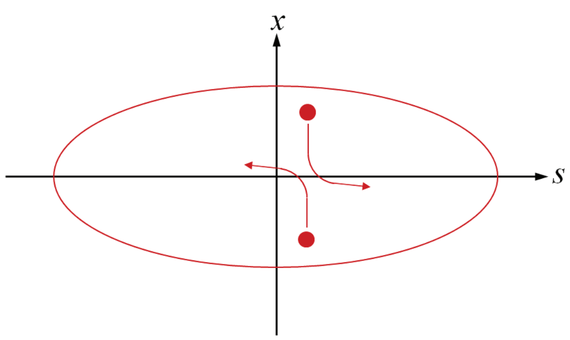
For a detailed analysis, particles are launched on the closed orbit at different locations around the ring. At each initial location, the particles are assigned a range of energy deviations (to represent particles following a Touschek scattering event).

The particles are tracked for a number of turns corresponding to one or more synchrotron radiation damping times. Those particles that exceed some specified bound are assumed to be lost from the beam.

A thorough analysis of the dynamic energy acceptance, computed at small intervals around the ring circumference, can be computationally expensive.

Touschek scattering is generally the dominant lifetime limitation in low emittance storage rings (for example, in third generation synchrotron light sources).

Particles within a bunch are continually making betatron and synchrotron oscillations. If two particles within a bunch collide, there can be a large momentum transfer from the transverse to the longitudinal directions.



If the energy deviation of either particle following the collision is outside the energy acceptance of the storage ring, then the particle will be lost from the beam.

A proper analysis of Touschek scattering is complex, so here we simply quote the standard formula for the beam lifetime:

$$\frac{1}{\tau} = -\frac{1}{N}\frac{dN}{dt} = \frac{r_e^2 cN}{8\pi\sigma_x\sigma_y\sigma_z}\frac{1}{\gamma^2|\delta|^3_{\mathrm{max}}} \cdot D(\theta^2), \tag{54}$$

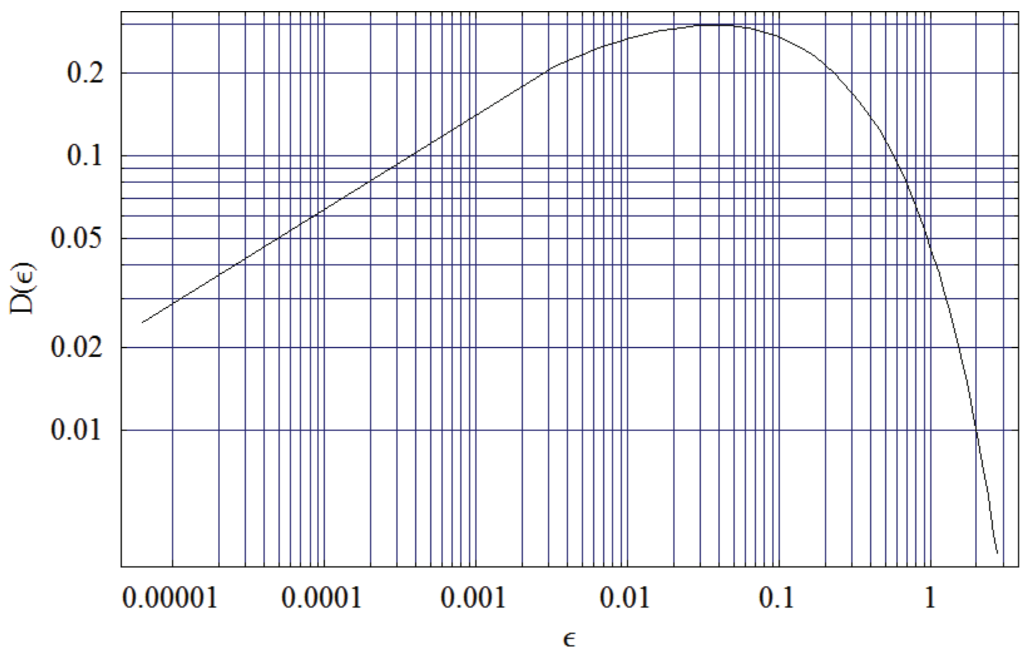
where:

$$\theta = \frac{|\delta_{\mathrm{max}}|\beta_x}{\gamma\sigma_x}, \tag{55}$$

and:

$$D(\epsilon) = \sqrt{\epsilon}\left[-\frac{3}{2}e^{-\epsilon} + \frac{\epsilon}{2}\int_\epsilon^\infty \frac{\ln u}{u}e^{-u}\,du + \frac{1}{2}(3\epsilon - \epsilon\ln\epsilon + 2)\int_\epsilon^\infty \frac{e^{-u}}{u}\,du\right]. \tag{56}$$

Note that the energy acceptance $|\delta|_{\mathrm{max}}$ is the smaller of the dynamic acceptance (which can vary around the ring) and the RF acceptance.



Note that:

- The decay rate is proporational to the bunch population. The decay is not exactly exponential: higher current means shorter lifetime.
- Neglecting the dependence on $D(\theta)$, the decay rate is inversely proportional to the bunch volume, $\sigma_x\sigma_y\sigma_z$. The Touschek lifetime is shorter in rings with lower emittance. Sometimes, third-harmonic cavities are used to "flatten" the RF focusing, and increase the bunch length to improve the lifetime without compromising the brightness.
- The lifetime is proportional to the square of the beam energy (but the cost of the storage ring increases with energy).
- There is a strong scaling of the lifetime with the energy acceptance: neglecting the dependence on $D(\theta)$ the lifetime increases with the *cube* of the energy acceptance.

- Chromaticity (dependence of the optics on particle energy) is an intrinsic property of accelerator beamlines.
- In a storage ring, chromaticity must be controlled so that the trajectories of particles with significant energy deviation do not cross harmful resonances in tune space.
- Sextupoles provide an effective way for controlling chromaticity in storage rings. In low-emittance rings, strong sextupoles are needed that produce strong nonlinear "side-effects".
- Understanding and controlling the nonlinear effects of magnets in storage rings is necessary for achieving good beam lifetime.