Tutorial 4

These questions cover Parts 8 and 9 of the Lecture Course.

Please give your solutions to your tutor by noon on Wednesday 7th May.

Both of these questions are from the 2007 exam.

Question 1

a) Explain with the aid of a diagram what is meant by a *retarded potential*, and why it is necessary to use retarded potentials in evaluating the fields from time-dependent electromagnetic sources.

b) A Hertzian dipole with dipole moment $\vec{p} = p_0 \exp(j\omega t)\,\hat{z}$ is situated in free space at the centre of a spherical coordinate system. The vector potential $\vec{A}$ at the field point $P(r,\theta,\phi)$ at time t has components (in spherical polar coordinates) given by:

$$A_r(r,\theta,\phi) = A\cos\theta$$

$$A_\theta(r,\theta,\phi) = -A\sin\theta$$

$$A_\phi(r,\theta,\phi) = 0$$

where

$$A(r,\theta,\phi) = \frac{j\omega\mu_0 p_0}{4\pi}\,\frac{\exp\, j(\omega t - kr)}{r}$$

$k = \dfrac{\omega}{c} = \omega\sqrt{\mu_0\varepsilon_0}$, and μ_0 and ε_0 are, respectively, the permeability and permittivity of free space.

Show that the azimuthal component of the magnetic intensity is given by:

$$H_\phi = -\frac{k^2 c p_0}{4\pi}\left(1 - \frac{j}{kr}\right)\frac{\exp\, j(\omega t - kr)}{r}\sin\theta$$

[You are given that, in the usual notation:

$$\nabla \times \vec{F} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial\phi} \\ F_r & rF_\theta & r\sin\theta F_\phi \end{vmatrix}$$

]

Question 1 *(continued)*

c) The other components of the electric and magnetic fields at point P are given by:

$$H_r = H_\theta = E_\phi = 0$$

$$E_r = \frac{jkp_0}{4\pi\varepsilon_0}\left(1-\frac{j}{kr}\right)2\frac{\exp j(\omega t - kr)}{r^2}\cos\theta$$

$$E_\theta = -\frac{k^2 p_0}{4\pi\varepsilon_0}\left(1-\frac{j}{kr}-\frac{1}{k^2r^2}\right)\frac{\exp j(\omega t - kr)}{r}\sin\theta$$

Show that the dominant fields in the 'far field' region ($kr >> 1$) are given by:

$$E_\theta = -\frac{k^2 p_0}{4\pi\varepsilon_0}\frac{\exp j(\omega t - kr)}{r}\sin\theta$$

$$H_\phi = -\frac{k^2 c p_0}{4\pi}\frac{\exp j(\omega t - kr)}{r}\sin\theta$$

Discuss briefly the spatial dependence and relative phases of the field components in the far field region.

d) Use the above expressions to show that the time-average Poynting vector $\langle \vec{S} \rangle$ at the field point P is given by

$$\langle \vec{S} \rangle = \frac{k^4 c p_0^2}{32\pi^2\varepsilon_0}\frac{\sin^2\theta}{r^2}\hat{r}\,.$$

e) Sketch the radiation power distribution of the dipole in the plane $y = 0$, and comment on how this distribution is modified for a real half-wave ($\lambda/2$) dipole.

Question 2

The Lorentz transformation of a four-vector p^μ can be written:

$$p'^\nu = \Lambda^\nu_\mu p^\mu$$

where p'^ν represents the four-vector when measured in a frame moving with constant velocity u along the x-axis relative to the frame in which p^μ is measured, and $\Lambda^\nu{}_\mu$ can be written as a matrix:

$$\Lambda^\nu_\mu = \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix}$$

a) Define the quantities β and γ appearing in the matrix form of $\Lambda^\nu{}_\mu$, in terms of the relative velocity u between the two frames.

Question 2 *(continued)*

b) A region of space has electric charge density ρ and current density $\vec{J}$. Show that under the Lorentz transformation given above, the equation:

$$\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$$

is invariant. Comment on the physical significance of this equation and on the significance of its invariance under Lorentz transformations.

c) The electromagnetic field $F^{\mu\nu}$ may be derived from the potential four-vector A^{μ} by:

$$F^{\mu\nu} = \partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu}.$$

Show that, in matrix form, $F^{\mu\nu}$ may be written:

$$F^{\mu\nu} = \begin{pmatrix} 0 & B_z & -B_y & -E_x/c \\ -B_z & 0 & B_x & -E_y/c \\ B_y & -B_x & 0 & -E_z/c \\ E_x/c & E_y/c & E_z/c & 0 \end{pmatrix}$$

where the electric field is $\vec{E} = (E_x, E_y, E_z)$ and the magnetic field is $\vec{B} = (B_x, B_y, B_z)$.

d) Maxwell's equations relating the electric and magnetic fields to a charge and current density in a non-magnetic material with relative permittivity $\varepsilon_r = 1$ are:

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \qquad \nabla \times \vec{B} = \mu_0 \vec{J} + \frac{1}{c}\dot{\vec{E}} \cdot$$

Show that these equations may be derived from the four-vector relationship:

$$\partial_{\mu} F^{\mu\nu} = -\mu_0 J^{\nu}.$$

e) The Lorentz transformations for the components of the electric field $\vec{E}$ and the magnetic field $\vec{B}$ are:

$$E'_x = E_x \qquad B'_x = B_x$$

$$E'_y = \gamma(E_y - uB_z) \qquad B'_y = \gamma\left(B_y + \frac{u}{c^2}E_z\right)$$

$$E'_z = \gamma(E_z + uB_y) \qquad B'_z = \gamma\left(B_z - \frac{u}{c^2}E_y\right)$$

A neutral hydrogen atom moves with kinetic energy 50 keV in the laboratory frame. Suppose that the atom enters a magnetic field of strength 0.5 T perpendicular to its direction of motion. What fields will the atom experience in its rest frame?

[You are given that the rest mass of a hydrogen atom is 0.938271 GeV/c^2.]