PHYS370 – Advanced Electromagnetism

Part 9: Electromagnetism and Special Relativity

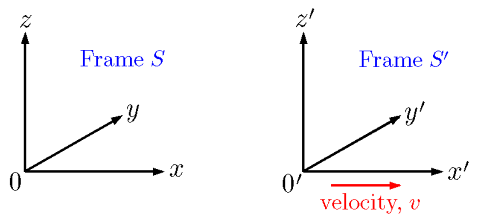
Review of Special Relativity

Special relativity is developed from two fundamental principles:

- Physical laws have the same form in all inertial frames of reference.
- All observers find the same value, c, for the speed of light in a vacuum.

Lorentz Transformations

Consider two inertial frames S and S' (i.e. two non-accelerating reference frames). Suppose that the two frames have a common origin ($x = y = z = 0$) at time $t = 0$, and that the coordinates are oriented so that the relative velocity of the frames is parallel to the x axis.



A given event occurs at time t and coordinates (x, y, z) in frame S, and at time t' and coordinates (x', y', z') in frame S'. The relationship between the times t and t', and the coordinates (x, y, z) and (x', y', z') is given by a *Lorentz transformation*.

Lorentz Transformations

For a given event, the time and coordinates of the event in the frame S' are found from the time and coordinates of the even in the frame S using a Lorentz transformation:

$$x' = \gamma(x - vt) \tag{1}$$

$$y' = y \tag{2}$$

$$z' = z \tag{3}$$

$$t' = \gamma\left(t - \frac{vx}{c^2}\right) \tag{4}$$

where:

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{5}$$

and v is the relative speed of S' with respect to S.

The Inverse Lorentz Transformations

The "inverse" transformation gives the time and coordinates of an event in S, in terms of the time and coordinates of the same event in S':

$$\begin{aligned} x &= \gamma(x' + vt') && (6) \\ y &= y' && (7) \\ z &= z' && (8) \\ t &= \gamma\left(t' + \frac{vx'}{c^2}\right) && (9) \end{aligned}$$

where, as before:

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad (10)$$

Time Dilation

The Lorentz transformations have two immediate consquences. The first is that the time interval $t_2 - t_1$ between two events in frame S is *greater than* the time interval $t'_2 - t'_1$ between the same two events, occuring at a given point x' in frame S'.

Since:

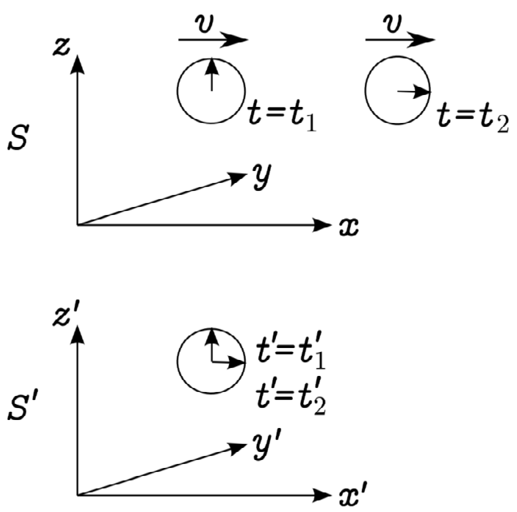
$$\begin{aligned} t_1 &= \gamma\left(t'_1 + \frac{vx'}{c^2}\right) && (11) \\ t_2 &= \gamma\left(t'_2 + \frac{vx'}{c^2}\right) && (12) \end{aligned}$$

it follows that:

$$t_2 - t_1 = \gamma(t'_2 - t'_1) \qquad (13)$$

Note that $\gamma > 1$ for all v; therefore, "moving clocks run slow".

Time Dilation



Length Contraction

The second immediate consequence of the Lorentz transformation is that the distance $x'_2 - x'_1$ between two events in frame S' is *less than* the distance $x_2 - x_1$ between the same two events, occuring at a given time t in frame S.

Since:

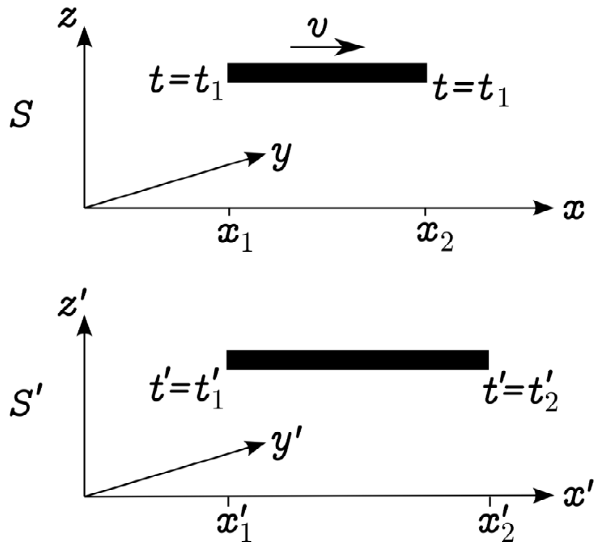
$$\begin{aligned} x'_1 &= \gamma\,(x_1 - vt) && (14) \\ x'_2 &= \gamma\,(x_2 - vt) && (15) \end{aligned}$$

it follows that:

$$x_2 - x_1 = \frac{1}{\gamma}\left(x'_2 - x'_1\right) \qquad (16)$$

The dimension along the x axis of an object moving parallel to the x axis appears to be shorter than if the same measurement was made on the same object at rest.

Time Dilation



Lorentz Invariance

A physical quantity that is unchanged under the Lorentz transformation is said to be Lorentz invariant. For example, consider a pulse of light that leaves the origin at time $t = t' = 0$, and propagates as a spherical wave. An observer at rest in S describes the locus of the spherical wavefront at time t by the equation:

$$x^2 + y^2 + z^2 = c^2t^2 \quad \Rightarrow \quad x^2 + y^2 + z^2 - c^2t^2 = 0 \tag{17}$$

But, from the fundamental principles of special relativity, an observer at rest in S' sees the light pulse travel at the same speed c, so writes a similar equation for the locus of the spherical wavefront in S':

$$x'^2 + y'^2 + z'^2 - c^2t'^2 = 0 \tag{18}$$

The quantity $x^2 + y^2 + z^2 - c^2t^2$ has the same value (zero) for all inertial observers: it is said to be *Lorentz invariant*.

Electric Charges Seen by Different Observers

Electric charge does not depend on time or position: therefore, the net charge carried by an object is Lorentz invariant.

However, from Maxwell's equations, an electric field is generated by a charge density, ρ:

$$\nabla \cdot \vec{E} = \rho \tag{19}$$

The charge density is the charge per unit volume. Since the volume of an object is not Lorentz invariant (because of Lorentz contraction), charge density is not Lorentz invariant.

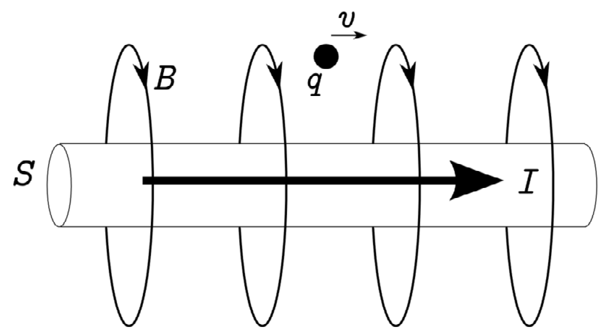
This suggests that electric (and magnetic) fields are not Lorentz invariant. Observers in different inertial frames will agree on how an electromagnetic system behaves, but will give different explanations for its behaviour.

We can illustrate this with a simple example...

Electromagnetic Forces Seen by Different Observers

Consider a long straight wire at rest in a frame S with zero net charge, but carrying a current I.

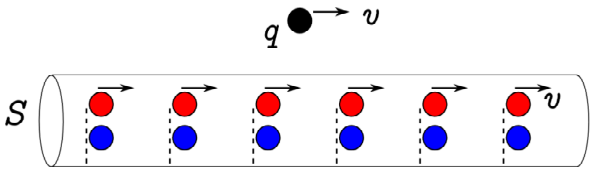
A charge q moving in the same direction as the current in the wire feels a magnetic force pushing it towards the wire.



An observer in S sees an electrically neutral wire, with the same number of negative and positive charges per unit length.

Let us suppose that the current arises from positive charges moving with speed v in the same direction as the charge q.

Since the wire is electrically neutral, the charge line densities of the stationary negative charges and the moving positive charges are the same.



At a radial distance r from the wire, the observer in S sees a magnetic field:

$$B = \frac{\mu_0 I}{2\pi r} \tag{20}$$

The charge q moving at speed v parallel to the wire at a distance r from the wire experiences a force:

$$F = qvB = -qv\frac{\mu_0 I}{2\pi r} \tag{21}$$

where the minus sign indicates a force towards the wire for positive q, v and I.

Using Newton's second law of motion, the acceleration of the charge resulting from the magnetic force is:

$$\frac{d^2 r}{dt^2} = \frac{F}{\gamma m} = -\frac{q}{\gamma m} v \frac{\mu_0 I}{2\pi r} \tag{22}$$

where m is the mass of the charge in its rest frame.

Now consider an observer in S', the rest frame of the charge q. This observer should also see the charge accelerate towards the wire.

Using time dilation, $dt = \gamma\, dt'$; so the rate of acceleration in S' should be:

$$\frac{d^2 r}{dt'^2} = \gamma^2 \frac{d^2 r}{dt^2} = -\gamma \frac{q}{m} v \frac{\mu_0 I}{2\pi r} \tag{23}$$

But in S', the charge is at rest: this means that it will feel no force from the magnetic field around the wire.

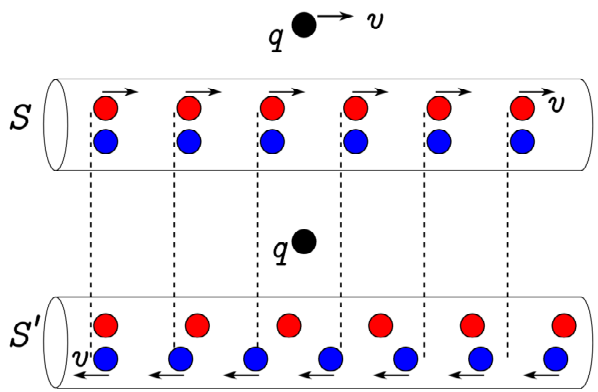
So why does the charge accelerate in S'?

Consider the wire as viewed by the observer in S'.

Suppose there are N charged particles per unit length of the wire when viewed in S.

The number of negative charges per unit length when viewed in S' is γN (since the negative charges were at rest in S, and are moving with speed v in S').

The number of positive charges per unit length when viewed in S' is N/γ (since the positive charges were moving with speed v in S, and are stationary when viewed in S').



The densities of the negative and positive charges do not cancel in S': the net charge line density λ' is:

$$\begin{aligned} \lambda' &= Ne\left(\frac{1}{\gamma} - \gamma\right) \\ &= -\gamma Ne\left(1 - \frac{1}{\gamma^2}\right) \\ &= -\gamma Ne\frac{v^2}{c^2} \end{aligned} \tag{24}$$

Since the current I comes from positive charges e with charge density Ne per unit length moving with speed v, we can write:

$$I = Nev \tag{25}$$

Hence the charge line density is:

$$\lambda' = -\gamma\frac{v}{c^2}I \tag{26}$$

Distances perpendicular to the direction of motion are not affected by relativistic contraction, so the charge q is at the same distance r from the wire in the frame S' as in the frame S.

The electric field at the position of the charge is:

$$E' = \frac{\lambda'}{2\pi\varepsilon_0 r} = -\gamma\frac{1}{2\pi\varepsilon_0 r}\frac{v}{c^2}I \tag{27}$$

Using $1/c^2 = \mu_0\varepsilon_0$, this can be written:

$$E' = -\gamma v\frac{\mu_0 I}{2\pi r} \tag{28}$$

The electrostatic force on the wire in S' is:

$$F' = qE' = -\gamma qv\frac{\mu_0 I}{2\pi r} \tag{29}$$

Hence, the acceleration of the charge in S' is:

$$\frac{d^2r}{dt'^2} = \frac{F'}{m} = -\gamma\frac{q}{m}v\frac{\mu_0 I}{2\pi r} \tag{30}$$

This is in agreement with equation (23) – even though we derived the result in a completely different way.

Electromagnetic Forces Seen by Different Observers

The force on the particle which was purely magnetic in S appears purely electrostatic in S'.

Observers in the two frames agree about the acceleration, but disagree about the origin of the force causing the acceleration.

Electric and magnetic fields are "interchangeable": whether one sees an electric or a magnetic field in a given situation depends on one's frame of reference.

This example represents a special case of the transformation of electric and magnetic fields in special relativity.

In what follows, we will first show that Maxwell's equations are compatible with special relativity, then derive the general form of the field transformations.

Four-Vectors and the Geometry of Space-Time

The Lorentz transformation is a linear transformation connecting space and time coordinates in one frame with those in another frame. Can we devise a more natural notation that treats space (x, y, z) and time t coordinates on an equal footing?

The answer is *Yes!* We simply extend the concept of a three-dimensional vector:

$$(x, y, z) \tag{31}$$

to four dimensions; thus we write a *four-vector*:

$$(x, y, z, ct) \tag{32}$$

Note that we write ct for the fourth component of a four-vector, so that it has the same units (i.e. units of length) as the other three components. Three-vectors obey certain rules of geometry. We need to be careful about how we extend these rules to four-vectors.

Three-Vectors and Rotations

The length (or rather, the length squared) of a three-vector is found by taking the scalar product:

$$r^2 = \vec{r} \cdot \vec{r} = x^2 + y^2 + z^2 \tag{33}$$

The quantity r^2 is invariant under rotations of the axes. For example, consider a rotation through angle ϕ about the z axis:

$$x \mapsto x' = x \cos\phi + y \sin\phi \tag{34}$$

$$y \mapsto y' = -x \sin\phi + y \cos\phi \tag{35}$$

$$z \mapsto z' = z \tag{36}$$

Three-Vectors and Rotations

We can write the rotation about the z axis as a matrix:

$$\vec{r} \mapsto \vec{r}' = R(\phi) \cdot \vec{r} \tag{37}$$

where:

$$\vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad R(\phi) = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{38}$$

We observe that the rotation matrices are orthonormal, i.e.:

$$R(\phi)^{\mathsf{T}} \cdot R(\phi) = I_3 \tag{39}$$

where I_3 is the 3×3 identity matrix.

Another way of saying this, is that the rotation matrices preserve the identity matrix, i.e.:

$$R(\phi)^{\mathsf{T}} \cdot I_3 \cdot R(\phi) = I_3 \tag{40}$$

This is true for rotations around the x axis and around the y axis, as well as rotations around the z axis.

Note that the scalar product of two three-vectors can be written as a matrix multiplication:

$$r^2 = \vec{r}^{\mathsf{T}} \cdot I_3 \cdot \vec{r} \tag{41}$$

Under a rotation R, we have:

$$\vec{r} \mapsto R \cdot \vec{r} \tag{42}$$

and the length of the vector is transformed:

$$r^2 \mapsto r'^2 = \vec{r}^{\mathsf{T}} \cdot R^{\mathsf{T}} \cdot I_3 \cdot R \cdot \vec{r} \tag{43}$$

But since the rotation matrix R preserves the identity matrix:

$$R^{\mathsf{T}} \cdot I_3 \cdot R = I_3 \tag{44}$$

the length of the vector $\vec{r}$ is invariant under R:

$$r'^2 = \vec{r}^{\mathsf{T}} \cdot I_3 \cdot \vec{r} = r^2 \tag{45}$$

The square of the length of a three-vector r^2 is invariant under rotations.

To extend this concept to four-vectors, we recall that the quantity:

$$r^2 = x^2 + y^2 + z^2 - c^2t^2 \tag{46}$$

is invariant under Lorentz transformations.

Let us write this as:

$$r^2 = \vec{r}^{\mathsf{T}} \cdot g \cdot \vec{r} \tag{47}$$

where $\vec{r}$ is now a four-vector, and g is a four-by-four matrix:

$$\vec{r} = \begin{pmatrix} x \\ y \\ z \\ ct \end{pmatrix} \qquad g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{48}$$

We note that, like the identity matrix I_3 in three dimensions, the matrix g is invariant under rotations.

For example if we write the rotation about the z axis as:

$$R(\phi) = \begin{pmatrix} \cos\phi & \sin\phi & 0 & 0 \\ -\sin\phi & \cos\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{49}$$

then we have:

$$R(\phi)^{\mathsf{T}} \cdot g \cdot R(\phi) = g \tag{50}$$

The fourth dimension gives us an extra set of transformations under which the matrix g is invariant.

The minus sign on the $(4,4)$ component of g means that these transformations look a little different from normal transformations.

An example of one of these transformations is:

$$\Lambda(\theta) = \begin{pmatrix} \cosh\theta & 0 & 0 & -\sinh\theta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh\theta & 0 & 0 & \cosh\theta \end{pmatrix} \tag{51}$$

Let us write:

$$\gamma = \cosh\theta \tag{52}$$

where θ is the parameter in one of the transformations $\Lambda(\theta)$.

Using the identity:

$$\cosh^2\theta - \sinh^2\theta \equiv 1 \tag{53}$$

we can write:

$$\sinh\theta = \beta\gamma \tag{54}$$

where

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \tag{55}$$

Then the transformation $\Lambda(\theta)$ becomes:

$$\Lambda(\theta) = \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \tag{56}$$

With $\beta = v/c$, the transformation $\Lambda(\theta)$ gives the Lorentz transformation (1) - (4):

$$\vec{r}' = \Lambda(\theta) \cdot \vec{r} \tag{57}$$

Summary: By combining the spatial coordinates and the time coordinate into a single four-vector:

$$\vec{r} = \begin{pmatrix} x \\ y \\ z \\ ct \end{pmatrix} \tag{58}$$

and considering transformations $\Lambda(\theta)$ that leave the matrix g invariant:

$$\Lambda(\theta)^{\mathsf{T}} \cdot g \cdot \Lambda(\theta) = g \qquad g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{59}$$

we have obtained the Lorentz transformations:

$$\Lambda(\theta) = \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix}, \qquad \gamma = \cosh\theta, \quad \beta\gamma = \sinh\theta \tag{60}$$

A Lorentz boost is just a kind of "rotation" in space-time. The matrix g, sometimes called the *metric*, is invariant under normal rotations (in three-dimensional space) and under Lorentz boost "rotations" in space-time.

The metric provides a rule for constructing invariant quantities. We have already seen that for $\vec{r} = (x, y, z, ct)$ describing the motion of a spherical wavefront of a light wave, the quantity:

$$r^2 = \vec{r}^{\mathsf{T}} \cdot g \cdot \vec{r} = x^2 + y^2 + z^2 - c^2t^2 = 0 \tag{61}$$

is invariant under Lorentz transformations.

In general, if $\vec{p}$ and $\vec{q}$ are four-vectors, then the quantity:

$$\vec{p}^{\mathsf{T}} \cdot g \cdot \vec{q} \tag{62}$$

is invariant under Lorentz transformations. This is because the metric g is preserved under Lorentz transformations.

The product of two four vectors:

$$\vec{p}^{\mathsf{T}} \cdot g \cdot \vec{q} \tag{63}$$

appears all the time in special relativity. To simplify things, we write the μth component ($\mu = 1 \ldots 4$) of a four-vector $\vec{p}$ as p^μ. Note that μ is written as a superscript.

We define a four-vector associated with $\vec{p}$ with components:

$$p_\mu = \sum_{\nu=1}^{4} p^\nu g_{\nu\mu} \tag{64}$$

The components of the new four-vector are distinguished from those of the original four-vector by writing the index μ as a subscript. The square of the "length" of the four-vector $\vec{p}$ is given by:

$$\vec{p}^{\mathsf{T}} \cdot g \cdot \vec{p} = \sum_{\mu,\nu=1}^{4} p^\nu g_{\nu\mu} p^\mu = \sum_{\mu=1}^{4} p_\mu p^\mu \tag{65}$$

In general, the "scalar product" of two four-vectors can be written as:

$$\vec{p}^{\mathsf{T}} \cdot g \cdot \vec{q} = \sum_{\mu,\nu=1}^{4} p^\nu g_{\nu\mu} q^\mu = \sum_{\mu=1}^{4} p_\mu q^\mu \tag{66}$$

A product of two four-vectors constructed in this way is Lorentz invariant.

Products such as these occur so frequently in special relativity, that we introduce a short-hand notation that avoids writing the summation symbol all the time.

The summation convention states: where a "down" index on one four-vector also appears as an "up" index on another four-vector, we sum over the components of the two four-vectors, thus:

$$p_\mu q^\mu = \sum_{\mu=1}^{4} p_\mu q^\mu = \sum_{\mu,\nu=1}^{4} p^\nu g_{\nu\mu} q^\mu \tag{67}$$

In general, any index should appear a maximum of two times in any expression: once as a "down" index and once as an "up" index.

When an index appears twice in this way, summation over the index is implied.

If an index appears twice or more as either a "down" index or an "up" index, you are doing something wrong! Stop, go back, and check what you have written.

The transformation of a four-vector p^μ from one inertial frame S into a second intertial frame S' can be written very easily as:

$$p'^\mu = \Lambda^\mu{}_\nu \, p^\nu \tag{68}$$

where the summation convention applies, and the matrix $\Lambda^\mu{}_\nu$ has components (in the case of a boost along the x axis):

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \tag{69}$$

Note that to maintain consistency with the summation convention, the matrix $\Lambda^\mu{}_\nu$ is written with one index "up" and the other index "down".

A four-vector can be constructed from the energy of a particle and its momentum. If the energy (the sum of the mass energy and the kinetic energy) of a particle is E and its momentum is $\vec{p} = (p_x, p_y, p_z)$, then the vector:

$$p^\mu = \begin{pmatrix} p_x \\ p_y \\ p_z \\ \frac{E}{c} \end{pmatrix} \tag{70}$$

is a four-vector, called the momentum four-vector of the particle.

The "length" squared of the momentum four-vector is given by:

$$p_\mu p^\mu = p_x^2 + p_y^2 + p_z^2 - \frac{E^2}{c^2} = -m_0^2 c^2 \tag{71}$$

where m_0 is the rest mass of the particle.

We can re-write this as:

$$E^2 = \vec{p}^2 c^2 + m_0^2 c^4 \tag{72}$$

which is familiar from special relativity.

Since all observers agree on the rest mass of a particle, the rest mass is Lorentz invariant.

So the quantity $p_\mu p^\mu$ is Lorentz invariant; hence, p^μ must be a four-vector.

The differential operator ∂^μ is a four-vector whose components are:

$$\partial^\mu = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \\ -\frac{1}{c}\frac{\partial}{\partial t} \end{pmatrix} \tag{73}$$

To see that ∂^μ is indeed a four-vector, we must check its transformation rules.

The first component transforms as:

$$\partial'^1 = \frac{\partial}{\partial x'} = \frac{\partial x}{\partial x'}\frac{\partial}{\partial x} + \frac{\partial y}{\partial x'}\frac{\partial}{\partial y} + \frac{\partial z}{\partial x'}\frac{\partial}{\partial z} + \frac{\partial t}{\partial x'}\frac{\partial}{\partial t} \tag{74}$$

Using the inverse Lorentz transformation (6) - (9) for a boost in the $+x$ direction:

$$\partial'^1 = \gamma\frac{\partial}{\partial x} + \gamma\frac{v}{c}\frac{1}{c}\frac{\partial}{\partial t} = \gamma\partial^1 - \beta\gamma\partial^4 \tag{75}$$

Altogether, we find the components ∂^μ transform as:

$$\partial'^1 = \gamma\partial^1 - \beta\gamma\,\partial^4 \tag{76}$$
$$\partial'^2 = \partial^2 \tag{77}$$
$$\partial'^3 = \partial^3 \tag{78}$$
$$\partial'^4 = -\beta\gamma\,\partial^1 + \gamma\,\partial^4 \tag{79}$$

Hence, ∂^μ transforms the same way as x^μ under a Lorentz transformation, and is therefore a four-vector.

Associated with ∂^μ is the differential operator ∂_μ:

$$\partial_\mu = g_{\mu\nu}\partial^\nu = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{1}{c}\frac{\partial}{\partial t}\right) \tag{80}$$

We define the differential operator $\Box$ as:

$$\Box = \partial_\mu\partial^\mu \tag{81}$$

Note that we use the summation convention, so that a summation over the repeated index μ is implied.

From the components of the vectors, we can write:

$$\Box = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \tag{82}$$

The second-order differential operator $\Box$:

$$\Box = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \tag{83}$$

is called the D'Alembertian.

Since the D'Alembertian is the product of two four-vectors, we expect it to be Lorentz invariant.

This is indeed the case, as can be verified by calculating its transformation properties directly.

The current density (J_x, J_y, J_z) and the charge density ρ can be combined into a four-vector:

$$J^\mu = (J_x, J_y, J_z, c\rho)^\top \tag{84}$$

The correct transformation properties for a four-vector follow from the Lorentz invariance of electric charge, together with time dilation and length contraction.

The continuity equation can be written:

$$\partial_\mu J^\mu = \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z} + \frac{\partial\rho}{\partial t} = \nabla\cdot\vec{J} + \frac{\partial\rho}{\partial t} = 0 \tag{85}$$

The left-hand side is the product of two four-vectors, and so should be Lorentz invariant. The right-hand side is a constant (zero) which is obviously Lorentz invariant.

Covariant Form

An equation expressed purely in terms of four-vectors and Lorentz invariants is said to be in *covariant form*.

If an equation can be put into covariant form, it means that the equation will still have the same form (i.e. will look the same) if all the quantities involved undergo a Lorentz transformation.

An equation that is in covariant form will be consistent with the first principle of special relativity.

We expect to be able to express the laws of physics (in so far as they are compatible with special relativity) in covariant form.

The Electromagnetic Potential Four-Vector

Consider the wave equations for the magnetic vector and electric scalar potential:

$$\Box \vec{A} = -\mu_0 \vec{J} \tag{86}$$

$$\Box \phi = -\frac{\rho}{\varepsilon_0} \tag{87}$$

Write the second equation as:

$$\Box \frac{\phi}{c} = -\frac{c\rho}{\varepsilon_0 c^2} = -\mu_0 c\rho \tag{88}$$

We can combine equations (86) and (88) as follows:

$$\Box A^\mu = -\mu_0 J^\mu \tag{89}$$

where $J^\mu = (J_x, J_y, J_z, c\rho)^\mathsf{T}$ is the current density four-vector, and we have defined the quantity A^μ as:

$$A^\mu = (A_x, A_y, A_z, \frac{\phi}{c})^\mathsf{T} \tag{90}$$

The Electromagnetic Potential Four-Vector

The operator $\Box$ is Lorentz invariant, as is the physical constant μ_0, and the current density J^μ is a four-vector. We assume that A^μ is a four-vector, called the electromagnetic potential four-vector. Then, the wave equation (89):

$$\Box A^\mu = -\mu_0 J^\mu$$

involves only Lorentz invariants and four-vectors, and hence is in covariant form.

The Lorenz gauge condition:

$$\nabla \cdot \vec{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} = 0 \tag{91}$$

can be written in covariant form:

$$\partial_\mu A^\mu = 0 \tag{92}$$

A Moving Point Charge: The Liénard-Wiechert Potentials

We can apply a Lorentz transformation to the potentials around a stationary point charge to find the potentials around a point charge moving at a constant velocity.

The resulting potentials are known as the Liénard-Wiechert potentials.

We start with the familiar Coulomb potential around a stationary point charge q:

$$\phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{|\vec{r} - \vec{r}_q|} \tag{93}$$

$$\vec{A}(\vec{r}) = 0 \tag{94}$$

where $\vec{r}_q$ is the location of the point charge.

We now make a Lorentz transformation from a frame in which the point charge is at rest, to one in which it is moving with some non-zero velocity.

Let us choose a coordinate system in which the charge is at rest, and the charge and the observation point lie on the x-axis.

We shall first consider a boost along the x-axis, and then generalise our result to include boosts in other directions.

In the inertial frame S, the point charge is at rest.

In the inertial frame S', the charge is moving with velocity v along the x'-axis.

Therefore, frame S' is moving with velocity $-v$ along the x-axis with respect to frame S.

The Lorentz transformations of the potentials are then:

$$\phi' = \gamma(\phi + vA_x) = \gamma\phi \tag{95}$$

$$A'_x = \gamma\left(A_x + \frac{v\phi}{c^2}\right) = \gamma\frac{v}{c^2}\phi = \frac{v}{c^2}\phi' \tag{96}$$

Since the vector potential A'_x is readily expressed in terms of the scalar potential ϕ', we concentrate on finding the scalar potential in frame S'.

Substituting from equations (93) and (94) into equation (95), we have for the scalar potential:

$$\phi' = \gamma\phi = \gamma\frac{1}{4\pi\varepsilon_0}\cdot\frac{q}{|x - x_q|} \tag{97}$$

To find an expression for ϕ' in terms of coordinate in S', we use the Lorentz transformations of the coordinates:

$$x = \gamma\left(x' - vt'\right) \tag{98}$$

Substituting from (98) into (97) gives:

$$\phi' = \gamma\frac{1}{4\pi\varepsilon_0}\cdot\frac{q}{\left|\gamma(x' - vt') - \gamma\left(x'_q - vt'_q\right)\right|} \tag{99}$$

$$= \frac{1}{4\pi\varepsilon_0}\cdot\frac{q}{\left|(x' - vt') - \left(x'_q - vt'_q\right)\right|} \tag{100}$$

Note that the charge is at coordinate x'_q at time t'_q (as measured in frame S'), and that the potentials ϕ' and A'_x are measured at coordinate x' and time t' (again, as measured in frame S').

Since any change in the source takes time $\Delta x'/c$ to propagate a distance $\Delta x'$, we must have:

$$t' - t'_q = \frac{\left|x' - x'_q\right|}{c} \tag{101}$$

Therefore, we can write equation (100) for the potential in frame S':

$$\phi' = \frac{1}{4\pi\varepsilon_0}\cdot\frac{q}{\left|x' - x'_q\right|(1 \mp v/c)} \tag{102}$$

where the minus sign holds for $x' > x'_q$ (charge moving towards the observer) and the plus sign holds for $x' < x'_q$ (charge moving away from the observer).

Since coordinates in directions transverse to the boost are not changed by the Lorentz transformation, we can generalise equation (102) to a boost in an arbitrary direction:

$$\phi'\left(\vec{r}', t'\right) = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{\left|\vec{r}' - \vec{r}'_q\right| \left(1 - \vec{\beta} \cdot \vec{n}'\right)} \tag{103}$$

where:

$$\vec{\beta} = \frac{\vec{v}}{c}, \qquad \vec{n}' = \frac{\vec{r}' - \vec{r}'_q}{\left|\vec{r}' - \vec{r}'_q\right|}, \qquad t' = t'_q + \frac{\left|\vec{r}' - \vec{r}'_q\right|}{c} \tag{104}$$

Dropping the prime, we can use equations (103) and (96) to write expressions for the potentials around a point charge moving with constant velocity $\vec{v} = \vec{\beta}c$:

$$\phi(\vec{r}, t) = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{|\vec{r} - \vec{r}_q| \left(1 - \vec{\beta} \cdot \vec{n}\right)} \tag{105}$$

$$\vec{A}(\vec{r}, t) = \frac{\vec{\beta}}{c}\phi(\vec{r}, t) \tag{106}$$

where $\vec{n}$ is a unit vector from the charge at $\vec{r}_q$ to the observer at r, and the charge is at $\vec{r}_q$ at time t_q, given by:

$$t = t_q + \frac{|\vec{r} - \vec{r}_q|}{c} \tag{107}$$

Equations (105) and (106) give the Liénard-Wiechert potentials for a point charge moving at constant velocity.

From the Liénard-Wiechert potentials (105) and (106):

$$\phi(\vec{r}, t) = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{|\vec{r} - \vec{r}_q| \left(1 - \vec{\beta} \cdot \vec{n}\right)}$$

$$\vec{A}(\vec{r}, t) = \frac{\vec{\beta}}{c}\phi(\vec{r}, t)$$

we see that there is a relativistic *enhancement* of the potential for a charge moving *towards* an observer, and a relativistic *reduction* of the potential for a charge moving *away* from an observer. The enhancement or reduction compared to the static case is a relativistic effect, since it vanishes in the limit $c \to \infty$ (in which case, the expressions for the potentials around a moving point charge are the same as those for the potentials around a static charge).

Finally, note that the Liénard-Wiechert potentials satisfy the Lorenz gauge condition:

$$\nabla \cdot \vec{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} = 0. \tag{108}$$

The components of the magnetic field (B_x, B_y, B_z) and the electric field (E_x, E_y, E_z) cannot be combined into a four-vector. However, they can be combined into a matrix that will allow us to write Maxwell's equations in explicitly covariant form.

Recall that the electromagnetic field is obtained from the derivatives of the potential. Let us define the matrix $F^{\mu\nu}$:

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \tag{109}$$

where A^μ is the four-vector electromagnetic potential, and ∂^μ is the four-vector differential operator. Since the right-hand side of equation (109) involves only four-vectors, it transforms under a Lorentz transformation as:

$$\partial'^\mu A'^\nu - \partial'^\nu A'^\mu = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \partial^\alpha A^\beta - \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \partial^\beta A^\alpha \tag{110}$$

Therefore, the matrix $F^{\mu\nu}$ transforms under a Lorentz transformation as:

$$F'^{\mu\nu} = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta F^{\alpha\beta} \tag{111}$$

Since $F^{\mu\nu}$ transforms appropriately under Lorentz transformations, this is a valid quantity to use in explicitly covariant expressions. Now we inspect the components of $F^{\mu\nu}$.

For example, we find that:

$$F^{1,2} = \partial^1 A^2 - \partial^2 A^1 = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = B_z \tag{112}$$

As another example, we find that:

$$F^{3,4} = \partial^3 A^4 - \partial^4 A^3 = \frac{1}{c}\frac{\partial \phi}{\partial z} + \frac{1}{c}\frac{\partial A_z}{\partial t} = -\frac{E_z}{c} \tag{113}$$

We also note that the diagonal components of $F^{\mu\nu}$ are zero:

$$F^{\mu\nu} = 0, \qquad \mu = \nu \tag{114}$$

and that $F^{\mu\nu}$ is antisymmetric:

$$F^{\nu\mu} = -F^{\mu\nu} \tag{115}$$

Overall, we find that the components of $F^{\mu\nu}$ are:

$$F^{\mu\nu} = \begin{pmatrix} 0 & B_z & -B_y & -E_x/c \\ -B_z & 0 & B_x & -E_y/c \\ B_y & -B_x & 0 & -E_z/c \\ E_x/c & E_y/c & E_z/c & 0 \end{pmatrix} \tag{116}$$

We observe that the six independent components of the 4×4 antisymmetric matrix $F^{\mu\nu}$ are the six components of the electromagnetic field.

The transformation properties of the electromagnetic field under Lorentz transformations follow immediately from the transformation properties of the matrix $F^{\mu\nu}$:

$$F'^{\alpha\beta} = \Lambda^{\alpha}{}_{\mu} \Lambda^{\beta}{}_{\nu} F^{\mu\nu} \tag{117}$$

We now have an explicitly covariant quantity $F^{\mu\nu}$ that contains the components of the electromagnetic field.

If we are able to write Maxwell's equations purely in terms of $F^{\mu\nu}$ and other quantities (four-vectors and Lorentz invariants) with the proper transformation properties, then we will have shown that Maxwell's equations are consistent with special relativity.

Explicitly Covariant Form of Maxwell's Equations

First, consider the expression:

$$\partial_\mu F^{\mu\nu} \tag{118}$$

This may be evaluated explicitly using equation (116); but note that we can also write it using (109):

$$\partial_\mu F^{\mu\nu} = \partial_\mu \partial^\mu A^\nu - \partial_\mu \partial^\nu A^\mu \tag{119}$$

Note that:

$$\partial_\mu \partial^\mu = \Box \tag{120}$$

In the Lorenz gauge, the four-vector potential A^μ satisfies the wave equation:

$$\Box A^\nu = -\mu_0 J^\nu \tag{121}$$

We can also choose the Lorenz gauge condition:

$$\partial_\mu A^\mu = 0 \tag{122}$$

and hence:

$$\partial_\mu F^{\mu\nu} = -\mu_0 J^\nu \tag{123}$$

Consider the explicitly covariant equation (123):

$$\partial_\mu F^{\mu\nu} = -\mu_0 J^\nu \tag{124}$$

If we take $\nu = 4$, we find that:

$$\frac{\partial}{\partial x}\frac{E_x}{c} + \frac{\partial}{\partial y}\frac{E_y}{c} + \frac{\partial}{\partial z}\frac{E_z}{c} = \mu_0 c \rho \tag{125}$$

which can be written:

$$\nabla \cdot \vec{E} = \mu_0 c^2 \rho \tag{126}$$

Using $c^2 = 1/\mu_0\varepsilon_0$, we obtain the familiar form of Maxwell's equation:

$$\nabla \cdot \vec{D} = \rho \tag{127}$$

Now consider the case $\nu = 1$ in the explicitly covariant equation (123):

$$\partial_\mu F^{\mu\nu} = -\mu_0 J^\nu \tag{128}$$

This gives:

$$-\frac{\partial}{\partial y}B_z + \frac{\partial}{\partial z}B_y + \frac{1}{c^2}\frac{\partial}{\partial t}E_x = -\mu_0 J_x \tag{129}$$

which can be written:

$$[\nabla \times \vec{B}]_x - \frac{1}{c^2}\frac{\partial E_x}{\partial t} = \mu_0 J_x \tag{130}$$

We obtain similar expressions from the cases $\nu = 2$ and $\nu = 3$; combining the equations from all the cases $\nu = 1, 2, 3$, we obtain Maxwell's equation:

$$\nabla \times \vec{H} - \frac{\partial \vec{D}}{\partial t} = \vec{J} \tag{131}$$

We find that the explicitly covariant equation (123):

$$\partial_\mu F^{\mu\nu} = -\mu_0 J^\nu \tag{132}$$

gives (by considering different values of the index ν), the inhomogeneous Maxwell's equations:

$$\nabla \cdot \vec{D} = \rho \tag{133}$$

$$\nabla \times \vec{H} - \frac{\partial \vec{D}}{\partial t} = \vec{J} \tag{134}$$

Now consider the definition of the matrix $F^{\mu\nu}$:

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \tag{135}$$

Using this definition, we find that:

$$\partial^\lambda F^{\mu\nu} + \partial^\nu F^{\lambda\mu} + \partial^\mu F^{\nu\lambda} \equiv 0 \tag{136}$$

Note that this is an *identity* for any values of the indices λ, μ and ν: it holds for any components of the matrix $F^{\mu\nu}$.

If we choose:

$$\mu = 1, \qquad \nu = 2, \qquad \lambda = 3 \tag{137}$$

we find:

$$\frac{\partial}{\partial x}B_x + \frac{\partial}{\partial y}B_y + \frac{\partial}{\partial z}B_z = 0 \tag{138}$$

which can be written in the form familiar from Maxwell's equations:

$$\nabla \cdot \vec{B} = 0 \tag{139}$$

Explicitly Covariant Form of Maxwell's Equations

Now let us take the equation (136):

$$\partial^\lambda F^{\mu\nu} + \partial^\nu F^{\lambda\mu} + \partial^\mu F^{\nu\lambda} \equiv 0 \tag{140}$$

with the values for the indices:

$$\mu = 1, \qquad \nu = 2, \qquad \lambda = 4 \tag{141}$$

We find that:

$$\frac{1}{c}\frac{\partial}{\partial t}B_z + \frac{1}{c}\frac{\partial}{\partial y}E_x - \frac{1}{c}\frac{\partial}{\partial x}E_y = 0 \tag{142}$$

which can be written:

$$[\nabla \times \vec{E}]_z + \frac{\partial}{\partial t}B_z = 0 \tag{143}$$

We find similar equations for $\mu = 1$, $\nu = 3$ and $\lambda = 4$; and for $\mu = 2$, $\nu = 3$ and $\lambda = 4$. Combining the equations together, we obtain the familiar Maxwell's equation:

$$\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \tag{144}$$

Explicitly Covariant Form of Maxwell's Equations

To summarise, the explicitly covariant equation (123):

$$\partial_\mu F^{\mu\nu} = -\mu_0 J^\nu \tag{145}$$

can be written using three-vectors:

$$\nabla \cdot \vec{D} = \rho, \qquad \nabla \times \vec{H} - \frac{\partial \vec{D}}{\partial t} = \vec{J} \tag{146}$$

The explicitly covariant equation (136):

$$\partial^\lambda F^{\mu\nu} + \partial^\nu F^{\lambda\mu} + \partial^\mu F^{\nu\lambda} = 0 \tag{147}$$

can be written using three-vectors:

$$\nabla \cdot \vec{B} = 0, \qquad \nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \tag{148}$$

Lorentz Transformation of the Electromagnetic Field

Explicit expressions for the transformations of the electromagnetic field can be found from equation (111):

$$F'^{\mu\nu} = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta F^{\alpha\beta} \tag{149}$$

Since the electromagnetic field $F^{\alpha\beta}$ is represented by a matrix, and the Lorentz transformation $\Lambda^\mu{}_\alpha$ is also represented by a matrix, applying the transformation just involves matrix multiplication.

Lorentz Transformation of the Electromagnetic Field

For a Lorentz boost of velocity v along the x axis, we find for the electric field:

$$E'_x = E_x \tag{150}$$

$$E'_y = \gamma(E_y - vB_z) \tag{151}$$

$$E'_z = \gamma(E_z + vB_y) \tag{152}$$

And for the magnetic field:

$$B'_x = B_x \tag{153}$$

$$B'_y = \gamma\left(B_y + \frac{v}{c^2}E_z\right) \tag{154}$$

$$B'_z = \gamma\left(B_z - \frac{v}{c^2}E_y\right) \tag{155}$$

The inverse transformations are obtained simply by replacing v by $-v$.

Note that the electric field in the S' frame depends on the magnetic field in the S frame; and that the magnetic field in S' depends on the electric field in S.

Lorentz Transformation of the EM Field: Example 1

An stationary observer measures the Earth's magnetic field to be 30 μT. What field would be measured by an observer in an aeroplane flying past the stationary observer at 900 km/h (250 m/s) perpendicular to the direction of the Earth's field?

Choose the x axis to be the direction of motion of the aeroplane, relative to the stationary observer, and the z axis to be in the direction of the magnetic field. For the stationary observer, the magnetic field is:

$$\begin{aligned} B_x &= 0 \\ B_y &= 0 \\ B_z &= 30\mu\text{T} \end{aligned}$$

and the electric field is:

$$\begin{aligned} E_x &= 0 \\ E_y &= 0 \\ E_z &= 0 \end{aligned}$$

Lorentz Transformation of the EM Field: Example 1

For the moving observer, $\beta = 8.3 \times 10^{-7}$ and $\gamma \approx 1$. The fields measured by the moving observer are:

$$\begin{aligned} B'_x &= 0 \\ B'_y &= 0 \\ B'_z &= \gamma B_z \approx 30\mu\text{T} \end{aligned}$$

and the electric field is:

$$\begin{aligned} E'_x &= 0 \\ E'_y &= -\gamma v B_z \approx -7.5 \text{ mV/m} \\ E'_z &= 0 \end{aligned}$$

Lorentz Transformation of the EM Field: Example 2

A neutral hydrogen atom moves with kinetic energy 100 keV in a laboratory frame. Suppose the atom enters a magnetic field of strength 1 T perpendicular to its direction of motion. What fields will the atom experience in its rest frame?

First, we calculate the velocity of the hydrogen atom. The rest mass of the hydrogen atom is $m = 0.938271$ GeV/c^2. So the total energy of the hydrogen atom is:

$$\gamma mc^2 = 0.938271 \text{ GeV} + 100 \text{ keV} = 0.938371 \text{ GeV} \tag{156}$$

Hence:

$$\gamma = \frac{0.938371}{0.938271} \approx 1.000107 \tag{157}$$

Hence:

$$\beta = \sqrt{1 - \frac{1}{\gamma^2}} \approx 0.0146 \tag{158}$$

and:

$$v = \beta c \approx 4.38 \times 10^6 \text{ m/s} \tag{159}$$

Lorentz Transformation of the EM Field: Example 2

Let the hydrogen atom be moving along the x axis, and the magnetic field be parallel to the z axis. The initial electric field seen by the hydrogen atom is zero; the magnetic field is:

$$\begin{aligned} B_x &= 0 \\ B_y &= 0 \\ B_z &= 1\text{T} \end{aligned}$$

The magnetic field seen by the hydrogen atom in its rest frame is:

$$\begin{aligned} B'_x &= 0 \\ B'_y &= 0 \\ B'_z &= \gamma B_z \approx 1.000107\text{T} \end{aligned}$$

The electric field seen by the hydrogen atom in its rest frame is:

$$\begin{aligned} E'_x &= 0 \\ E'_y &= -\gamma v B_z \approx -4.38 \text{ MV/m} \\ E'_z &= 0 \end{aligned}$$

The hydrogen atom sees an electric field of over 4 *megavolts* per meter! This is an extremely strong electric field, and can result in ionisation of the hydrogen atom (an effect called Lorentz ionisation).

What are the fields around a moving point charge?

Let the charge q be moving along the x axis with velocity v. In the rest frame S' of the charge, there is no magnetic field, and the electric field is given by:

$$\vec{E}' = \frac{q}{4\pi\varepsilon_0}\frac{\vec{r}'}{r'^3} \tag{160}$$

In cartesian coordinates, the field components are:

$$E'_x = \frac{q}{4\pi\varepsilon_0}\frac{x'}{(x'^2+y'^2+z'^2)^{3/2}} \tag{161}$$

$$E'_y = \frac{q}{4\pi\varepsilon_0}\frac{y'}{(x'^2+y'^2+z'^2)^{3/2}} \tag{162}$$

$$E'_z = \frac{q}{4\pi\varepsilon_0}\frac{z'}{(x'^2+y'^2+z'^2)^{3/2}} \tag{163}$$

Now we apply the inverse Lorentz transformations to find the fields in the laboratory frame. Note that we have to transform the coordinates as well as the fields.

$$x' = \gamma(x - vt) \tag{164}$$

$$y' = y \tag{165}$$

$$z' = z \tag{166}$$

With zero magnetic field in S', the electric field transforms as:

$$E_x = E'_x \tag{167}$$

$$E_y = \gamma E'_y \tag{168}$$

$$E_z = \gamma E'_z \tag{169}$$

and the magnetic field transformations are:

$$B_x = 0 \tag{170}$$

$$B_y = -\gamma\frac{v}{c^2}E'_z \tag{171}$$

$$B_z = \gamma\frac{v}{c^2}E'_y \tag{172}$$

We find that the electric field in the frame S is given by:

$$E_x = \frac{q}{4\pi\varepsilon_0}\frac{\gamma(x-vt)}{(\gamma^2(x-vt)^2+y^2+z^2)^{3/2}} \tag{173}$$

$$E_y = \frac{q}{4\pi\varepsilon_0}\frac{\gamma y}{(\gamma^2(x-vt)^2+y^2+z^2)^{3/2}} \tag{174}$$

$$E_z = \frac{q}{4\pi\varepsilon_0}\frac{\gamma z}{(\gamma^2(x-vt)^2+y^2+z^2)^{3/2}} \tag{175}$$

Notice the factor γ that appears in the x-dependence of the fields. This means that with increasing velocity, the fields become "flattened" towards the plane perpendicular to the direction of motion of the charge.

Lorentz Transformation of the EM Field: Example 3

The magnetic field is given by:

$$B_x = 0 \tag{176}$$
$$B_y = -\frac{v}{c^2}E_z \tag{177}$$
$$B_z = \frac{v}{c^2}E_y \tag{178}$$

The magnetic field is "flattened" at high particle velocities, in the same way as the electric field. There is also a direct dependence of the size of the magnetic field on the velocity (as we expect): at $v = 0$, the magnetic field vanishes altogether.

Lorentz Transformation of the EM Field: Example 3

To visualise the fields, consider the fields along the axes for the case $t = 0$:

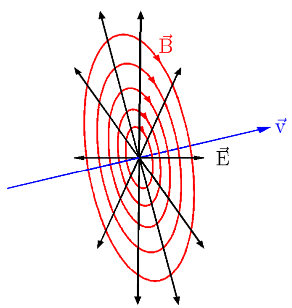
$$E_x(y = z = 0) = \frac{q}{4\pi\varepsilon_0}\frac{1}{\gamma^2 x^2} \tag{179}$$
$$E_y(x = z = 0) = \frac{q}{4\pi\varepsilon_0}\frac{\gamma}{y^2} \tag{180}$$
$$E_z(x = y = 0) = \frac{q}{4\pi\varepsilon_0}\frac{\gamma}{z^2} \tag{181}$$

and the magnetic field is given by:

$$B_x = 0 \tag{182}$$
$$B_y(x = y = 0) = -\frac{v}{c^2}\frac{q}{4\pi\varepsilon_0}\frac{\gamma}{z^2} \tag{183}$$
$$B_z(x = z = 0) = \frac{v}{c^2}\frac{q}{4\pi\varepsilon_0}\frac{\gamma}{y^2} \tag{184}$$

Lorentz Transformation of the EM Field: Example 3

The electric and magnetic fields around a relativistic charged particle are "flattened" towards a plane perpendicular to the direction of motion of the charged particle.



Summary of Part 9: Electromagnetism and Special Relativity

You should be able to:

- Explain what is meant by a "Lorentz invariant".
- State that electric charge is a Lorentz invariant, and show that electric and magnetic fields are not Lorentz invariants.
- Write the Lorentz transformations using four-vector index notation.
- Write down and use the four-vector equivalents of the grad, div and laplacian differential operators.
- Write down the components of the four-vectors representing current density and electromagnetic potentials.
- Derive a 4×4 matrix representing the electromagnetic fields, by taking the "grad" of the electromagnetic potential four-vector.
- Write down Maxwell's equations and the continuity equation using four-vector notation, and show the equivalence of the equations in this form to the equations written in the usual three-vector notation.
- Perform Lorentz transformations of the current density, electromagnetic potentials and electric and magnetic fields.