UNIVERSITY OF LIVERPOOL

PHYS370 – Advanced Electromagnetism

Part 8: Sources of Electromagnetic Radiation

Generation of Electromagnetic Waves

In this section, we consider the generation of electromagnetic waves.

In general, electromagnetic radiation arises when sources of the electromagnetic fields vary in time.

Conceptually, the simplest case would be that of a stationary, isolated charge, for which the size of the charge varies with time.

The radiation in such a case would be termed monopole radiation; however, such a situation violates Maxwell's equations (specifically, the continuity equation which expresses the local conservation of electric charge), so cannot occur in reality.

Dipole Radiation

The simplest way to generate electromagnetic radiation is from a system of electric charges whose electric dipole moment or magnetic dipole moment varies with time.

For example, we might have two conducting spheres, separated by a distance l (but joined by a wire) and carrying charges $+q$ and $-q$ that oscillate so that:

$$q = q_0 \cos \omega t \tag{1}$$

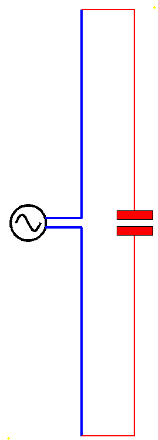
The dipole moment is $p = ql$, so:

$$p = p_0 \cos \omega t \tag{2}$$

where $p_0 = q_0 l$.

Dipole Radiation

Dipole radiation can be generated by an antenna connected to an AC source.

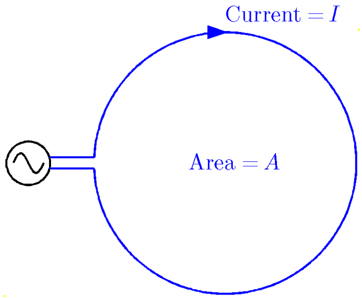


Dipole Radiation

Alternatively, we might have a magnetic dipole, consisting of a current loop of area A, in which the current I varies as:

$$I = I_0 \cos \omega t \tag{3}$$

The magnetic dipole moment is $\vec{M} = I\vec{A}$.



Fields and Potentials

Our goal is to find the electric field $\vec{E}$ and the magnetic field $\vec{B}$ in the presence of a time-dependent source (e.g. a time-dependent dipole).

We will take the approach of finding first the electric scalar potential ϕ and the magnetic vector potential $\vec{A}$, and then deriving the fields using:

$$\vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t} \tag{4}$$

$$\vec{B} = \nabla \times \vec{A} \tag{5}$$

Recall that, in the Lorenz gauge, the time-dependent potentials satisfy the wave equations:

$$\nabla^2 \vec{A} - \frac{1}{c^2}\ddot{\vec{A}} = -\mu \vec{J} \tag{6}$$

$$\nabla^2 \phi - \frac{1}{c^2}\ddot{\phi} = -\frac{\rho}{\varepsilon} \tag{7}$$

In general, the current density $\vec{J}$ and the charge density ρ can be functions of time.

Retarded Potentials

Recall that the wave equation for the vector potential (6):

$$\nabla^2 \vec{A} - \frac{1}{c^2}\ddot{\vec{A}} = -\mu \vec{J} \tag{8}$$

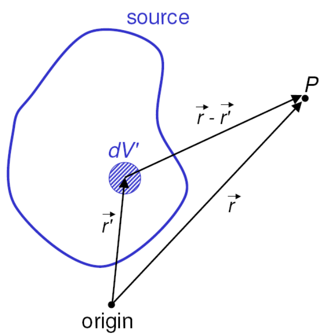
has the solution:

$$\vec{A}(\vec{r}, t) = \frac{\mu}{4\pi} \int \frac{\vec{J}(\vec{r}', t')}{|\vec{r} - \vec{r}'|} dV' \tag{9}$$

where

$$t = t' + \frac{|\vec{r} - \vec{r}'|}{c} \tag{10}$$

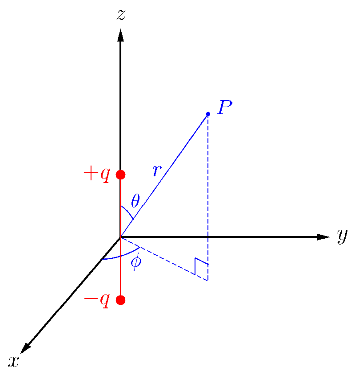
We shall apply this solution to find the potential, and hence the fields, around an oscillating electric dipole.



Hertzian Dipole

Let us consider a dipole constructed from two conducting spheres in free space, separated by a distance l along the z axis.

Our goal will be to find the electric and magnetic fields at some point P, a distance r from the dipole, in a direction specified by polar angle θ and azimuthal angle ϕ.



Hertzian Dipole

Let the charge on one sphere be $+q$, and that on the other sphere be $-q$. We suppose that the charge is oscillating between the two spheres, such that:

$$q = q_0 e^{j\omega t} \tag{11}$$

The dipole moment is given by:

$$\vec{p} = q\,\vec{l} = q_0\,\vec{l}\,e^{j\omega t} = \vec{p}_0 e^{j\omega t} \tag{12}$$

We can construct an infinitesimal dipole by taking the limit $|\vec{l}| \to 0$; but as we do so we allow q_0 to increase, so that the dipole moment $\vec{p}_0$ remains constant.

An infinitesimal electric dipole oscillating at fixed frequency is called a *Hertzian dipole*.

Electric Current in a Hertzian Dipole

The electric current flow at the midpoint of an oscillating dipole is:

$$I = \frac{dq}{dt} = j\omega q_0 e^{j\omega t} \tag{13}$$

where $I_0 = j\omega q_0$ is the peak current flow along the z axis.

For a Hertzian (infinitesimal) dipole at the origin, the current is located only at the origin. It follows that:

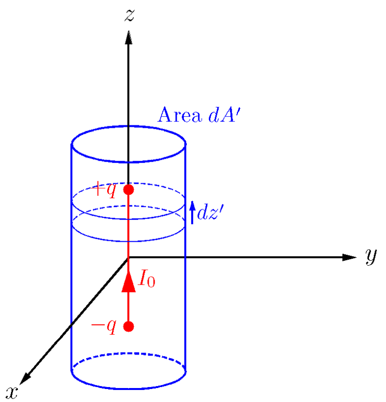
$$\int \frac{\vec{J}(\vec{r}', t')}{|\vec{r} - \vec{r}'|}\, dV' = \frac{I(t')\,\vec{l}}{r} = \frac{j\omega q_0\, e^{j\omega t'}\,\vec{l}}{r} \tag{14}$$

where $r = |\vec{r}|$.

Using $\vec{p}_0 = q_0 \vec{l}$, we can write the integral in terms of the dipole moment:

$$\int \frac{\vec{J}(\vec{r}', t')}{|\vec{r} - \vec{r}'|}\, dV' = j\omega \vec{p}_0 \frac{e^{j\omega t'}}{r} \tag{15}$$

Vector Potential Around a Hertzian Dipole



Vector Potential Around a Hertzian Dipole

Substituting the expression for the integral (15) into the expression for the vector potential (9) we find:

$$\vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} j\omega \vec{p}_0 \frac{e^{j\omega t'}}{r} = \frac{\mu_0}{4\pi} j\omega \vec{p}_0 \frac{e^{j(\omega t - kr)}}{r} \tag{16}$$

where we have used:

$$t' = t - \frac{|\vec{r} - \vec{r}'|}{c} \tag{17}$$

and we have defined:

$$k = \frac{\omega}{c} \tag{18}$$

Notice that the vector potential is always oriented parallel to the dipole moment, and takes the form of a spherical wave propagating away from the origin.

Magnetic Field Around a Hertzian Dipole

From the vector potential (16), we can calculate the fields.

It is convenient to work in spherical polar coordinates. Let us assume that the charge in the Hertzian dipole is oscillating parallel to the z axis.

In spherical polar coordinates, the vector potential $\vec{A}$ is:

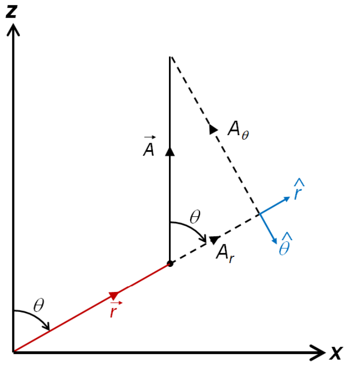
$$A_r = A\cos\theta \tag{19}$$

$$A_\theta = -A\sin\theta \tag{20}$$

$$A_\phi = 0 \tag{21}$$

where

$$A = \frac{\mu_0}{4\pi}\frac{e^{j(\omega t - kr)}}{r} j\omega p_0 \tag{22}$$



Magnetic Field Around a Hertzian Dipole

The magnetic field $\vec{B}$ is given by:

$$\vec{B} = \nabla \times \vec{A} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{vmatrix} \tag{23}$$

where $\hat{r}$, $\hat{\theta}$ and $\hat{\phi}$ are unit vectors along each of the axes.

The result for the magnetic field is:

$$B_r = 0 \tag{24}$$

$$B_\theta = 0 \tag{25}$$

$$B_\phi = -\frac{\mu_0}{4\pi}k^2 c p_0 \sin\theta \left(1 - \frac{j}{kr}\right)\frac{e^{j(\omega t - kr)}}{r} \tag{26}$$

Electric Field Around a Hertzian Dipole

The electric field around a Hertzian dipole can be obtained from Maxwell's equation:

$$\nabla \times \vec{H} = \vec{J} + \dot{\vec{D}} \tag{27}$$

where, in free space:

$$\vec{H} = \frac{\vec{B}}{\mu_0} \tag{28}$$

$$\vec{J} = 0 \tag{29}$$

$$\vec{D} = \varepsilon_0 \vec{E} \tag{30}$$

Since the time dependence of all quantities is given by the factor $e^{j\omega t}$, we can write:

$$\nabla \times \vec{H} = j\omega\varepsilon_0 \vec{E} \tag{31}$$

or:

$$\vec{E} = -\frac{j}{\omega\mu_0\varepsilon_0}\nabla \times \vec{B} = -j\frac{c^2}{\omega}\nabla \times \vec{B} \tag{32}$$

Electric Field Around a Hertzian Dipole

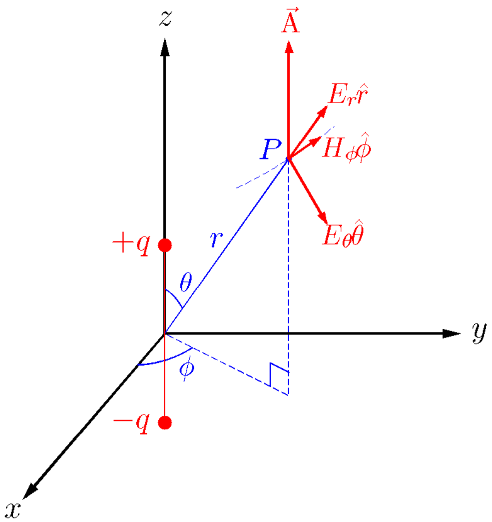
Continuing to work in spherical polar coordinates, taking the curl of the magnetic field (24), (25), (26), we find that:

$$E_r = \frac{j}{4\pi\varepsilon_0} 2kp_0 \cos\theta \left(1 - \frac{j}{kr}\right)\frac{e^{j(\omega t - kr)}}{r^2} \tag{33}$$

$$E_\theta = -\frac{1}{4\pi\varepsilon_0}k^2 p_0 \sin\theta \left(1 - \frac{j}{kr} - \frac{1}{k^2 r^2}\right)\frac{e^{j(\omega t - kr)}}{r} \tag{34}$$

$$E_\phi = 0 \tag{35}$$

Fields Around a Hertzian Dipole

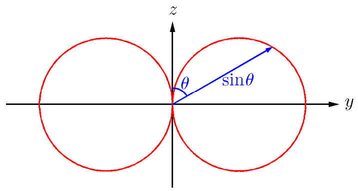


Angular Dependence of Fields Around a Hertzian Dipole

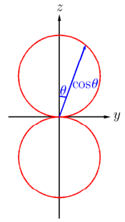
Neither the magnetic field $\vec{B}$ nor the electric field $\vec{E}$ have any dependence on the azimuthal coordinate ϕ. This is because of the rotational symmetry of the system around the z axis.

The dependence of the fields on the coordinate θ are as follows:

$$B_\phi \propto \sin\theta, \qquad E_r \propto \cos\theta, \qquad E_\theta \propto \sin\theta \tag{36}$$



$E_\theta,\ H_\phi$



E_r

Radial Dependence of Fields Around a Hertzian Dipole

The dependence of the fields on the radial coordinate r (distance from the dipole) is a little more complicated. We generally consider two approximations:

- the near field approximation, $kr \ll 1$, i.e. $r \ll \lambda/2\pi$
- the far field approximation, $kr \gg 1$, i.e. $r \gg \lambda/2\pi$

Fields Around a Hertzian Dipole: Near Field Approximation

In the near field approximation, $kr \ll 1$:

$$B_\phi \approx j\frac{\mu_0}{4\pi}kcp_0 \sin\theta \frac{e^{j(\omega t - kr)}}{r^2} \tag{37}$$

$$E_r \approx \frac{1}{4\pi\varepsilon_0}2p_0 \cos\theta \frac{e^{j(\omega t - kr)}}{r^3} \tag{38}$$

$$E_\theta \approx \frac{1}{4\pi\varepsilon_0}p_0 \sin\theta \frac{e^{j(\omega t - kr)}}{r^3} \tag{39}$$

Note that in the near field approximation, at a given point in space, the radial and polar components of the electric field oscillate in phase with each other.

However, the oscillation of the magnetic field leads the oscillation of the electric field by a phase of 90°.

Fields Around a Hertzian Dipole: Far Field Approximation

In the far field approximation, $kr \gg 1$:

$$B_\phi \approx -\frac{\mu_0}{4\pi} k^2 c p_0 \sin\theta \frac{e^{j(\omega t - kr)}}{r} \quad (40)$$

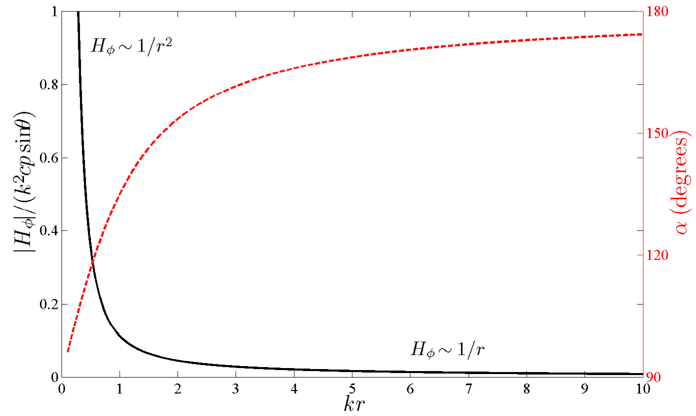
$$E_r \approx \frac{j}{4\pi\varepsilon_0} 2kp_0 \cos\theta \frac{e^{j(\omega t - kr)}}{r^2} \quad (41)$$

$$E_\theta \approx -\frac{1}{4\pi\varepsilon_0} k^2 p_0 \sin\theta \frac{e^{j(\omega t - kr)}}{r} \quad (42)$$

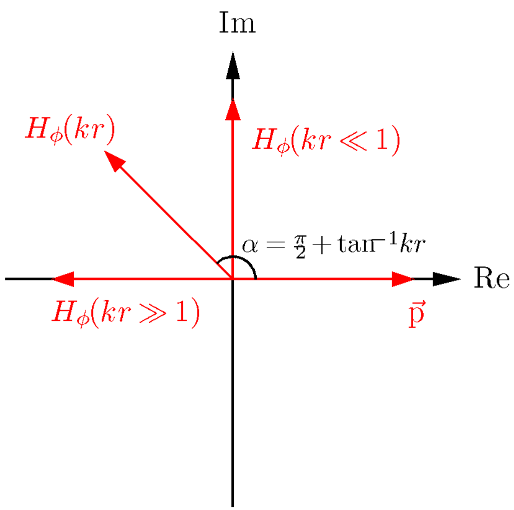
Now the azimuthal component of the magnetic field is in phase with the polar component of the electric field; and these field components lead the radial component of the electric field by a phase of 90°.

Note that the amplitude of the radial component of the electric field falls off as $1/r^2$, whereas the magnetic field and the polar component of the electric field fall off more slowly, as $1/r$.

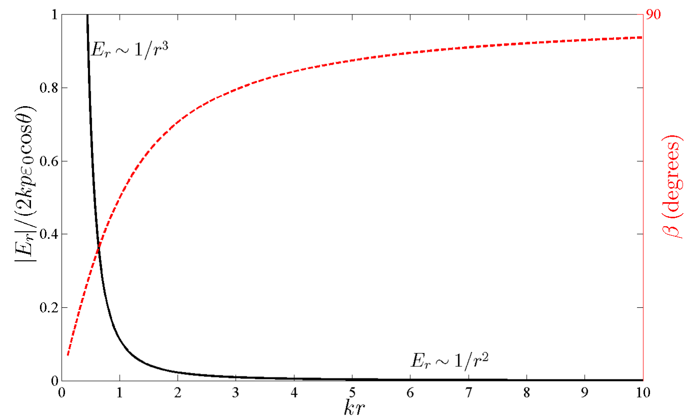
Fields Around a Hertzian Dipole



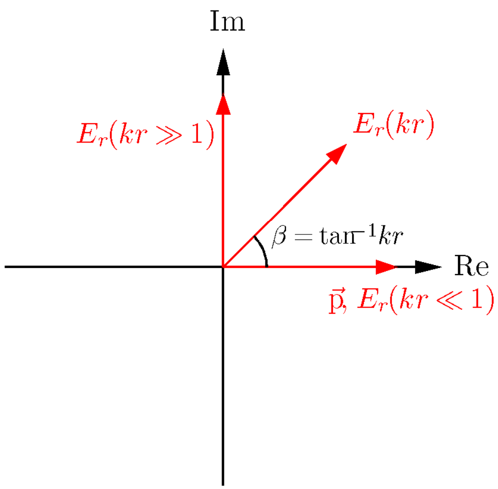
Fields Around a Hertzian Dipole



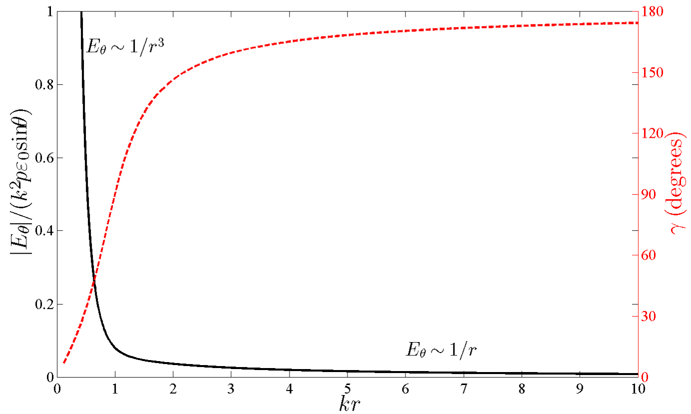
Fields Around a Hertzian Dipole



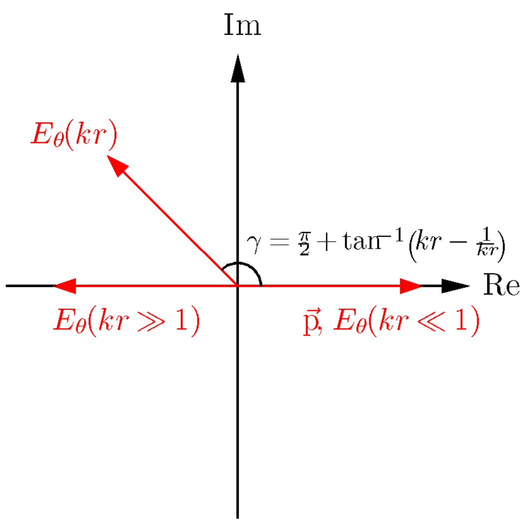
Fields Around a Hertzian Dipole



Fields Around a Hertzian Dipole



Fields Around a Hertzian Dipole



Fields Around a Hertzian Dipole: Far Field Approximation

In the far field approximation, $kr \gg 1$:

$$B_\phi \approx -\frac{\mu_0}{4\pi} k^2 c p_0 \sin\theta \frac{e^{j(\omega t - kr)}}{r} \quad (43)$$

$$E_r \approx \frac{j}{4\pi\varepsilon_0} 2kp_0 \cos\theta \frac{e^{j(\omega t - kr)}}{r^2} \quad (44)$$

$$E_\theta \approx -\frac{1}{4\pi\varepsilon_0} k^2 p_0 \sin\theta \frac{e^{j(\omega t - kr)}}{r} \quad (45)$$

In the limit of large r, the components B_ϕ and E_θ dominate.

We essentially have a travelling plane wave, with orthogonal magnetic and electric field components, and with greatest intensity on the mid-plane of the dipole.

The Poynting vector $\vec{S} = \vec{E} \times \vec{H}$ only has a component in the radial direction, so there is a flux of electromagnetic energy away from the dipole.

Fields Around a Hertzian Dipole: Far Field Approximation

In the far field approximation, $kr \gg 1$:

$$B_\phi \approx -\frac{\mu_0}{4\pi}k^2 c p_0 \sin\theta \frac{e^{j(\omega t - kr)}}{r} \tag{46}$$

$$E_r \approx \frac{j}{4\pi\varepsilon_0} 2kp_0 \cos\theta \frac{e^{j(\omega t - kr)}}{r^2} \tag{47}$$

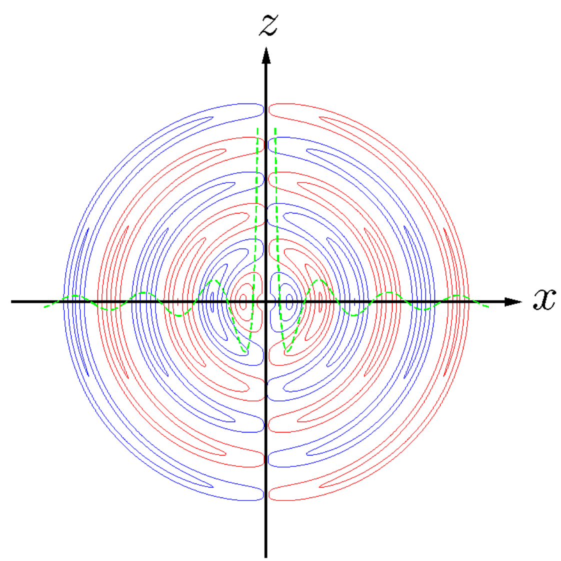
$$E_\theta \approx -\frac{1}{4\pi\varepsilon_0} k^2 p_0 \sin\theta \frac{e^{j(\omega t - kr)}}{r} \tag{48}$$

Finally, we note that for large r, the ratio of the electric and magnetic field strengths are given by:

$$\lim_{r\to\infty}\frac{E}{B} = \lim_{r\to\infty}\frac{E_\theta}{B_\phi} = \frac{1}{\mu_0\varepsilon_0 c} = c \tag{49}$$

as expected for a plane electromagnetic wave.

Fields Around a Hertzian Dipole at a Fixed Instant



Power Radiated by a Hertzian Dipole

Let us now calculate the power radiated by a Hertzian dipole.

The energy flux is given by the Poynting vector:

$$\vec{S} = \vec{E} \times \vec{H} \tag{50}$$

The Poynting vector is second-order in the fields, so we first have to take the real parts of the complex expressions for the fields.

Taking the real parts of the non-zero field components, (26), (33) and (34), we find:

$$E_r = -\frac{2kp_0}{4\pi\varepsilon_0}\frac{\cos\theta}{r^2}\left[\sin(\omega t - kr) - \frac{\cos(\omega t - kr)}{kr}\right] \tag{51}$$

$$E_\theta = -\frac{k^2 p_0}{4\pi\varepsilon_0}\frac{\sin\theta}{r}\left[\left(1 - \frac{1}{k^2r^2}\right)\cos(\omega t - kr) + \frac{\sin(\omega t - kr)}{kr}\right] \tag{52}$$

$$H_\phi = -\frac{k^2 p_0 c}{4\pi}\frac{\sin\theta}{r}\left[\cos(\omega t - kr) + \frac{\sin(\omega t - kr)}{kr}\right] \tag{53}$$

Power Radiated by a Hertzian Dipole

We can write the vector product as:

$$\vec{S} = \begin{vmatrix} \hat{r} & \hat{\theta} & \hat{\phi} \\ E_r & E_\theta & 0 \\ 0 & 0 & H_\phi \end{vmatrix} \tag{54}$$

Multiplying the appropriate vector components, and taking the average over time, we find:

$$\langle S_\theta \rangle_t = \langle S_\phi \rangle_t = 0 \tag{55}$$

and:

$$\langle S_r \rangle_t = \frac{k^4 c p_0^2}{32\pi^2\varepsilon_0}\frac{\sin^2\theta}{r^2} \tag{56}$$

Only the radial component of the Poynting vector has a non-zero average over time: electromagnetic radiation is emitted radially away from the dipole.

Note that we have performed the calculation exactly (not using either the near-field or far-field approximations).

Power Radiated by a Hertzian Dipole

The average energy flux from a Hertzian dipole is given by (56):

$$\langle \vec{S} \rangle_t = \frac{k^4 c p_0^2}{32\pi^2 \varepsilon_0} \frac{\sin^2\theta}{r^2} \hat{r} \qquad (57)$$

Since $\langle S_r \rangle_t \propto \sin^2\theta$, the energy flux is largest at $\theta = 90°$, and zero at $\theta = 0°$ and $\theta = 180°$. There is no energy radiated along the axis of the dipole; and the radiation is largest in the mid-plane perpendicular to the axis.

Finally, note that the energy flux is proportional to k^4: energy is radiated much more effectively at shorter wavelengths. This explains why the sky is blue.

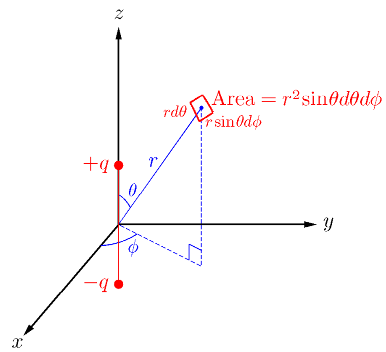
Gas molecules in the atmosphere behave as Hertzian dipoles excited by radiation from the sun. Shorter wavelengths (blue light) are scattered much more effectively than longer wavelengths (red light) which penetrate more directly to the surface of the Earth.

Total Power Radiated by a Hertzian Dipole

To calculate the total (time-average) power, W radiated by a Hertzian dipole, we must integrate the energy flux (56) over a sphere enclosing the dipole.

In spherical polar coordinates, the area element is:

$$dS = r^2 \sin\theta \, d\theta \, d\phi \qquad (58)$$



Total Power Radiated by a Hertzian Dipole

Hence:

$$W = \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \frac{k^4 c p_0^2}{32\pi^2 \varepsilon_0} \frac{\sin^2\theta}{r^2} r^2 \sin\theta \, d\theta \, d\phi \qquad (59)$$

Since:

$$\int_{\phi=0}^{2\pi} d\phi = 2\pi, \qquad \int_{\theta=0}^{\pi} \sin^3\theta \, d\theta = \frac{4}{3} \qquad (60)$$

we find that:

$$W = \frac{k^4 c p_0^2}{12\pi\varepsilon_0} \qquad (61)$$

Total Power Radiated by a Hertzian Dipole

The total power radiated by a Hertzian dipole is:

$$W = \frac{k^4 c p_0^2}{12\pi\varepsilon_0} \qquad (62)$$

Note that:

- $W \propto p_0^2$, therefore $W \propto I_0^2$ where I_0 is the peak current in the dipole.
- $W \propto k^4$, therefore $W \propto \omega^4$ where ω is the frequency of the dipole oscillations.

Total Power Radiated by a Hertzian Dipole

Recall that the dipole amplitude p_0 is given by:

$$p_0 = q_0 \, l \tag{63}$$

where $\pm q_0$ is the maximum charge on each of two small conducting spheres with separation l.

Since the current is the rate of flow of charge, we have for charge oscillations of frequency ω:

$$j\omega p_0 = I_0 \, l \tag{64}$$

Hence, using equation (61) we can express the total power W radiated by the dipole in terms of the peak current I_0 and dipole length l:

$$W = \frac{(kl)^2}{12\pi\varepsilon_0 c} I_0^2 = \frac{1}{2}\frac{Z_0(kl)^2}{6\pi} I_0^2 \tag{65}$$

Power Radiated by a Hertzian Dipole: Radiation Resistance

Compare equation (65):

$$W = \frac{1}{2}\frac{Z_0(kl)^2}{6\pi} I_0^2 \tag{66}$$

with the expression for the average electrical power P dissipated by an alternating current in a resistance R:

$$P = \frac{1}{2} R I_0^2 \tag{67}$$

We define the *radiation resistance* R:

$$R = \frac{Z_0(kl)^2}{6\pi} \tag{68}$$

The radiation resistance is the resistance that would dissipate (in the form of heat) the same power that the dipole radiates in the form of electromagnetic waves, if it carried the same current.

Radiation from a Half-Wave Antenna

A half-wave antenna is a wire of length equal to half the wavelength of the electromagnetic waves that it generates. The length has to be correctly matched to the frequency of the radiation.

When an alternating current $I_0 e^{j\omega t}$ is driven from the centre of the antenna, standing waves are set up in the antenna, with zero current flow at the ends.

The exact variation of the current along the wire is affected by the radiation losses, and (in practical situations) by the presence of the earth.

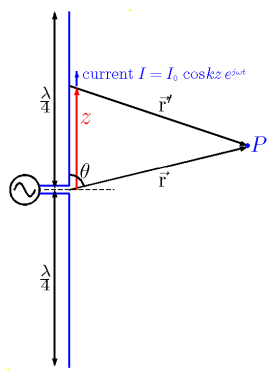
To a good approximation, the current at a point z along the antenna (with $z = 0$ at the centre) is given by:

$$I(z,t) = I_0 \cos kz \, e^{j\omega t} \tag{69}$$

Radiation from a Half-Wave Antenna

To find the radiation field, we can treat each small section of the wire as a Hertzian dipole, and sum the contributions from every point along the wire.

It is important to take account of the phase differences.



We can write the current as the sum of two travelling waves (in opposite directions):

$$I(z,t) = \frac{1}{2} I_0 \left(e^{jkz} + e^{-jkz} \right) e^{j\omega t} \tag{70}$$

For an element dz of the wire, the dipole moment is given by:

$$p = \frac{1}{j\omega} I\,dz = \frac{1}{2j\omega} I_0 \left(e^{jkz} + e^{-jkz} \right) e^{j\omega t}\,dz \tag{71}$$

Let us consider the field at a point distant from the antenna, i.e. in the radiation region, with $r \gg \lambda$.

For the electric field, the radial component vanishes, and the polar component is given by:

$$E_\theta = -\frac{k^2 p \sin\theta}{4\pi\varepsilon_0} \frac{e^{-jkr}}{r} \tag{72}$$

Combining the expressions for the dipole moment (71) and the electric field (72), we can write:

$$dE_\theta = -\frac{k^2}{4\pi\varepsilon_0} \left(\frac{1}{2j\omega} I_0 (e^{jkz} + e^{-jkz}) \right) \sin\theta \frac{e^{-jkr'}}{r'} e^{j\omega t} dz \tag{73}$$

where

$$r' = r - z\cos\theta \tag{74}$$

Since $r \gg \lambda$, we can approximate $1/r' \approx 1/r$.

At a given point in space (far from the antenna) we can take the contributions dE_θ as parallel, but with different phases.

Then we have:

$$dE_\theta = -\frac{k^2}{4\pi\varepsilon_0} \frac{1}{2j\omega} \frac{e^{j(\omega t - kr)}}{r} I_0 \sin\theta \left(e^{jkz(1+\cos\theta)} + e^{-jkz(1-\cos\theta)} \right) dz \tag{75}$$

Integrating equation (75) over all points along the antenna gives the total electric field:

$$\begin{aligned} E_\theta &= \int dE_\theta \\ &= -\int_{-\lambda/4}^{\lambda/4} \frac{k^2}{4\pi\varepsilon_0} \frac{1}{2j\omega} \frac{e^{j(\omega t - kr)}}{r} I_0 \sin\theta \left(e^{jkz(1+\cos\theta)} + e^{-jkz(1-\cos\theta)} \right) dz \\ &= -\frac{k^2}{4\pi\varepsilon_0} \frac{1}{2j\omega} \frac{e^{j(\omega t - kr)}}{r} I_0 \sin\theta \left[\frac{e^{jkz(1+\cos\theta)}}{jk(1+\cos\theta)} - \frac{e^{-jkz(1-\cos\theta)}}{jk(1-\cos\theta)} \right]_{-\lambda/4}^{\lambda/4} \\ &= \frac{k}{4\pi\varepsilon_0} \frac{j}{\omega} \frac{e^{j(\omega t - kr)}}{r} I_0 \sin\theta \left[\frac{\sin(\frac{\pi}{2}(1+\cos\theta))}{(1+\cos\theta)} - \frac{\sin(\frac{\pi}{2}(1-\cos\theta))}{(1-\cos\theta)} \right] \end{aligned} \tag{76}$$

Now we use the results:

$$\sin(\frac{\pi}{2}(1 \pm \cos\theta)) = \cos(\frac{\pi}{2}\cos\theta) \tag{77}$$

and:

$$\frac{x}{1-\cos\theta} + \frac{x}{1+\cos\theta} = \frac{2x}{1-\cos^2\theta} = \frac{2x}{\sin^2\theta} \tag{78}$$

Then, using also $\omega/k = c$, we find from (76):

$$E_\theta = \frac{2j}{4\pi\varepsilon_0 c} I_0 \frac{\cos(\frac{\pi}{2}\cos\theta)}{\sin\theta} \frac{e^{j(\omega t - kr)}}{r} \tag{79}$$

In the far-field region, the radial and azimuthal components of the electric field are zero, i.e. for $1/r \approx 0$:

$$E_r \approx 0, \qquad E_\phi \approx 0 \tag{80}$$

Equation (76) gives the electric field far from the antenna. Note that at $\theta \to 0$ and $\theta \to \pi$, the angular dependence gives $E_\theta \to 0/0$.

We can use l'Hopital's rule to find the limit:

$$\begin{aligned}
\lim_{\theta \to 0,\pi} \frac{\cos(\frac{\pi}{2}\cos\theta)}{\sin\theta} &= \left\{ \frac{\frac{d}{d\theta}\cos(\frac{\pi}{2}\cos\theta)}{\frac{d}{d\theta}\sin\theta} \right\}_{\theta=0,\pi} \\
&= \left\{ \frac{\sin(\frac{\pi}{2}\cos\theta)\,\frac{\pi}{2}\sin\theta}{\cos\theta} \right\}_{\theta=0,\pi} \\
&= 0
\end{aligned}$$

So at $\theta = 0$ and $\theta = \pi$, the electric field is zero: there is no radiation along the axis of the antenna.

Note also that, for the half-wave antenna, the field intensity is independent of frequency (as long as the length is correctly matched to the frequency). This contrasts with the case of the Hertzian dipole, where the field intensity $\propto k^2$.

We can find the magnetic field associated with the time-dependent electric field using Maxwell's equation:

$$\nabla \times \vec{E} = -\dot{\vec{B}} = -j\omega\vec{B} \tag{81}$$

We find that, in the far-field region, the magnetic field is:

$$B_\phi = 2j\frac{\mu_0}{4\pi}I_0\frac{\cos(\frac{\pi}{2}\cos\theta)}{\sin\theta}\frac{e^{j(\omega t - kr)}}{r} \tag{82}$$

The radial and polar components are small:

$$H_r \approx 0, \qquad H_\theta \approx 0 \tag{83}$$

Note that, as expected, we find:

$$\frac{E_\theta}{B_\phi} = c \tag{84}$$

The energy flux is given by the Poynting vector. As usual, we have to take the real parts of the complex expressions for the fields, before calculating the vector product.

The time-averaged flux is:

$$\langle\vec{S}\rangle_t = \frac{I_0^2}{8\pi^2\varepsilon_0 c}\frac{\cos^2(\frac{\pi}{2}\cos\theta)}{r^2\sin^2\theta}\hat{r} \tag{85}$$

Note that the flux $\propto 1/r^2$, so the power across the surface of a sphere centered on the antenna is independent of the radius of the sphere.

The total power W radiated by the antenna is found by integrating over a spherical surface:

$$W = \int_0^\pi \int_0^{2\pi} \langle\vec{S}\rangle_t\, r^2 \sin\theta\, d\theta\, d\phi \tag{86}$$

Evaluating the integral gives:

$$W \approx \frac{1.22}{4\pi\varepsilon_0 c}I_0^2 \approx 0.194 Z_0\, I_{RMS}^2 \tag{87}$$

where $Z_0 \approx 376.7\,\Omega$ is the impedance of free space.

Note that the power W is independent of the frequency.

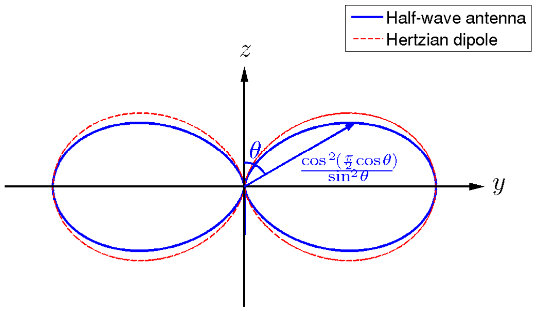
The radiation resistance is $R \approx 73.1\Omega$. The antenna is easily matched to a coaxial line with characteristic impedance $\approx 75\,\Omega$.

Radiation Distribution from a Half-Wave Antenna

The power radiated by a half-wave antenna is symmetric in the azimuthal angle ϕ, but has a variation with the polar angle θ:

$$\langle S_r \rangle_t \propto \frac{\cos^2(\frac{\pi}{2}\cos\theta)}{\sin^2\theta} \qquad (88)$$

The distribution of radiation from a half-wave antenna is similar to that from a Hertzian dipole, but is "more directional".



Radiation Distribution from a Half-Wave Antenna

Note:

- The energy flux $\vec{S}$ and total radiated power W are independent of frequency: this is a common feature of many antenna systems.
- A vertical half-wave antenna will radiate power equally in all horizontal directions. This is a property often suitable for broadcasting applications (radio and television).
- Radio and television broadcasting antennae often consist of a single vertical mast, with the current driven from the base. The ground (often covered with a conducting screen) acts as a mirror.

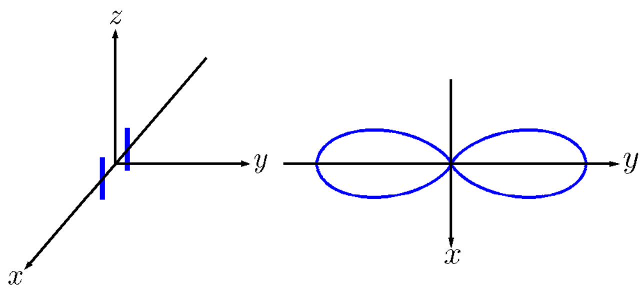
Antenna Arrays

Hertzian dipoles and half-wave antennae radiate isotropically in the plane perpendicular to the orientation of the dipole or antenna.

It is often desirable to radiate (or receive) energy in a particular direction, i.e. to make the transmitter or receiver *directional*. This can be achieved using interference effects in arrays of antennae.

Antenna Arrays

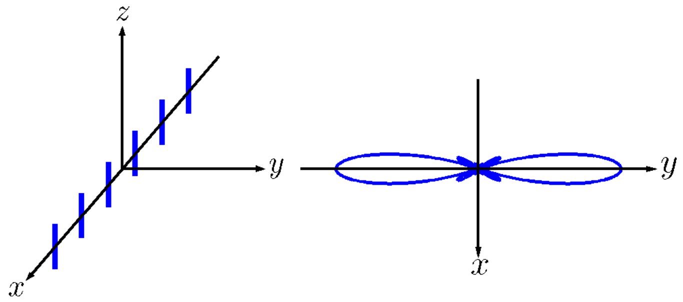
For example, we can use two parallel half-wave antennae, space a distance of $\lambda/2$ apart, and excited in phase from the same source. This gives directionality in the plane perpendicular to the antennae.



Antenna Arrays

By adding more half-wave antennae in a row (all spaced by $\lambda/2$) we can further improve the directionality.

For example, with six antennae in a row:

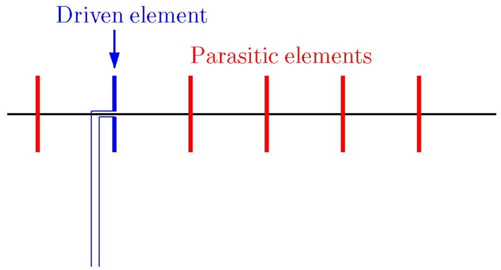


The small "secondary" maxima can be reduced by decreasing the current in the antennae away from the centre; but the principal maxima become wider. The final design is a compromise between the two effects.

Antenna Arrays

A further possibility is the use of "parasitic elements".

In this case, we have a single antenna in which the current is driven and adjacent (parasitic) antenna that are excited by the radiation from the driven antenna.



Antenna Arrays

The resultant field of an antenna array is again a superposition of the fields from all the elements, with the directionality provided by interference effects.

The lengths of the parasitic elements can be tuned to control the directionality further, over some bandwidth (i.e. over some frequency range).

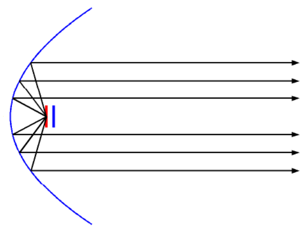
In general, a very narrow beam (i.e. a highly directional antenna) requires an array of antennae that is large compared with the wavelength.

Parabolic Reflectors

A parabolic reflector can be used to direct the waves from an antenna placed at the focus of the reflector.

The reflector must be large compared to the wavelength of the radiation.

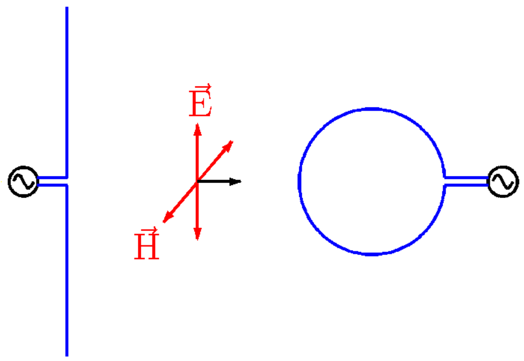
This technique is frequently used with microwaves, since the wavelength of microwave radiation is of the order of 1 cm or less. This makes the sizes of the components (i.e. the antenna and the reflector) manageable.



The Reciprocity Theorem and Receiving Antenna

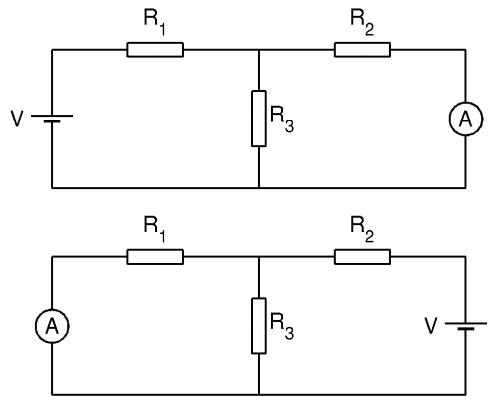
Consider an antenna (a source) with some current driven in it, so that it radiates electromagnetic waves.

If there is a second antenna (a detector) within the field of the emitted radiation, the radiation will drive some current in the detector.



The Reciprocity Theorem and Receiving Antenna

The *reciprocity theorem* states that: the ratio of the current in the detector to the voltage across the source remains the same if the source and detector are interchanged.



The Reciprocity Theorem and Receiving Antenna

The reciprocity theorem implies that a receiving antenna with a given geometry has similar directional properties to a transmitting antenna with the same geometry.

Thus, the theory we have already developed gives the directional properties of an antenna array used for domestic TV reception.

A loop of wire carrying an oscillating electric current acts as an oscillating magnetic dipole, and will radiate electromagnetic waves in a plane parallel to the loop.

The analysis is similar to that of the electric dipole that we have already covered, so we do not present the analysis of the magnetic dipole explicitly. However, we note that from the reciprocity theorem, a loop of wire makes a good receiving antenna. "Loop aerials" are often used on portable TV sets.

Satellite Communications

Consider a satellite following a circular orbit around the earth.

If the satellite's orbit lies in the equatorial plane, and the period of the orbit is 24 hours, then the satellite will stay above a fixed point on the Earth's surface. Such an orbit is called a *geostationary orbit*.

Geostationary orbits are useful for communication satellites because directional detectors on the Earth can be aimed at a fixed point in the sky.

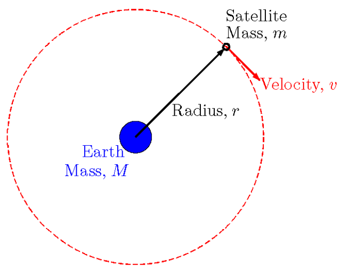
Geostationary orbits have a fixed radius, and this affects the signal power.

For a given strength and angular divergence of the transmitted signal, the recieved signal strength gets weaker as the radius of the orbit increases.

The centripetal force F required for the circular motion is provided by the gravitational attraction between the earth and the satellite:

$$F = \frac{mv^2}{r} = \frac{GMm}{r^2} \tag{89}$$

where m is the mass of the satellite, v is the speed of the satellite, r is the radius of the orbit, M is the mass of the Earth, and G is the gravitational constant.



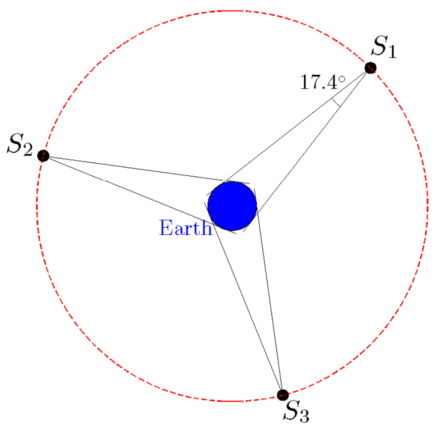
The period T of the satellite's motion (time taken to complete one orbit) is given by:

$$T = \frac{2\pi}{\omega} = 2\pi\frac{r}{v} = 2\pi\sqrt{\frac{r^3}{GM}} \tag{90}$$

where ω is the angular frequency of the satellite's motion.

For a geostationary orbit, $T = 24$ hours; with the known values for the mass of the Earth and the gravitational constant, we find that the radius of a geostationary orbit is 42,250 km, i.e. the satellite should be 36,000 km above the Earth's surface.

To provide complete coverage of the Earth's surface, three geostationary satellites are needed.



The output power of a signal from a satellite is limited by the fact that satellites generally obtain power from solar cells (with efficiency of order 10%).

The power received is determined by the output power, and the area of the reciever divided by the area covered by the signal:

$$P_{\text{received}} = P_{\text{output}} \times \frac{A_{\text{receiver}}}{A_{\text{signal}}} \tag{91}$$

Satellite Communications

As an example, consider a 100 W signal output uniformly over a cone with half-angle 5°, from a geostationary satellite.

The power received by a detector consisting of a parabolic reflector with radius 200 mm is:

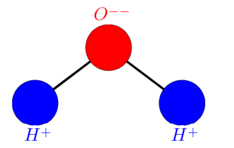
$$P_{\text{received}} = 100\ \text{W} \times \frac{\pi \times 0.2^2}{\pi \times (36 \times 10^6 \times \tan 5°)^2} \approx 4 \times 10^{-13}\ \text{W} \tag{92}$$

Early satellite reception needed large dishes.

Because the power received by a detector with a small parabolic reflector is tiny, a good detector must have narrow bandwidth and use high-quality electronics to avoid noise.

Microwave Ovens

Microwave ovens utilise the fact that water molecules have a small electric dipole moment. Most other substances have zero electric dipole moment.



In water at room temperature, the individual dipole moment vectors point in random directions, but when an electric field is applied, the molecules become partially aligned along the field direction. Thus, an alternating electric field will drive oscillations of the molecules. If the electric field oscillates at the natural frequency of the molecular oscillations, then the resonance between the field oscillations and the molecular oscillations will lead to the transfer of significant amounts of energy from the electric field to the kinetic (heat) energy of the water molecules.

Microwave Ovens

Microwave ovens use a *magnetron* to generate electromagnetic waves at 2.45 GHz (12.2 cm wavelength); the energy is transmitted from an antenna into a shielded cavity.

- Since the power distribution in the cavity is not uniform, the food or the antenna must be rotated to ensure good coverage.
- The absorption length of the radiation is a few cm, so the food must be in small pieces, or the power must be applied in short bursts with periods in between to allow for thermal conduction.
- Microwaves are not efficient for thawing frozen food, since water molecules within ice cannot respond to the radiation in the same way as the molecules within liquid water.

Microwave Ovens: Safety Issues

- The shielding must be efficient to avoid cooking objects (including people) outside the oven. Shielding can consist of metal sheets, possibly with holes much smaller than the wavelength of the radiation.
- Effective interlocks are required to prevent the door being opened while the oven is on.
- The electromagnetic radiation will induce large currents in metal objects placed inside the oven: this has bad effects!
- Water inside the oven damps (attenuates!) the electromagnetic radiation. Field gradients inside an empty oven can become very large, and give potential for damage.

Electromagnetic waves cannot penetrate metal surfaces with thickness much larger than the skin depth δ:

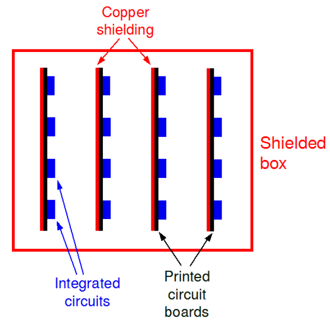
$$\delta \approx \sqrt{\frac{2}{\omega\mu\sigma}} \tag{93}$$

where ω is the frequency of the wave, μ is the magnetic permeability, and σ is the conductivity.

For high conductivity and high frequency, the skin depth is very short.

Metals with high conductivity, such as silver, copper or aluminium, can be used to construct closed boxes that exclude radiation from external sources. Such boxes are useful for protecting devices, such as integrated circuits, that are sensitive to electromagnetic radiation.

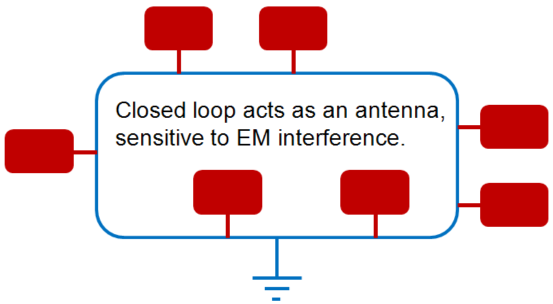
High density electronic circuits are usually protected in shielded boxes. The circuits are mounted on boards, with shielding in-between the boards to protect the circuits on each board from radiation generated by the other boards.



In practice, shielding cannot be complete, because of the need to allow gaps for cooling, and to provide routes for electrical power and signals.

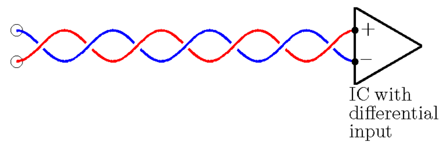
Circuits with more than one connection to "ground" can act as loop aerials, picking up external electromagnetic radiation (noise).

Circuit design should provide single connections to ground to minimise noise levels.



Parallel wires carrying signals can act as transmission lines, also picking up external electromagnetic radiation. By using wires in "twisted pairs", the induced signals on alternate loops cancel, reducing the noise levels.

Using wires in twisted pairs only reduces the induction of currents flowing in opposite directions in the two wires. Electromagnetic radiation can still induce currents flowing in the same direction in each wire. However, noise from such effects can be minimised by using a "differential input" on the device connected by the wires. Such an input responds only to differences in the voltage level on each wire.



Summary of Part 8: Electromagnetic Radiation

You should be able to:

- Explain what is meant by "retarded potential".
- Write expressions for the retarded scalar and vector potentials in terms of integrals over the charge and current densities.
- Describe a "Hertzian dipole", and perform the integrals for the retarded vector potential in the space around a Hertzian dipole.
- Starting from the expression for the vector potential, derive expressions for the magnetic and electric fields around a Hertzian dipole.
- Discuss the phase and amplitude relationships between the various components of the electric and magnetic fields around a Hertzian dipole, in the near-field and far-field approximations.
- Derive an expression for the power radiated by a Hertzian dipole.
- Explain what is meant by the "radiation resistance" for a source of electromagnetic radiation.

Summary of Part 8 (continued): Electromagnetic Radiation

You should be able to:

- By treating a half-wave antenna as a sequence of Hertzian dipoles, derive the properties of the radiation (field amplitudes and phases) around a half-wave antenna.
- Derive an expression for the power radiated by a half-wave antenna, and compare the spatial distribution of the power with that from a Hertzian dipole.
- Explain how antenna arrays can be used to improve the "directionality" of a half-wave antenna.
- State the reciprocity theorem, and explain how it leads to relationships between the properties of transmitting and receiving antennae.
- Discuss applications of transmitters and receivers, including satellite communications, microwave ovens, and shielding of electronic devices.