UNIVERSITY OF LIVERPOOL

PHYS370 – Advanced Electromagnetism

Part 7: Electromagnetic Potentials

In this section, we consider:

- The electric scalar and magnetic vector potentials.
- The wave equations for the electromagnetic potentials.

Potentials

A potential is a function whose derivative gives a field. Fields are associated with forces; potentials are associated with energy.

The magnetic vector potential $\vec{A}$ is defined so that the magnetic field $\vec{B}$ is given by:

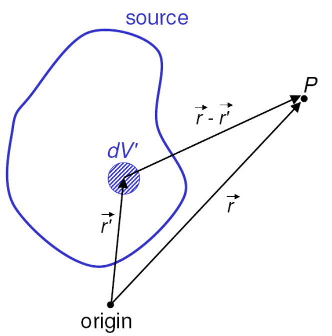
$$\vec{B} = \nabla \times \vec{A} \tag{1}$$

The electric scalar potential ϕ is defined so that the electric field $\vec{E}$ is given by:

$$\vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t} \tag{2}$$

Note that in general, the scalar and vector potentials are functions of position and time.

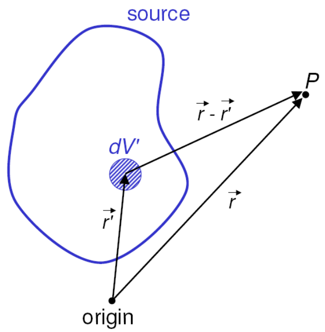
Electrostatic Potential



The electric field in the presence of a static charge distribution $\rho(\vec{r})$ is found from Coulomb's law:

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}')(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}\,dV' \tag{3}$$

where the integral extends over all space. Note that the prime on the coordinates indicates that the coordinate is associated with the charge.



In terms of the scalar potential, for a static charge distribution, we have:

$$\phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|} dV' \quad (4)$$

Calculating the potential is simpler than calculating the field directly; and one can then use $\vec{E} = -\nabla\phi$ to find the electric field.

Since we have from Maxwells' equations:

$$\nabla \cdot \vec{D} = \rho \quad (5)$$

where

$$\vec{D} = \varepsilon \vec{E} \quad (6)$$

it follows that in an homogeneous, isotropic medium:

$$\nabla \cdot \vec{E} = -\nabla \cdot \nabla\phi = \frac{\rho}{\varepsilon} \quad (7)$$

and so:

$$\nabla^2 \phi(\vec{r}) = -\frac{\rho(\vec{r})}{\varepsilon} \quad (8)$$

Equation (8) is called Poisson's equation.

Equation (4) is the solution to Poisson's equation, expressed as an integral.

The behaviour of a charged particle in an electric field is determined by the field $\vec{E}$, rather than by the potential.

Since $\vec{E} = -\nabla\phi$ for an electrostatic field, we can add any function with vanishing gradient to the potential ϕ, and obtain the same physics. In other words, the behaviour of any electrostatic system is the same under the transformation:

$$\phi(\vec{r}) \mapsto \phi(\vec{r}) + \phi_0 \quad (9)$$

where ϕ_0 is a constant (independent of position).

The freedom that we have in choosing the potential is called *gauge invariance*.

This allows us to choose arbitrarily the point at which $\phi(\vec{r}) = 0$.

Note that if we write the solution to Poisson's equation (4):

$$\phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|} dV' \quad (10)$$

then implicitly (assuming that all charges are within a finite distance from the origin), we make the gauge choice:

$$\phi(\vec{r}) \to 0 \quad \text{as} \quad |\vec{r}| \to \infty \quad (11)$$

In the presence of sources for the magnetic field (i.e. a current distribution), the magnetic field $\vec{B}$ can be found from the Biot-Savart law:

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}') \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} \, dV' \tag{12}$$

where the integral extends over all space.

Generally, the Biot-Savart law is difficult to apply.

It is often easier to first calculate the magnetic vector potential; but first, we need to derive the differential equation for the vector potential.

In a static case (constant fields, charges and currents), the magnetic field is related to the current density by:

$$\nabla \times \vec{B} = \mu \vec{J} \tag{13}$$

Substituting $\vec{B} = \nabla \times \vec{A}$, and using the vector identity:

$$\nabla \times \nabla \times \vec{A} \equiv \nabla \left(\nabla \cdot \vec{A} \right) - \nabla^2 \vec{A} \tag{14}$$

we find:

$$\nabla^2 \vec{A} - \nabla \left(\nabla \cdot \vec{A} \right) = -\mu \vec{J} \tag{15}$$

This looks like a complicated equation; but there is a way to simplify it...

Suppose that:

$$\nabla \cdot \vec{A} = f \tag{16}$$

where f is some function of position. Let us define a new vector potential $\vec{A}'$:

$$\vec{A}' = \vec{A} + \nabla \psi_0 \tag{17}$$

Since:

$$\nabla \times \nabla \psi_0 \equiv 0 \tag{18}$$

for *any* function ψ_0, the new vector potential $\vec{A}'$ gives exactly the same magnetic field as the old vector potential $\vec{A}$.

However, if we choose ψ_0 such that:

$$\nabla^2 \psi_0 = -f \tag{19}$$

then:

$$\nabla \cdot \vec{A}' = \nabla \cdot \vec{A} + \nabla^2 \psi_0 = 0 \tag{20}$$

In other words, given a vector potential, we can always choose to work with another vector potential that gives the same field as the original one, but that has zero divergence.

Assuming that we make such a choice, then equation (15) for the vector potential becomes:

$$\nabla^2 \vec{A} = -\mu \vec{J} \tag{21}$$

This is again Poisson's equation – or rather, three Poisson equations, one for each component of the vectors involved.

Magnetic Vector Potential

Since we already know the solution to Poisson's equation for the scalar potential, we can immediately write down the solution to Poisson's equation for the vector potential:

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|}\,dV' \qquad (22)$$

This integral is generally easier to perform than the one involved in the Biot-Savart law.

Once we have obtained the vector potential, we can derive the magnetic field from $\vec{B} = \nabla \times \vec{A}$.

Gauge Transformations of the Magnetic Vector Potential

Notice that to derive equation (22), we made use of a *gauge transformation* of the vector potential: the magnetic field is unchanged if we make the transformation:

$$\vec{A} \mapsto \vec{A} + \nabla\psi_0 \qquad (23)$$

for *any* scalar function ψ_0.

In particular, we made a gauge transformation so that:

$$\nabla \cdot \vec{A} = 0 \qquad (24)$$

The *gauge condition* (24) is known as the Coulomb gauge, and is implicit in equation (22).

The Coulomb gauge is a convenient choice for static systems; but as we shall see later, there is a better choice for dynamic systems.

Summary (so far)

For time-independent fields, we can perform calculations more simply using the electric scalar and magnetic vector potentials. The potentials obey Poisson's equation in the presence of sources:

$$\nabla^2\phi(\vec{r}) = -\frac{\rho(\vec{r})}{\varepsilon}, \qquad \nabla^2\vec{A}(\vec{r}) = -\mu\vec{J}(\vec{r}) \qquad (25)$$

The physics is invariant under gauge transformations of the scalar and vector potentials:

$$\phi \mapsto \phi + \phi_0, \qquad \vec{A} \mapsto \vec{A} + \nabla\psi_0 \qquad (26)$$

where ϕ_0 is a constant, and ψ_0 is any scalar field. One possible choice of gauge is such that:

$$\phi(|\vec{r}| = \infty) = 0, \qquad \nabla\cdot\vec{A} = 0 \qquad (27)$$

The potentials can be calculated directly from the sources:

$$\phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}\,dV', \qquad \vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int\frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|}\,dV' \qquad (28)$$

Wave Equations with Sources

We are interested in the case of electromagnetic waves produced by time-dependent sources, i.e. charge and current distributions that vary with time.

Note that we can (somewhat artificially) divide electric currents into two sorts:

- "External" currents that cause electromagnetic waves.
- "Induced" currents caused by electromagnetic waves.

We will consider only the case where the conductivity σ of the medium is zero; then we can neglect induced currents, and include only external sources of power.

As usual, we start from Maxwell's equations:

$$\nabla \cdot \vec{D} = \rho \qquad \nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\dot{\vec{B}} \qquad \nabla \times \vec{H} = \vec{J} + \dot{\vec{D}}$$

Taking the curl of $\nabla \times \vec{E}$ we find:

$$\nabla \times \nabla \times \vec{E} = \nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E} = -\mu \nabla \times \dot{\vec{H}} = -\mu \dot{\vec{J}} - \mu\varepsilon \ddot{\vec{E}} \qquad (29)$$

We find that the wave equation for the electric field, in the presence of sources, is:

$$\nabla^2 \vec{E} - \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} = \mu \frac{\partial \vec{J}}{\partial t} + \nabla(\rho/\varepsilon) \qquad (30)$$

Following a similar procedure, starting by taking the curl of $\nabla \times \vec{H}$, we obtain the wave equation for the magnetic field $\vec{H}$:

$$\nabla^2 \vec{H} - \mu\varepsilon \frac{\partial^2 \vec{H}}{\partial t^2} = -\nabla \times \vec{J} \qquad (31)$$

Note that as the sources ρ and $\vec{J}$ approach zero, we obtain the usual source-free wave equations.

Also note that the wave equation for the magnetic field has only a source term dependent on the electric current $\vec{J}$, whereas the wave equation for the electric field includes a source term for the electric charge ρ.

This is a consequence of the fact that there are no magnetic monopoles.

Equations (30) and (31) are best solved using the electric scalar and magnetic vector potentials, and with an appropriate gauge choice. For static electromagnetic fields, we have already seen the Coulomb gauge:

$$\nabla \cdot \vec{A} = 0 \qquad (32)$$

Recall that by making an appropriate gauge transformation:

$$\vec{A} \mapsto \vec{A} + \nabla \psi_0 \qquad (33)$$

we can fix $\nabla \cdot \vec{A}$ to be anything we like, while leaving $\nabla \times \vec{A}$ unchanged. For time-dependent fields, we make the choice of gauge:

$$\nabla \cdot \vec{A} + \mu\varepsilon \frac{\partial \phi}{\partial t} = 0 \qquad (34)$$

The condition (34) is called the Lorenz gauge. Working in the Lorenz gauge simplifies the solution of the wave equations.

Let us substitute:

$$\vec{B} = \nabla \times \vec{A}, \qquad \vec{E} = -\nabla\phi - \dot{\vec{A}} \qquad (35)$$

into Maxwell's equation:

$$\nabla \times \vec{B} = \mu \vec{J} + \mu\varepsilon \dot{\vec{E}} \qquad (36)$$

We obtain:

$$\nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A} = \mu \vec{J} - \mu\varepsilon \nabla\dot{\phi} - \mu\varepsilon \ddot{\vec{A}} \qquad (37)$$

Imposing the Lorenz gauge (34):

$$\nabla \cdot \vec{A} + \mu\varepsilon \dot{\phi} = 0 \qquad (38)$$

we obtain:

$$\nabla^2 \vec{A} - \mu\varepsilon \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu \vec{J} \qquad (39)$$

This is the wave equation for the vector potential $\vec{A}$. Note that in the static case, it reduces to equation (21).

Now we substitute:

$$\vec{E} = -\nabla\phi - \dot{\vec{A}} \tag{40}$$

into Maxwell's equation:

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon} \tag{41}$$

to obtain:

$$-\nabla^2\phi - \nabla \cdot \dot{\vec{A}} = \frac{\rho}{\varepsilon} \tag{42}$$

Imposing the Lorenz gauge (34):

$$\nabla \cdot \vec{A} + \mu\varepsilon\dot{\phi} = 0 \tag{43}$$

we find:

$$\nabla^2\phi - \mu\varepsilon\frac{\partial^2\phi}{\partial t^2} = -\frac{\rho}{\varepsilon} \tag{44}$$

Equation (44) is the wave equation for the electric scalar potential, with sources. In the static case, it reduces to Poisson's equation (8).

We have derived, using Maxwell's equations and the Lorenz gauge condition, the wave equations (39) and (44) for the magnetic vector and electric scalar potentials:

$$\left(\nabla^2 - \mu\varepsilon\frac{\partial^2}{\partial t^2}\right)\vec{A} = -\mu\vec{J} \tag{45}$$

$$\left(\nabla^2 - \mu\varepsilon\frac{\partial^2}{\partial t^2}\right)\phi = -\frac{\rho}{\varepsilon} \tag{46}$$

Let us take the divergence of equation (45), plus $\mu\varepsilon$ multiplied by the time derivative of equation (46):

$$\left(\nabla^2 - \mu\varepsilon\frac{\partial^2}{\partial t^2}\right)\left(\nabla \cdot \vec{A} + \mu\varepsilon\frac{\partial\phi}{\partial t}\right) = -\mu\left(\nabla \cdot \vec{J} + \frac{\partial\rho}{\partial t}\right) \tag{47}$$

But from the Lorenz gauge condition (34), the left hand side of equation (47) must be zero. Hence:

$$\nabla \cdot \vec{J} + \frac{\partial\rho}{\partial t} = 0 \tag{48}$$

Equation (48) is the continuity equation, that expresses the local conservation of electric charge.

In free space, $\mu = \mu_0$ and $\varepsilon = \varepsilon_0$. Recall that:

$$c = \frac{1}{\sqrt{\mu_0\varepsilon_0}} \tag{49}$$

where c is the speed of light in a vacuum. We define the *d'Alembertian operator* $\Box$:

$$\Box = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \tag{50}$$

In terms of the d'Alembertian operator, the wave equations for the potentials in free space can be written:

$$\Box\vec{A} = -\mu_0\vec{J} \tag{51}$$

$$\Box\phi = -\frac{\rho}{\varepsilon_0} \tag{52}$$

The final step is to write down the solution to the wave equations (39) and (44) for the vector and scalar potentials in the presence of sources.

The equations have essentially the same form, so let us consider just the equation for the scalar potential:

$$\nabla^2\phi - \mu\varepsilon\frac{\partial^2\phi}{\partial t^2} = -\frac{\rho}{\varepsilon} \tag{53}$$

In the absence of any charge, we know there are solutions in which changes in ϕ propagate through space at speed $v = 1/\sqrt{\mu\varepsilon}$.

In the presence of a static charge, we know that a solution can be written:

$$\phi(\vec{r}) = \frac{1}{4\pi\varepsilon}\int \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|}\, dV' \tag{54}$$

Solution to the Wave Equation with Sources

Putting aside conservation of charge for a moment, imagine a system in which a point charge q appears at some point $\vec{r}'$ in space at time t', and then disappears a moment later.

An observer measuring the potential at point $\vec{r}$ and time t will find a potential:

$$\phi = \frac{q}{4\pi\varepsilon|\vec{r}-\vec{r}'|} \tag{55}$$

if:

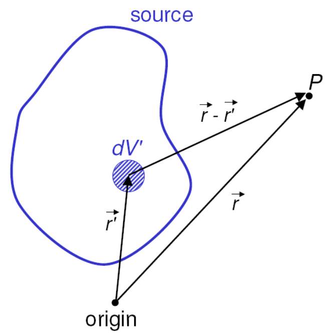
$$t = t' + \frac{|\vec{r}-\vec{r}'|}{v} \tag{56}$$

This suggests the solution to the wave equation with sources:

$$\phi(\vec{r},t) = \frac{1}{4\pi\varepsilon}\int \frac{\rho(\vec{r}',t')}{|\vec{r}-\vec{r}'|}\,dV' \tag{57}$$

where t and t' are related by (56).

Solution to the Wave Equation with Sources



Equation (57) is in fact a correct solution to the wave equation with sources.

The integral is very similar to the static case; but the finite speed of propagation of signals through space is taken into account by evaluating the charge density at the source at an earlier time than the observation time.

Solution to the Wave Equation with Sources

Having obtained the solution to the wave equation for the scalar potential, we can immediately write down the solution to the wave equation for the vector potential:

$$\vec{A}(\vec{r},t) = \frac{\mu}{4\pi}\int \frac{\vec{J}(\vec{r}',t')}{|\vec{r}-\vec{r}'|}\,dV' \tag{58}$$

where, as before:

$$t = t' + \frac{|\vec{r}-\vec{r}'|}{v} \tag{59}$$

In the next part of the course, we shall apply these equations to find the electric and magnetic fields generated by an oscillating dipole. In other words, we shall investigate the generation of electromagnetic waves.

Electromagnetic Potentials: Summary

You should be able to:

- Write expressions for the electric and magnetic fields in terms of the scalar and vector potentials.
- Explain that under a *gauge transformation* of the scalar and vector potentials, the electric and magnetic fields remain unchanged.
- Starting from Maxwell's equations and the expressions for the fields in terms of the potentials, derive wave equations for the potentials in the Lorenz gauge.
- Write integral expressions for the scalar and vector potentials in terms of the source charges and currents, for both static and dynamic systems.