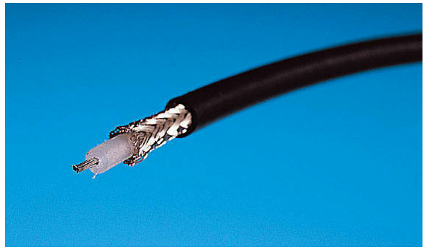
PHYS370 – Advanced Electromagnetism

Part 6: Transmission Lines

Transmission Lines



Transmission lines are used (as are waveguides) to guide electromagnetic waves from one place to another. A coaxial cable (used, for example, to connect a radio or television to an aerial) is an example of a transmission line.

Transmission lines may be less bulky and less expensive than waveguides; but they generally have higher losses, so are more appropriate for carrying low-power signals over short distances.

Transmission Lines

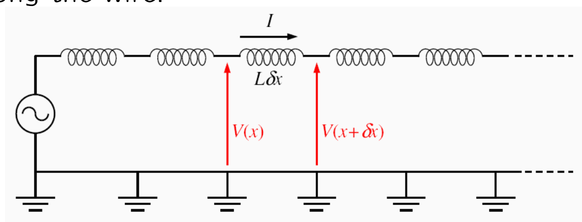
In this part of the course, we shall consider:

- a simple LC model of a general transmission line;
- the speed of propagation of a wave in a transmission line;
- the characteristic impedance of a transmission line;
- impedance matching at the termination of a transmission line;
- practical transmission lines (parallel wires; coaxial cable).

LC Model of a Transmission Line

Consider an infinitely long, parallel wire with zero resistance.

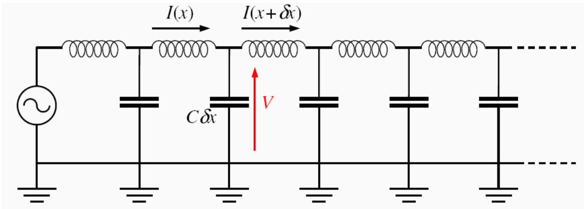
In general, the wire will have some inductance per unit length, L, which means that when an alternating current I flows in the wire, there will be a potential difference between different points along the wire.



If V is the potential at some point along the wire with respect to earth, then the potential difference between two points along the wire is given by:

$$\Delta V = \frac{\partial V}{\partial x}\delta x = -L\delta x \frac{\partial I}{\partial t} \tag{1}$$

In general, as well as the inductance, there will also be some capacitance per unit length, C, between the wire and earth.



This means that the current in the wire can vary with position:

$$\frac{\partial I}{\partial x}\delta x = -C\delta x\frac{\partial V}{\partial t} \tag{2}$$

Let us take Equations (1) and (2) above:

$$\frac{\partial V}{\partial x} = -L\frac{\partial I}{\partial t} \tag{3}$$

$$\frac{\partial I}{\partial x} = -C\frac{\partial V}{\partial t} \tag{4}$$

Differentiate (3) with respect to t:

$$\frac{\partial^2 V}{\partial x \partial t} = -L\frac{\partial^2 I}{\partial t^2} \tag{5}$$

and (4) with respect to x:

$$\frac{\partial^2 I}{\partial x^2} = -C\frac{\partial^2 V}{\partial t \partial x} \tag{6}$$

Hence:

$$\frac{\partial^2 I}{\partial x^2} = LC\frac{\partial^2 I}{\partial t^2} \tag{7}$$

Similarly (by differentiating (3) with respect to x and (4) with respect to t), we find:

$$\frac{\partial^2 V}{\partial x^2} = LC\frac{\partial^2 V}{\partial t^2} \tag{8}$$

Equations (7) and (8) are wave equations for the current in the wire, and the voltage between the wire and earth.

The waves travel with speed v, given by:

$$v = \frac{1}{\sqrt{LC}} \tag{9}$$

The solutions to the wave equations may be written:

$$V = V_0 e^{j(\omega t - kx)} \tag{10}$$

$$I = I_0 e^{j(\omega t - kx)} \tag{11}$$

where the phase velocity is:

$$v = \frac{\omega}{k} = \frac{1}{\sqrt{LC}} \tag{12}$$

Note that the inductance per unit length L and the capacitance per unit length C are real and positive.

Therefore, if the frequency ω is real, the wave number k will also be real: this implies that waves propagate along the transmission line with constant amplitude. This is expected, given our assumption about the line having zero resistance.

Phase Velocity and Impedance

The solutions must also satisfy the first-order equations (1) and (2). Substituting the above solutions into these equations, we find:

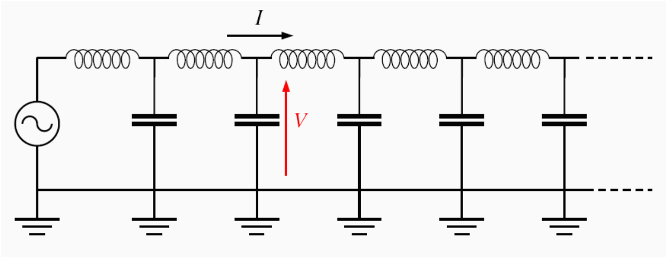
$$kV_0 = \omega L I_0 \tag{13}$$
$$kI_0 = \omega C V_0 \tag{14}$$

Hence:

$$\frac{V_0}{I_0} = \sqrt{\frac{L}{C}} = Z \tag{15}$$

The ratio of the voltage to the current is called the *characteristic impedance*, Z, of the transmission line. Z is measured in ohms, Ω. Note that, since L and C are real and positive, the impedance is a real number: this means that the voltage and current are in phase.

Summary (1)

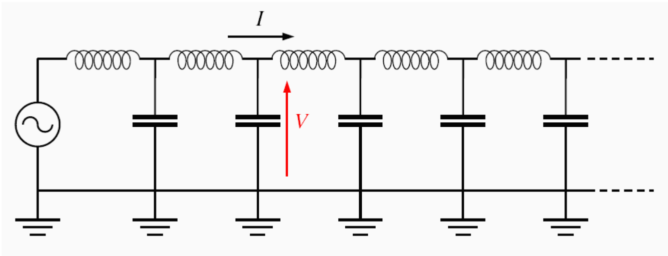


The current and voltage in a transmission line propagate as waves:

$$V = V_0 e^{j(\omega t - kx)} \tag{16}$$

$$I = I_0 e^{j(\omega t - kx)} \tag{17}$$

Summary (2)



The phase velocity v is given by:

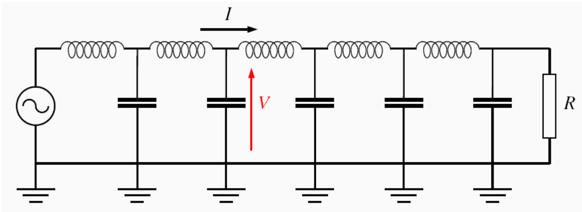
$$v = \frac{\omega}{k} = \frac{1}{\sqrt{LC}} \tag{18}$$

and the ratio of the voltage to the current is given by the characteristic impedance Z:

$$Z = \frac{V_0}{I_0} = \sqrt{\frac{L}{C}} \tag{19}$$

Here, L is the inductance per unit length, and C is the capacitance per unit length.

Impedance Matching



So far, we have assumed that the transmission line has infinite length. Obviously, this cannot be achieved in practice.

We can terminate the transmission line using a "load" that dissipates the energy in the wave while maintaining the same ratio of voltage to current as exists all along the transmission line.

In that case, our above analysis for the infinite line will remain valid for the finite line, and we say that the impedances of the line and the load are properly *matched*.

What happens if the impedance of the load, Z_L, is not properly matched to the characteristic impedance of the transmission line, Z?

In that case, we need to consider a solution consisting of a superposition of waves travelling in opposite directions:

$$V = V_0 e^{j(\omega t - kx)} + K V_0 e^{j(\omega t + kx)} \quad (20)$$

The corresponding current is given by:

$$I = \frac{V_0}{Z} e^{j(\omega t - kx)} - K \frac{V_0}{Z} e^{j(\omega t + kx)} \quad (21)$$

Note the minus sign in the second term in the expression for the current: this comes from equations (3) and (4).

Let us take the end of the transmission line, where the load is located, to be at $x = 0$. At this position, we have:

$$V = V_0 e^{j\omega t}(1 + K) \quad (22)$$

$$I = \frac{V_0}{Z} e^{j\omega t}(1 - K) \quad (23)$$

If the impedance of the load is Z_L, then:

$$Z_L = \frac{V}{I} = Z \frac{1 + K}{1 - K} \quad (24)$$

Solving this equation for K (which gives the relative amplitude and phase of the "reflected" wave), we find:

$$K = \frac{Z_L/Z - 1}{Z_L/Z + 1} = \frac{Z_L - Z}{Z_L + Z} \quad (25)$$

If a transmission line of impedance Z is terminated by a load of impedance $Z_L \neq Z$, then current and voltage signals incident on the load are reflected back down the transmission line.

The ratio K of the reflected wave amplitude to the incident wave amplitude is given by (25):

$$K = \frac{Z_L - Z}{Z_L + Z} \quad (26)$$

If $Z_L = Z$, then there is no reflected wave, and the impedance of the load is correctly matched to the impedance of the transmission line.

Note that for a lossless transmission line, Z is real, which implies that to match the impedances correctly, the load must be a pure resistance. In that case, all the energy in the wave is dissipated in the load.

Voltage Standing Wave Ratio

Note that the voltage in a terminated transmission line can be written, from equation (20):

$$V = V_0 e^{j\omega t}\left(e^{-jkx} + K e^{jkx}\right) \quad (27)$$

If we write:

$$K = |K| e^{j\phi} \quad (28)$$

the voltage can be written:

$$V = V_0 e^{j\left(\omega t + \frac{\phi}{2}\right)} \left[e^{-j\left(kx + \frac{\phi}{2}\right)} + |K| e^{j\left(kx + \frac{\phi}{2}\right)} \right] \quad (29)$$

The physical voltage is the real part of this expression:

$$\begin{aligned} \mathrm{Re}\, V = {} & V_0 \Big[(1 + |K|) \cos\left(\omega t + \frac{\phi}{2}\right) \cos\left(kx + \frac{\phi}{2}\right) \\ & + (1 - |K|) \sin\left(\omega t + \frac{\phi}{2}\right) \sin\left(kx + \frac{\phi}{2}\right) \Big] \end{aligned} \quad (30)$$

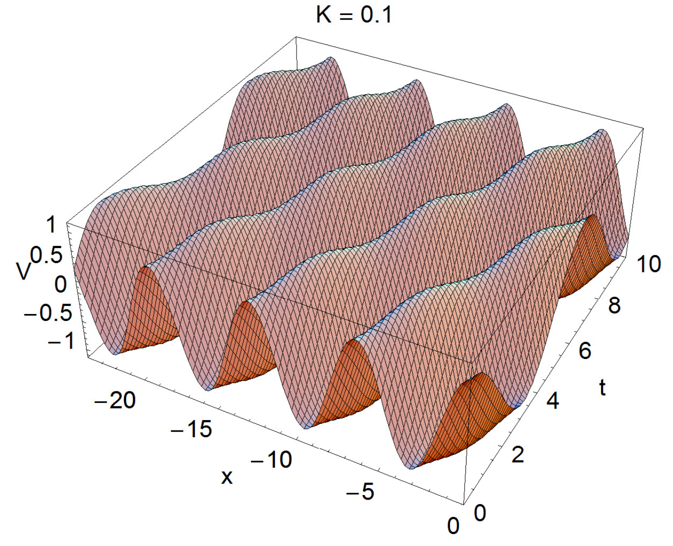
At any given time, the voltage varies sinusoidally with position along the transmission line.

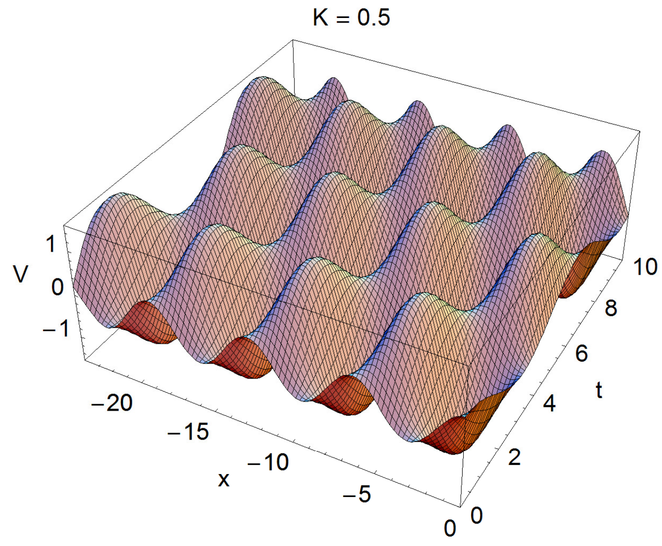
But the amplitude of the variation varies (with time) from $1 - |K|$ to $1 + |K|$.

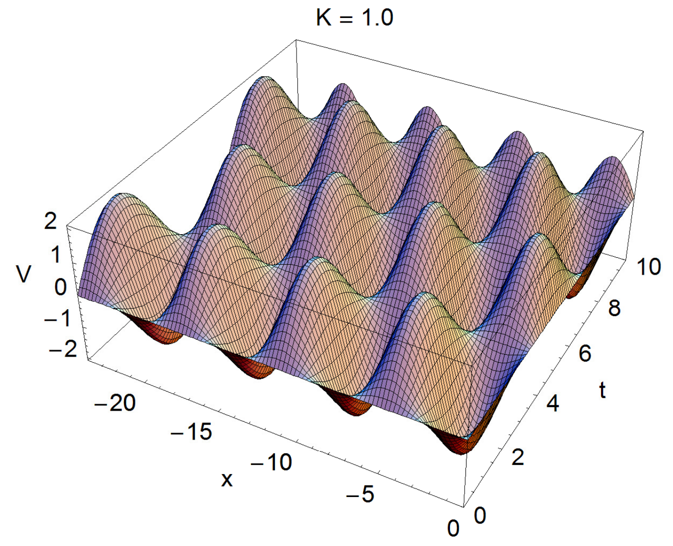
The *voltage standing wave ratio* (VSWR) is defined as the ratio of the maximum to the minimum voltage amplitude:

$$\text{VSWR} = \frac{1 + |K|}{1 - |K|} \tag{31}$$

The VSWR is frequently used to characterise the impedance mismatch at the termination of a transmission line.





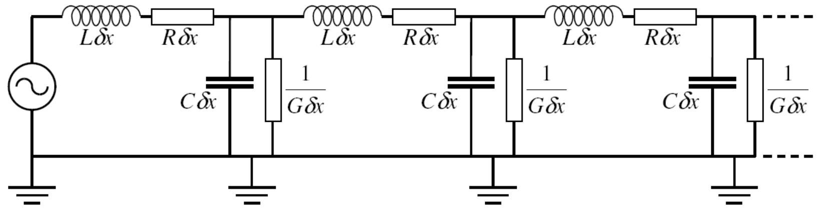


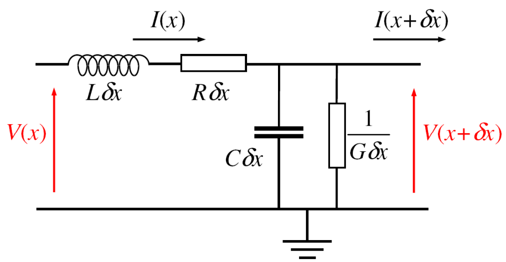
So far, we have assumed that the conductors in the transmission line have zero resistance, and are separated by a perfect insulator.

Usually, though, the conductors will have finite conductivity; and the insulator will have some finite resistance.

To understand the impact that this has, we need to modify our transmission line model to include:

- a resistance per unit length R in series with the inductance;
- a conductance per unit length G in parallel with the capacitance.





The equations for the current and voltage are then:

$$\frac{\partial V}{\partial x} = -L\frac{\partial I}{\partial t} - RI \qquad (32)$$

$$\frac{\partial I}{\partial x} = -C\frac{\partial V}{\partial t} - GV \qquad (33)$$

We can find solutions to the equations (32) and (33) for the voltage and current in the lossy transmission line by considering the case that we propagate a wave with a single, well-defined frequency ω.

In that case, we can replace each time derivative by a factor $j\omega$. The equations become:

$$\frac{\partial V}{\partial x} = -j\omega LI - RI = -\tilde{L}\frac{\partial I}{\partial t} \qquad (34)$$

$$\frac{\partial I}{\partial x} = -j\omega CV - GV = -\tilde{C}\frac{\partial V}{\partial t} \qquad (35)$$

where

$$\tilde{L} = L + \frac{R}{j\omega} \quad \text{and} \quad \tilde{C} = C + \frac{G}{j\omega} \qquad (36)$$

The new equations (34) and (35) for the lossy transmission line look exactly like the original equations (3) and (4) for a lossless transmission line, but with the capacitance C and inductance L replaced by (complex) quantities $\tilde{C}$ and $\tilde{L}$.

The imaginary parts of $\tilde{C}$ and $\tilde{L}$ characterise the losses in the lossy transmission line.

Mathematically, we can solve the equations for a lossy transmission line in exactly the same way as we did for the lossless line. In particular, we find for the phase velocity:

$$v = \frac{1}{\sqrt{\tilde{L}\tilde{C}}} = \frac{1}{\sqrt{\left(L - j\frac{R}{\omega}\right)\left(C - j\frac{G}{\omega}\right)}} \tag{37}$$

and for the impedance:

$$Z = \sqrt{\frac{\tilde{L}}{\tilde{C}}} = \sqrt{\frac{L - j\frac{R}{\omega}}{C - j\frac{G}{\omega}}} \tag{38}$$

Since the impedance (38) is now a complex number, there will be a phase difference (given by the complex phase of the impedance) between the current and voltage in the transmission line.

Note that the phase velocity (37) depends explicitly on the frequency. That means that a lossy transmission line will exhibit dispersion: waves of different frequencies will travel at different speeds, and a the shape of a wave "pulse" composed of different frequencies will change as it travels along the transmission line.

This is one reason why it is important to keep losses in a transmission line as small as possible (for example, by using high-quality materials). The other reason is that in a lossy transmission line, the wave amplitude will attenuate, much like an electromagnetic wave propagating in a conductor.

Recall that we can write the phase velocity:

$$v = \frac{\omega}{k} \tag{39}$$

where k is the wave number appearing in the solution to the wave equation:

$$V = V_0 e^{j(\omega t - kx)} \tag{40}$$

(and similarly for the current I). Using equation (37) for the phase velocity, we have:

$$k = \omega\sqrt{LC}\sqrt{\left(1 - \frac{jR}{\omega L}\right)\left(1 - \frac{jG}{\omega C}\right)} \tag{41}$$

Let us assume $R \ll \omega L$ (i.e. good conductivity along the transmission line) and $G \ll \omega C$ (i.e. poor conductivity between the lines); then we can make a Taylor series expansion, to find:

$$k \approx \omega\sqrt{LC}\left[1 - \frac{j}{2\omega}\left(\frac{R}{L} + \frac{G}{C}\right)\right] \tag{42}$$

Finally, we write:

$$k = \alpha - j\beta \tag{43}$$

and equate real and imaginary parts in equation (42) to give:

$$\alpha \approx \omega\sqrt{LC} \tag{44}$$

and:

$$\beta \approx \frac{1}{2}\left(\frac{R}{Z_0} + GZ_0\right) \tag{45}$$

where $Z_0 = \sqrt{L/C}$ is the impedance with $R = G = 0$ (not to be confused with the impedance of free space). Note that since:

$$V = V_0 e^{j(\omega t - kx)} = V_0 e^{-\beta x} e^{j(\omega t - \alpha x)} \tag{46}$$

the value of α gives the wavelength $\lambda = 2\pi/\alpha$, and the value of β gives the attenuation length $\delta = 1/\beta$.

Attenuation in a "Lossy" Transmission Line

Note that:

- The real part of the wavenumber is given by $\alpha \approx \omega\sqrt{LC}$, which is independent of R and G. Therefore, the wavelength of a wave propagating down the transmission line will not be significantly affected by losses in the transmission line.

- Slow attenuation corresponds to a low value of β, for which we want to keep both the conductance between the lines and the resistance along the lines as small as possible.

- The attenuation length depends on the losses in such a way that for a high impedance, the conductance between the lines becomes more important; while for a low impedance, the resistance long the lines becomes more important.

Inductance and Capacitance in a Transmission Line

A lossless transmission line has two key properties: the phase velocity v, and the characteristic impedance Z.

These are given in terms of the inductance per unit length L, and the capacitance per unit length C:

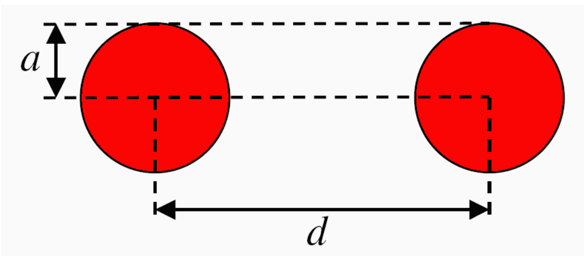
$$v = \frac{1}{\sqrt{LC}} \qquad Z = \sqrt{\frac{L}{C}} \tag{47}$$

The problem, when designing or analysing a transmission line, is to calculate the values of L and C. These are determined by the geometry of the transmission line, and are calculated by solving Maxwell's equations.

We shall consider two important (commonly used) examples:

- a parallel wire transmission line;
- a coaxial cable transmission line.

Parallel Wire Transmission Line



Let us first consider the case of two infinite parallel wires of radius a, with the centres of the wires separated by distance d.

We shall first calculate the capacitance per unit length, and then the inductance per unit length.

Parallel Wire Transmission Line: Capacitance

The capacitance per unit length is given by the charge per unit length on each wire, divided by the potential difference between them:

$$C = \frac{\lambda}{V} \tag{48}$$

where one wire carries charge $+\lambda$ per unit length, and the other $-\lambda$ per unit length.

If the system consisted of a single charged wire, then the electric field around the wire would be the same as that around a line of charge. From Maxwell's equations, we have:

$$\nabla \cdot \vec{D} = \rho \tag{49}$$

Integrating around a cylinder of length l enclosing the line charge and coaxial with it, and applying Gauss' theorem we have:

$$\int_V \nabla \cdot \vec{D}\, dV = \int_S \vec{D} \cdot d\vec{S} = \lambda l \tag{50}$$

By symmetry, the electric field is normal to the curved surface of the cylinder, and has uniform intensity over the curved surface.

If the cylinder has radius r then the surface area is $2\pi rl$, and hence:

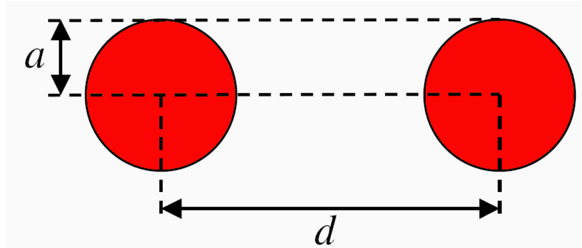
$$\int_S \vec{D} \cdot d\vec{S} = 2\pi rl|\vec{D}| = \lambda l \tag{51}$$

If the wire is in a medium with permittivity ε:

$$\vec{D} = \varepsilon \vec{E} \tag{52}$$

and hence:

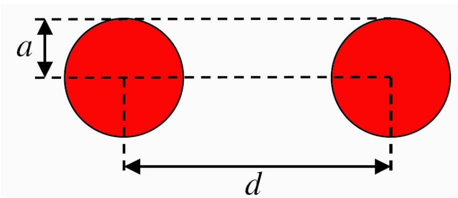
$$|\vec{E}| = \frac{\lambda}{2\pi\varepsilon r} \tag{53}$$



In the case of two parallel charged wires, the field is modified. It is *not* simply the superposition of the fields from two line charge densities at the centres of the wires.

The surface of each wire must be an equipotential surface.

This can be achieved if the line densities representing the charged wires are "displaced" slightly from the centres of the wires.



The voltage between the wires is found by integrating the field:

$$V = \int_A^B \vec{E} \cdot d\vec{r} \tag{54}$$

If $d \gg a$, then we can neglect the small displacement of the line densities from the centres of the wires, and write:

$$V \approx \int_a^{d-2a} |\vec{E}| dr \approx \frac{\lambda}{\pi\varepsilon} \int_a^d \frac{dr}{r} \tag{55}$$

Hence:

$$V \approx \frac{\lambda}{\pi\varepsilon} \ln\left(\frac{d}{a}\right) \tag{56}$$

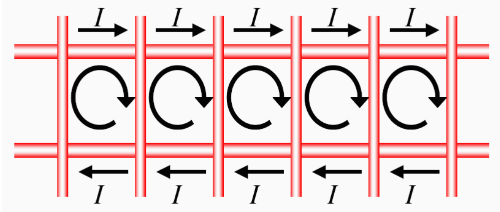
Hence, the capacitance per unit length between the parallel wires is:

$$C = \frac{\lambda}{V} \approx \frac{\pi\varepsilon}{\ln\left(\frac{d}{a}\right)} \tag{57}$$

where the approximation is valid for $d \gg a$.

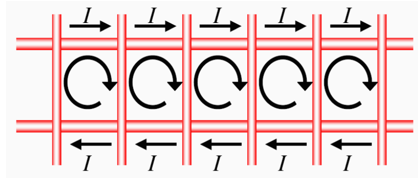
We now have an expression for the capacitance per unit length in terms of the geometry of the transmission line.

The next step is to calculate the inductance per unit length.



To calculate the inductance, we imagine completing a set of "current loops" by bridging the wires in the transmission lines with lengths of conductor at regular intervals.

Since the current is (approximately) constant along the transmission line, the "imaginary" lengths of conductor actually carry zero current – so it makes no difference whether they are there or not.



However, we can now calculate the total flux in each current loop. From Maxwell's equations, we have:

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \tag{58}$$

Integrating over a circular surface perpendicular to one wire and normal to it, we have:

$$\int_S \nabla \times \vec{H} \cdot d\vec{S} = \int_S \vec{J} \cdot d\vec{S} = I \tag{59}$$

Applying Stokes' theorem:

$$\int_S \nabla \times \vec{H} \cdot d\vec{S} = \int_C \vec{H} \cdot d\vec{l} = I \tag{60}$$

where the circle C bounds the circular disc S.

By symmetry, the magnetic field strength must be the same at all points around C, and tangential to C.

Hence:

$$2\pi r|\vec{B}| = 2\pi r \mu |\vec{H}| = \mu I \tag{61}$$

Considering a current loop of length l, and taking into account the field generated by each of the two wires in the transmission line, the total flux through the current loop is:

$$\Phi = 2\int |\vec{B}| dS = \frac{\mu}{\pi} lI \int_a^{d-2a} \frac{dr}{r} \tag{62}$$

With the approximation $d \gg a$, we have:

$$\Phi \approx \frac{\mu}{\pi} lI \ln\left(\frac{d}{a}\right) \tag{63}$$

The inductance per unit length L is defined by:

$$\mathcal{E} = -\frac{d\Phi}{dt} = -Ll\frac{dI}{dt} \tag{64}$$

Hence, we write:

$$L = \frac{\Phi}{lI} \approx \frac{\mu}{\pi} \ln\left(\frac{d}{a}\right) \tag{65}$$

Parallel Wire Transmission Line

For the parallel wire transmission line, we find for the capacitance per unit length (57):

$$C \approx \frac{\pi\varepsilon}{\ln\left(\frac{d}{a}\right)} \qquad (66)$$

and for the inductance per unit length (65):

$$L \approx \frac{\mu}{\pi}\ln\left(\frac{d}{a}\right) \qquad (67)$$

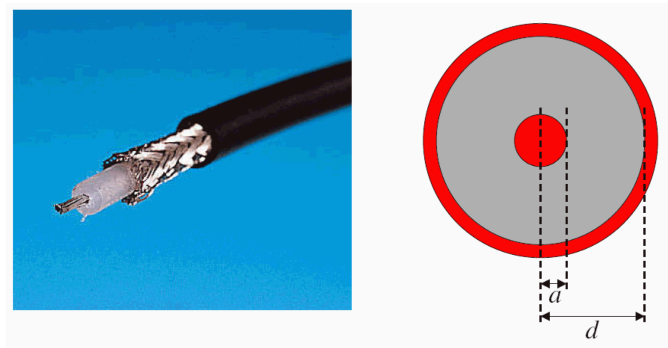
Both approximations are valid for $d \gg a$. Assuming this condition is true, we can write for the phase velocity of waves along the transmission line:

$$v = \frac{1}{\sqrt{LC}} \approx \frac{1}{\sqrt{\mu\varepsilon}} \qquad (68)$$

and for the characteristic impedance:

$$Z = \sqrt{\frac{L}{C}} \approx \frac{1}{\pi}\ln\left(\frac{d}{a}\right)\sqrt{\frac{\mu}{\varepsilon}} \qquad (69)$$

Coaxial Cable Transmission Line



As a second example, we consider a coaxial cable transmission line, consisting of a central wire of radius a, surrounded by a conducting "sheath" of internal radius d.

The central wire and surrounding sheath are separated by a dielectric of permittivity ε and permeability μ.

Coaxial Cable Transmission Line: Capacitance

Suppose that the central wire carries line charge $+\lambda$, and the surrounding sheath carries line charge $-\lambda$ (so that the sheath is at zero potential).

We can apply Maxwell's equations as before, to find that the electric field in the dielectric is given by:

$$|\vec{E}| = \frac{\lambda}{2\pi\varepsilon r} \qquad (70)$$

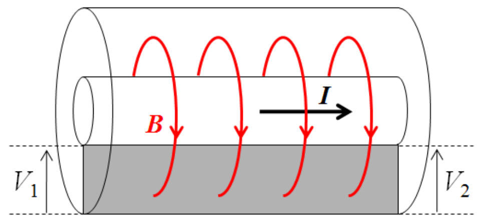
where r is the radial distance from the axis. The potential between the conductors is given by:

$$V = \int_a^d \vec{E}\cdot d\vec{r} = \frac{\lambda}{2\pi\varepsilon}\ln\left(\frac{d}{a}\right) \qquad (71)$$

Hence the capacitance per unit length of the coaxial cable is:

$$C = \frac{\lambda}{V} = \frac{2\pi\varepsilon}{\ln(d/a)} \qquad (72)$$

Coaxial Wire Transmission Line

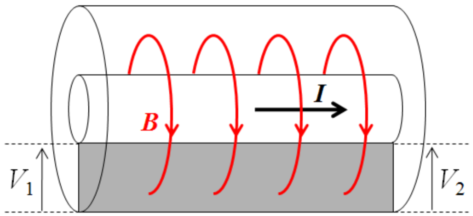


To find the inductance per unit length, we consider a length l of the cable. If the central wire carries a current I, then the magnetic field at a radius r from the axis is given by:

$$|\vec{B}| = \frac{\mu I}{2\pi r} \qquad (73)$$

The flux through the shaded area shown in the diagram is given by:

$$\Phi = l\int_a^d |\vec{B}|dr = \frac{\mu l I}{2\pi}\ln\left(\frac{d}{a}\right) \qquad (74)$$



The change in voltage between two points at either end of the section of cable will be (by Faraday's law):

$$\Delta V = V_2 - V_1 = -\frac{d\Phi}{dt} = -\frac{\mu l}{2\pi} \ln\left(\frac{d}{a}\right) \frac{dI}{dt} \tag{75}$$

The change in voltage can also be expressed in terms of the inductance per unit length of the cable:

$$\Delta V = -lL\frac{dI}{dt} \tag{76}$$

Hence the inductance per unit length is given by:

$$L = \frac{\mu}{2\pi} \ln\left(\frac{d}{a}\right) \tag{77}$$

With the expression (72) for the capacitance per unit length:

$$C = \frac{2\pi\varepsilon}{\ln(d/a)} \tag{78}$$

the phase velocity of waves along the coaxial cable is given by:

$$v = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{\mu\varepsilon}} \tag{79}$$

and the characteristic impedance of the cable is given by:

$$Z = \sqrt{\frac{L}{C}} = \frac{1}{2\pi} \ln\left(\frac{d}{a}\right) \sqrt{\frac{\mu}{\varepsilon}} \tag{80}$$

As we shall see in Part 8 of this lecture course, television aerials have an impedance of around 75 Ω.

This is determined by the geometry of the aerial, which is designed to maximise its interaction with electromagnetic radiation.

The impedance of the aerial means that the transmission line (usually a coaxial cable) used to connect the aerial to the television should also have an impedance of around 75 Ω. Of course, the "receiver" circuitry in the television should have the same impedance.

Items of scientific equipment in laboratories, such as signal generators and oscilloscopes, usually have an impedance of 50 Ω. These items are best connected using coaxial cable with the same impedance.

In a coaxial cable, the outer conductor is usually earthed. This means that:

- The fields outside the cable are very small, so the cable does not emit significant amounts of electromagnetic radiation.
- Fields from external sources cannot penetrate into the cable. As a result, the cable is effectively shielded from noise that may otherwise be induced on the signal that is being propagated.

By contrast, a parallel wire transmission line is not shielded, and is thus susceptible to noise. Also, a parallel wire transmission line will emit electromagnetic radiation, that will lead to attenuation of the signal as well as being a possible source of noise for other devices.

Summary of Part 6

You should be able to:

- Use the LC model to derive the wave equations for the voltage and current in an infinitely long transmission line.
- Derive expressions for the phase velocity and characteristic impedance of a transmission line, in terms of the capacitance per unit length and inductance per unit length.
- Explain the significance of impedance matching at the termination of a transmission line.
- Calculate the relative amplitude of the wave reflected from the end of a transmission line, given the characteristic impedance of the line and the impedance of the load terminating the line.
- Explain the sources of losses in a transmission line, and explain the impacts of such losses.
- Derive an expression for the attenuation length in a lossy transmission line.
- Derive expressions for the capacitance per unit length and inductance per unit length in parallel wire and coaxial transmission lines, in terms of the geometries of the lines.