PHYS370 – Advanced Electromagnetism

Part 5: Cavities and Waveguides

Cavities and Waveguides

In previous parts of this course, we have considered electromagnetic waves in unbounded media, and electromagnetic waves on plane boundaries between two media.

We have seen that electromagnetic waves are reflected almost completely from the surfaces of good conductors.

This suggests that we can use metal tubes to guide electromagnetic waves from one place to another; and metal boxes to store electromagnetic energy in the form of standing waves.

Electromagnetic cavities (boxes) and waveguides (tubes) do in fact have a number of important practical applications. In this part of the course, we shall investigate the properties of these devices.

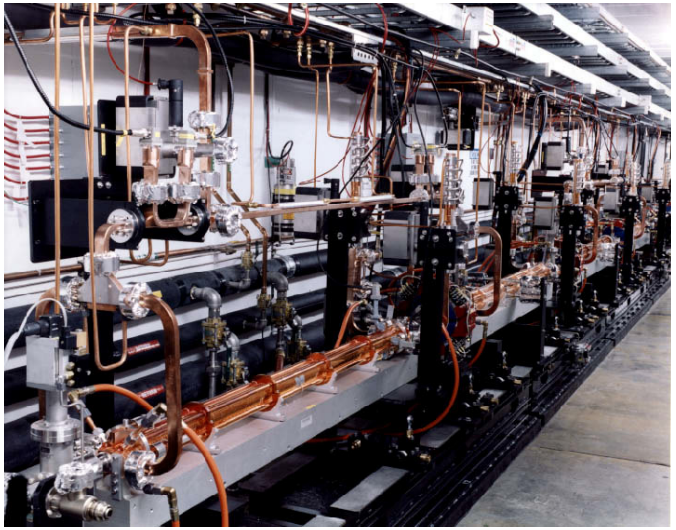
History of Waveguides

1884 Sir Oliver Lodge detected electromagnetic waves from a spark at the end of a cylinder, and found that the amplitude did not fall off as $1/r^2$.

1897 Lord Rayleigh showed that two classes of waves are possible, "transverse electric" (TE) and "transverse magnetic" (TM). For each class, there is a minimum frequency for propagation.

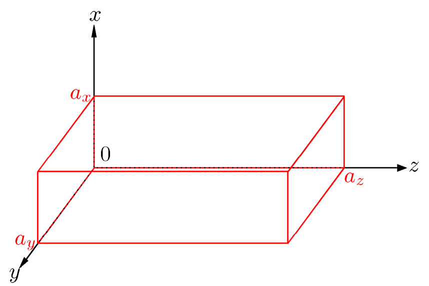
1936 Barrow-Southworth showed that for practical guides, the attenuation in waveguides was much less than in wires or coaxial cables.

An Application of Cavities and Waveguides



The Next Linear Collider Test Accelerator (NLCTA) at the Stanford Linear Accelerator Center (SLAC), California.

We consider first a rectangular cavity with perfectly conducting walls.



We assume that the interior of the cavity consists of a uniform, uncharged dielectric with no electric currents. The wave equation inside the cavity is:

$$\nabla^2 \vec{E} - \mu\varepsilon \ddot{\vec{E}} = 0 \qquad (1)$$

where μ is the permeability inside the cavity, and ε is the permittivity.

The boundary conditions at the walls of the cavity are:

$$E_t = 0 \qquad (2)$$

$$B_n = 0 \qquad (3)$$

where E_t is the component of the electric field tangential to the wall, and B_n is the component of the magnetic field normal to the wall.

Solutions to the wave equation must also satisfy Maxwell's equations.

Plane wave solutions will not satisfy the boundary conditions.

However, the situation is analogous to mechanical waves on a stretched wire.

On stretched wire, the wave equation for the displacement ζ as a function of position x and time t is:

$$\frac{\partial^2 \zeta}{\partial t^2} - \frac{1}{v^2}\frac{\partial^2 \zeta}{\partial t^2} = 0 \qquad (4)$$

On an infinite wire, the solution is given by (the real part of):

$$\zeta(x,t) = \zeta_0 e^{j(\omega t - kx)} \qquad (5)$$

where the frequency ω and wave vector k satisfy the dispersion relation:

$$\frac{\omega}{k} = v \qquad (6)$$

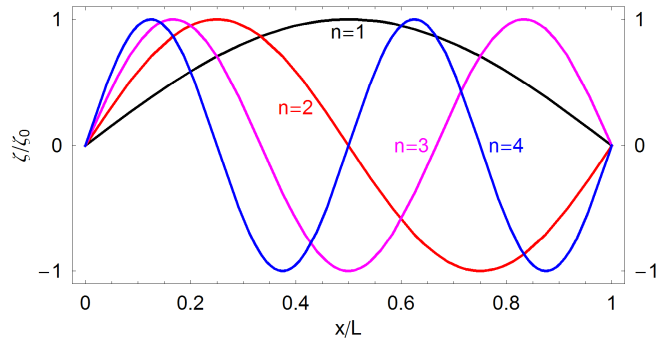
If the wire is of length L and is fixed at each end, we can satisfy the boundary conditions:

$$\zeta(0,t) = \zeta(L,t) = 0 \qquad \text{for all } t \qquad (7)$$

with a solution of the form:

$$\zeta(x,t) = \zeta_0 \sin\left(\pi n \frac{x}{L}\right) e^{j\omega t} \qquad (8)$$

The boundary conditions are satisfied if n is any integer.



Standing Waves

The standing wave solution (8) can be obtained from the travelling wave solution (5), by superposing waves with equal frequencies and amplitudes, travelling in opposite directions:

$$e^{j(\omega t - kx)} - e^{j(\omega t + kx)} \equiv -2j \sin(kx)\, e^{j\omega t} \tag{9}$$

We can satisfy the boundary conditions on electromagnetic waves in a conducting box in the same way...

Rectangular Cavity with Perfectly Conducting Walls

In free space, the wave equation for the electric field had the solution:

$$\vec{E}(\vec{r}, t) = \vec{E}_0 e^{j(\omega t - \vec{k}\cdot\vec{r})} \tag{10}$$

To satisfy the boundary conditions inside a conducting cavity, we write a solution of the form:

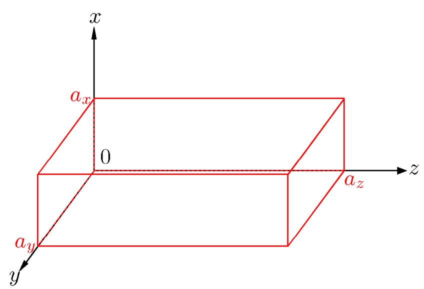
$$E_x = E_{x0} \cos k_x x \sin k_y y \sin k_z z\, e^{j\omega t} \tag{11}$$

$$E_y = E_{y0} \sin k_x x \cos k_y y \sin k_z z\, e^{j\omega t} \tag{12}$$

$$E_z = E_{z0} \sin k_x x \sin k_y y \cos k_z z\, e^{j\omega t} \tag{13}$$

Notice that we write a cosine dependence on the coordinate corresponding to the component of the field; and a sine dependence on the other coordinates.

Rectangular Cavity with Perfectly Conducting Walls



Consider the x component of the field:

$$E_x = E_{x0} \cos k_x x \sin k_y y \sin k_z z\, e^{j\omega t} \tag{14}$$

This component is parallel to the walls defined by:

$$y = 0, \qquad y = a_y, \qquad z = 0, \qquad z = a_z \tag{15}$$

For perfectly conducting walls, E_x must vanish on these surfaces.

Rectangular Cavity with Perfectly Conducting Walls

Since $E_x \sim \sin k_y y$ and $E_x \sim \sin k_z z$, the requirement that E_x vanishes on $y = 0$ (for all x and z) and on $z = 0$ (for all x and y) is automatically satisfied.

This would not be the case if E_x had cosine-like dependence on y and z.

We also need E_x to vanish on $y = a_y$ (for all x and z) and on $z = a_z$ (for all x and y).

These requirements can be satisfied if:

$$k_y = \frac{\pi n_y}{a_y}, \qquad \text{and} \qquad k_z = \frac{\pi n_z}{a_z} \tag{16}$$

for any integers n_y and n_z.

Similar considerations apply to the other field components, E_y and E_z.

Overall, the condition that the tangential component of the electric field vanishes at all the conducting walls imposes the constraints on the components of the wave vector:

$$k_x = \frac{\pi n_x}{a_x}, \qquad k_y = \frac{\pi n_y}{a_y}, \qquad k_z = \frac{\pi n_z}{a_z} \tag{17}$$

for any integers n_x, n_y and n_z.

n_x, n_y and n_z are called *mode numbers*: they specify the dependence of the electric field on the coordinates.

Note that at least two of the mode numbers must be non-zero, otherwise the field vanishes everywhere.

We have seen that the boundary conditions on the electric field in a rectangular cavity can be satisfied by an electric field of the form:

$$E_x = E_{x0} \cos k_x x \sin k_y y \sin k_z z \, e^{j\omega t} \tag{18}$$
$$E_y = E_{y0} \sin k_x x \cos k_y y \sin k_z z \, e^{j\omega t} \tag{19}$$
$$E_z = E_{z0} \sin k_x x \sin k_y y \cos k_z z \, e^{j\omega t} \tag{20}$$

if k_x, k_y and k_z satisfy certain constraints.

To satisfy Maxwell's equation:

$$\nabla \cdot \vec{E} = 0 \tag{21}$$

the wave vector components and the field amplitudes must be related:

$$k_x E_{x0} + k_y E_{y0} + k_z E_{z0} = 0 \tag{22}$$

To satisfy the wave equation:

$$\nabla^2 \vec{E} - \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} = 0 \tag{23}$$

the wave vector and the frequency must be related:

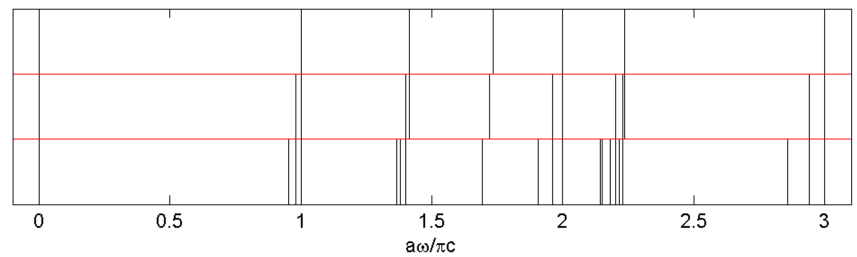
$$\vec{k}^2 = k_x^2 + k_y^2 + k_z^2 = \frac{\omega^2}{c^2} \tag{24}$$

Since the components of the wave vector are constrained to discrete values (since the mode numbers must be integers), the frequency is only allowed to take certain values:

$$\omega = \pi c \sqrt{\frac{n_x^2}{a_x^2} + \frac{n_y^2}{a_y^2} + \frac{n_z^2}{a_z^2}} \tag{25}$$

c is the speed of light in the cavity.

The possible values of ω are called the *resonant* frequencies.



In a cube-shaped cavity ($a_x = a_y = a_z = a$), many of the modes are degenerate, i.e. have the same frequency: see the top row in the diagram above.

If two sides are of different lengths (middle row), or all sides are different (bottom row) then the mode spectrum becomes more complicated.

Oscillation Frequencies in a Rectangular Cavity

An oscillation *mode* is specified by particular values of the integers n_x, n_y and n_z. Different modes may have the same frequency or different frequencies.

The *lowest allowed frequency* ω_{min} is found by setting to zero the integer n_x, n_y or n_z associated with the *smallest* dimension a_x, a_y or a_z; and setting the other mode numbers to 1.

For example, if $a_z < a_x$ and $a_z < a_y$, then:

$$\omega_{min} = \pi c \sqrt{\frac{1}{a_x^2} + \frac{1}{a_y^2}} \tag{26}$$

Magnetic and Electric Fields in a Rectangular Cavity

Given an electric field in a particular mode, the associated magnetic field may be found from Maxwell's equation:

$$\nabla \times \vec{E} = -\dot{\vec{B}} \tag{27}$$

To satisfy Maxwell's equations at all times, the time dependence of the magnetic field must be the same (to within a constant phase angle) as the time dependence of the electric field.

Therefore, equation (27) becomes:

$$\nabla \times \vec{E} = -j\omega \vec{B} \tag{28}$$

Magnetic and Electric Fields in a Rectangular Cavity

Since the spatial dependence of the electric field $\vec{E}$ is given by real-valued trigonometric functions, it follows from equation (28):

$$\nabla \times \vec{E} = -j\omega \vec{B} \tag{29}$$

that the electric and magnetic fields are 90° out of phase.

Taking the real part to find the physical field, if the electric field varies as:

$$\vec{E} \sim \cos(\omega t + \phi_0) \tag{30}$$

then the magnetic field varies as:

$$\vec{B} \sim \sin(\omega t + \phi_0) \tag{31}$$

Energy Flow in a Rectangular Cavity

Hence, the Poynting vector at any point varies in time as:

$$\vec{S} = \vec{E} \times \vec{H} \sim \cos(\omega t + \phi_0)\sin(\omega t + \phi_0) = \frac{1}{2}\sin 2(\omega t + \phi_0) \tag{32}$$

At any given point in the cavity, the energy flux oscillates at twice the frequency of the fields.

The time-averaged value of the Poynting vector is:

$$\langle \vec{S} \rangle_t \sim \langle \sin 2(\omega t + \phi_0) \rangle_t = 0 \tag{33}$$

There is no net energy flow within the cavity: the waves are standing waves. Energy is transferred between the electric and magnetic fields.

Example: (0,1,1) Mode in a Cube-shaped Cavity

The (0,1,1) mode is the mode with the lowest frequency in a cube-shaped cavity. The wave numbers are given by:

$$k_x = 0, \qquad k_y = k_z = \frac{\pi}{a} \tag{34}$$

where a is the length of the side of the cavity.

The electric field is given by:

$$E_x = E_0 \sin(k_y y) \sin(k_z z) \, e^{j\omega t} \tag{35}$$

$$E_y = 0 \tag{36}$$

$$E_z = 0 \tag{37}$$

Example: (0,1,1) Mode in a Cube-shaped Cavity

Solving Maxwell's equation:

$$\nabla \times \vec{E} = -\dot{\vec{B}} \tag{38}$$

gives for the magnetic field:

$$B_x = 0 \tag{39}$$

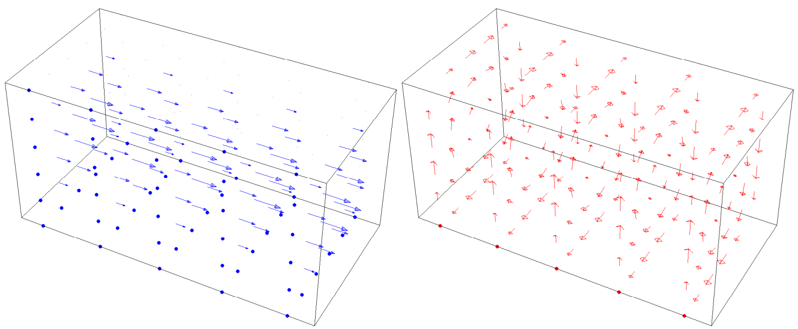
$$B_y = j\frac{k_z}{\omega} E_0 \sin(k_y y) \cos(k_z z) \, e^{j\omega t} \tag{40}$$

$$B_z = -j\frac{k_y}{\omega} E_0 \cos(k_y y) \sin(k_z z) \, e^{j\omega t} \tag{41}$$

Notice that the magnetic field is perpendicular to the electric field:

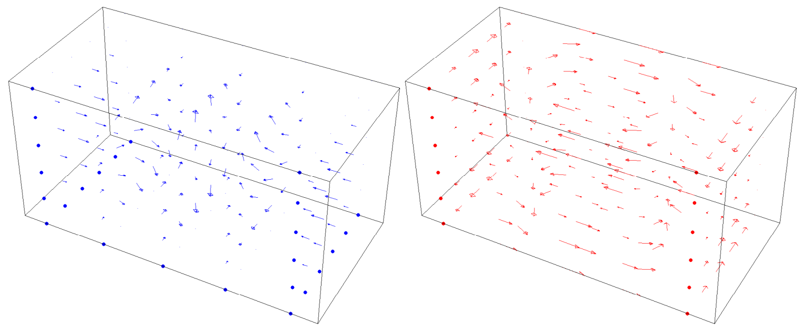
$$\vec{E} \cdot \vec{B} = 0 \tag{42}$$

Example: (0,1,1) Mode in a Cube-shaped Cavity

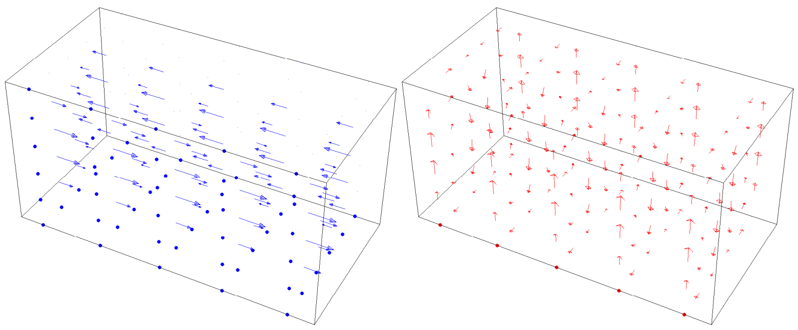


Electric field (left) and magnetic field (right) in the (0,1,1) mode in a rectangular cavity.

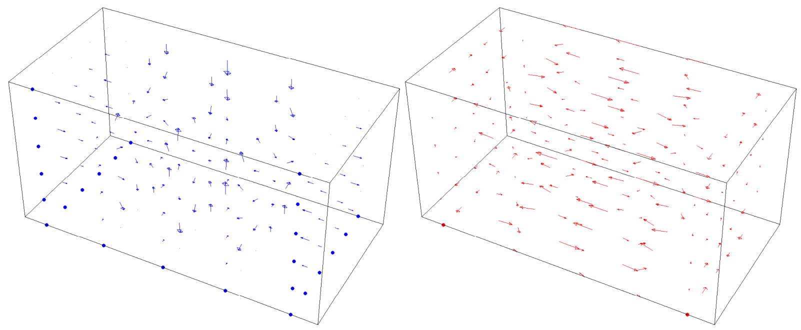
Higher modes: (1,1,1)



Higher modes: (0,2,1)



Higher modes: (1,2,1)



Example: (0,1,1) Mode in a Cube-shaped Cavity

The stored energy in the cavity can be found by integrating the energy density over the volume of the cavity.

The electric and magnetic energy densities (energy per unit volume) are given respectively by:

$$U_E = \frac{1}{2}\varepsilon\vec{E}^2 \tag{43}$$

$$U_H = \frac{1}{2}\mu\vec{H}^2 \tag{44}$$

First, consider the total energy stored in the electric field:

$$\int U_E\, dV = \frac{1}{2}\varepsilon E_0^2 \int_0^a dx \int_0^a \sin^2 k_y y\, dy \int_0^a \sin^2 k_z z\, dz\, \cos^2(\omega t) \tag{45}$$

$$= \frac{1}{8}\varepsilon E_0^2 a^3 \cos^2(\omega t) \tag{46}$$

Example: (0,1,1) Mode in a Cube-shaped Cavity

Now consider the total energy stored in the magnetic field:

$$\begin{aligned}\int U_H\, dV &= \frac{1}{2}\frac{k_z^2}{\mu\omega^2}E_0^2 \int_0^a dx \int_0^a \sin^2(ky)\, dy \int_0^a \cos^2(kz)\, dz\, \sin^2(\omega t)\\ &\quad + \frac{1}{2}\frac{k_y^2}{\mu\omega^2}E_0^2 \int_0^a dx \int_0^a \cos^2(ky)\, dy \int_0^a \sin^2(kz)\, dz\, \sin^2(\omega t)\\ &= \frac{1}{8}\frac{(k_y^2+k_z^2)}{\mu\omega^2}E_0^2 a^3 \sin^2(\omega t) \end{aligned} \tag{47}$$

Since:

$$k_y^2 + k_z^2 = \frac{\omega_2}{c^2} \tag{48}$$

the magnetic energy can be written:

$$\int U_H\, dV = \frac{1}{8}\frac{1}{\mu c^2}E_0^2 a^3 \sin^2(\omega t) = \frac{1}{8}\varepsilon E_0^2 a^3 \sin^2(\omega t) \tag{49}$$

where the last step follows from $1/c^2 = \mu\varepsilon$

The total electric energy in the cavity at time t is:

$$\mathcal{E}_E = \int U_E \, dV = \frac{1}{8}\varepsilon E_0^2 a^3 \cos^2(\omega t) \tag{50}$$

and the total magnetic energy in the cavity at time t is:

$$\mathcal{E}_H = \int U_H \, dV = \frac{1}{8}\varepsilon E_0^2 a^3 \sin^2(\omega t) \tag{51}$$

The electric and magnetic energies are out of phase; as a result, at *any* time t, the total electromagnetic energy in the cavity is:

$$\mathcal{E}_E + \mathcal{E}_H = \frac{1}{8}\varepsilon E_0^2 a^3 \tag{52}$$

which is independent of time: the total electromagnetic energy in the cavity is constant in time.

As an example, consider a cavity with side length $a = 0.5$ m, with a vacuum in the interior.

The lowest frequency mode is the (0,1,1) mode, which, from equation (26), has angular frequency:

$$\omega = \pi \frac{\sqrt{2}}{a} c \tag{53}$$

so the frequency f is:

$$f = \frac{\omega}{2\pi} = \frac{\sqrt{2}}{2a} c \approx 424\,\text{MHz} \tag{54}$$

If the peak field is 10 MV/m, then the stored energy is:

$$\mathcal{E} = \frac{1}{8}\varepsilon E_0^2 a^3 \approx 14\,\text{J} \tag{55}$$

Energy Dissipation and Quality Factor

Since the tangential component of the electric field vanishes at the walls of the cavity, there are no currents induced in the walls by the fields, and no mechanism for dissipating the energy.

In practice, there will always be some currents induced in the walls of the cavity, which will dissipate the energy.

The rate of energy dissipation is characterised by the quality factor, Q:

$$\mathcal{E}(t) = \mathcal{E}_0 \, e^{-\frac{\omega t}{Q}} \tag{56}$$

Q is the number of cycles made by an oscillator, before the energy falls by a factor $1/e$.

The resonant modes (integer mode numbers) in a cavity will have high Q values. Fields can exist in other modes (non-integer mode numbers), but the energy will be rapidly damped (low Q values), because the fields on the walls will be large.

Practical Applications of RF Cavities

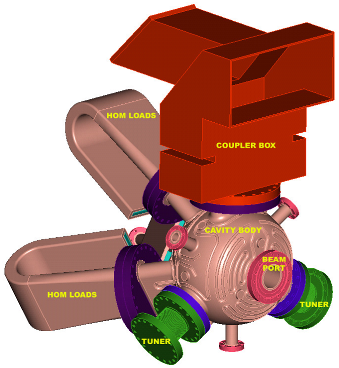
Cavity resonators display similar properties to LC circuits, but at higher frequencies (GHz), and with high Q (quality) factors.

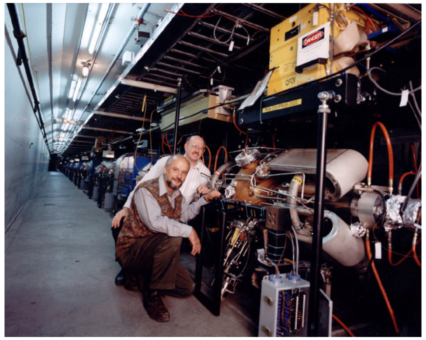
Cavity resonators have applications in:

- microwave ovens
- radar systems
- particle accelerators

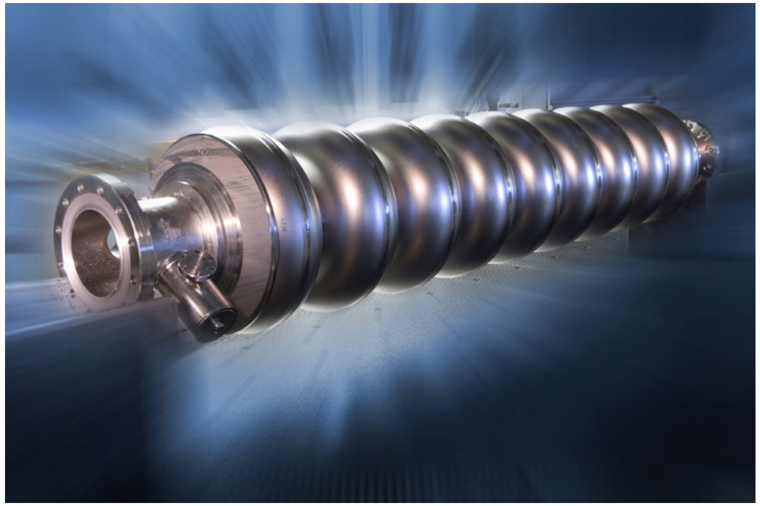
Superconducting cavities have very small losses, and can achieve Q factors of the order of 10^{10}; i.e. once excited, the fields will make of order 10^{10} oscillations before the energy is dissipated.

Normal-Conducting RF Cavity in PEP-II



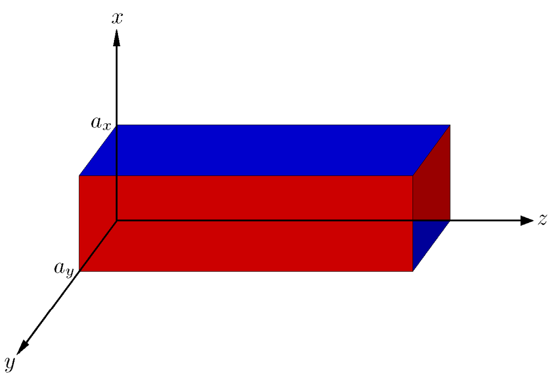


Superconducting RF Cavity for ILC



Electromagnetic Waves in Waveguides

Consider a perfectly conducting tube with rectangular cross-section, of height and width a_x and a_y. This is essentially a cavity resonator with length $a_z \to \infty$.



Electromagnetic Waves in Waveguides

The electric field must solve the wave equation:

$$\nabla^2 \vec{E} - \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} = 0 \tag{57}$$

together (as usual) with Maxwell's equations.

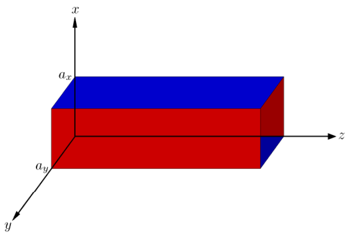
By comparison with the rectangular cavity case, we expect to find standing waves in x and y, with plane wave solution in z.

Therefore, we write a solution of the form:

$$E_x = E_{x0} \cos k_x x \sin k_y y \, e^{j(\omega t - k_z z)} \tag{58}$$

$$E_y = E_{y0} \sin k_x x \cos k_y y \, e^{j(\omega t - k_z z)} \tag{59}$$

$$E_z = -jE_{z0} \sin k_x x \sin k_y y \, e^{j(\omega t - k_z z)} \tag{60}$$



Now we apply the boundary conditions. Consider the field component E_x:

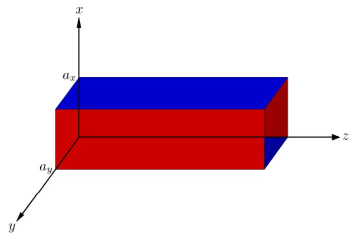
$$E_x = E_{x0} \cos k_x x \, \sin k_y y \, e^{j(\omega t - k_z z)} \tag{61}$$

This must vanish where it is tangential to a wall:

$$y = 0, \qquad y = a_y \tag{62}$$

Thus, we require, for any integer n_y:

$$k_y = \frac{\pi n_y}{a_y} \tag{63}$$



Similarly, for the field component E_y:

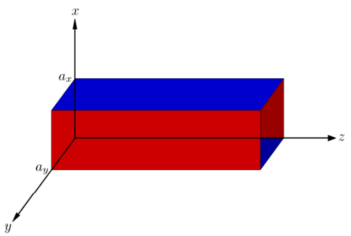
$$E_y = E_{y0} \sin k_x x \, \cos k_y y \, e^{j(\omega t - k_z z)} \tag{64}$$

This must vanish where it is tangential to a wall:

$$x = 0, \qquad x = a_x \tag{65}$$

Thus, we require, for any integer n_y:

$$k_y = \frac{\pi n_y}{a_y} \tag{66}$$



With the constraints on k_x and k_y:

$$k_x = \frac{\pi n_x}{a_x}, \qquad k_y = \frac{\pi n_y}{a_y} \tag{67}$$

(for any integers n_x and n_y) the longitudinal field component:

$$E_z = -jE_{z0} \sin k_x x \, \sin k_y y \, e^{j(\omega t - k_z z)} \tag{68}$$

always vanishes on the walls: there is no constraint on k_z.

As usual, we must satisfy Maxwell's equation:

$$\nabla \cdot \vec{E} = 0 \tag{69}$$

for all x, y and z.

This leads to a relation between the amplitudes and the components of the wave vector:

$$k_x E_{x0} + k_y E_{y0} + k_z E_{z0} = 0 \tag{70}$$

Electromagnetic Waves in Waveguides

We must also satisfy the wave equation:

$$\nabla^2\vec{E} - \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} = 0 \qquad (71)$$

This leads to the dispersion relation:

$$k_x^2 + k_y^2 + k_z^2 = \frac{\omega^2}{c^2} \qquad (72)$$

where $c = 1/\sqrt{\mu\varepsilon}$.

In a cavity, k_x, k_y and k_z were all constrained to take discrete values, so there were only certain "resonant" frequencies allowed.

However, in a waveguide, there is no constraint on k_z. This means that there is a continuous range of frequencies allowed in a waveguide.

The "Cut-Off" Frequency in a Rectangular Waveguide

Although there is a continuous range of frequencies allowed in a waveguide, there is still a *minimum* frequency allowed in any given mode.

For a travelling wave, k_z must be real. This means that $k_z^2 \geq 0$.

Hence, from the dispersion relation (72):

$$\omega \geq c\sqrt{k_x^2 + k_y^2} = \pi c\sqrt{\frac{n_x^2}{a_x^2} + \frac{n_y^2}{a_y^2}} \qquad (73)$$

The minimum frequency for a propagating wave is called the *cut-off frequency*, ω_{co}:

$$\omega_{\mathrm{co}} = \pi c\sqrt{\frac{n_x^2}{a_x^2} + \frac{n_y^2}{a_y^2}} \qquad (74)$$

The "Cut-Off" Frequency in a Rectangular Waveguide

It is possible for fields to oscillate in a waveguide at frequencies below the cut-off frequency. However, such fields do not constitute travelling waves.

If $\omega < \omega_{\mathrm{co}}$, then $k_z^2 < 0$, so k_z must be imaginary:

$$k_z = j\beta \qquad (75)$$

The horizontal field component (for example) in this case would be:

$$E_x = E_{x0}\cos k_x x\,\sin k_y y\, e^{-\beta z}\, e^{j\omega t} \qquad (76)$$

Note that there is an exponential decay of the field amplitude in the z direction, rather than an oscillation.

The fields in this case constitute an "evanescent" wave.

Phase Velocity of Waves in a Waveguide

For waves above the cut-off frequency, the variation of E_x with longitudinal position and time is:

$$E_x \sim e^{j(\omega t - k_z z)} \qquad (77)$$

and similarly for all other field components.

The phase velocity, v_p is the speed at which a particle would have to move along the waveguide to stay at constant phase with respect to the fields, i.e.:

$$\omega t - k_z z = \text{constant} \qquad (78)$$

Hence, the phase velocity is:

$$v_p = \frac{dz}{dt} = \frac{\omega}{k_z} \qquad (79)$$

Phase Velocity of Waves in a Waveguide

Using the dispersion relation (72) we can write the phase velocity:

$$v_p = \frac{\omega}{k_z} = c\frac{\sqrt{k_x^2 + k_y^2 + k_z^2}}{k_z} \tag{80}$$

For a travelling wave, k_x, k_y and k_z are all real. Hence:

$$\sqrt{k_x^2 + k_y^2 + k_z^2} > k_z \tag{81}$$

Therefore:

$$v_p > c \tag{82}$$

The phase velocity in the waveguide is greater than the speed of light.

Group Velocity of Waves in a Waveguide

The phase velocity of a wave in a waveguide is greater than the speed of light.

However, the energy travels with the *group velocity*, v_g:

$$v_g = \frac{d\omega}{dk_z} = \frac{k_z}{\sqrt{k_x^2 + k_y^2 + k_z^2}}c \tag{83}$$

so we have:

$$v_g < c \tag{84}$$

Note that for a rectangular waveguide, the phase and group velocities are related by:

$$v_p v_g = c^2 \tag{85}$$

Group Velocity of Waves in a Waveguide

Equation (83) expresses the group velocity, for a given mode, in terms of the longitudinal wave number, k_z.

It is sometimes convenient to express the group velocity in terms of the frequency.

Using the dispersion relation (72) we find:

$$v_g = c\sqrt{1 - \frac{c^2}{\omega^2}\left(k_x^2 + k_y^2\right)} \tag{86}$$

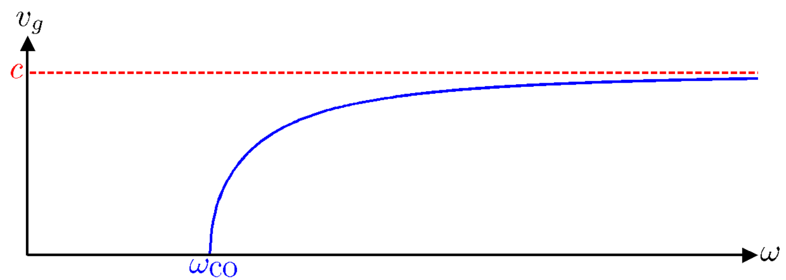
Group Velocity of Waves in a Waveguide

Energy in the wave propagates along the waveguide only for $\omega > \omega_{\text{co}}$ (where ω_{co} is the cut-off frequency).

Note the limiting behaviour of the group velocity, for $\omega > \omega_{\text{co}}$:

$$\lim_{\omega\to\infty} v_g = c \tag{87}$$

$$\lim_{\omega\to\omega_{\text{co}}} v_g = 0 \tag{88}$$



Modes of Transmission

In practice, waveguides are often used so that either the electric field or the magnetic field has no longitudinal component:

- Transverse electric, or TE modes: $E_z = 0$
- Transverse magnetic, or TM modes: $B_z = 0$

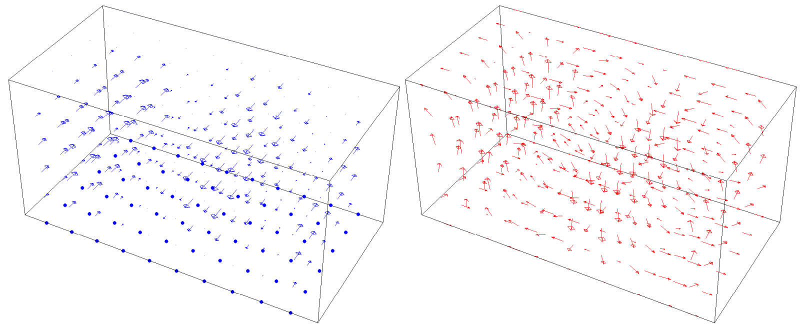
In the TE mode, $E_{z0} = 0$, so it follows from equation (70) that:

$$k_x E_{x0} + k_y E_{y0} = 0 \tag{89}$$

or:

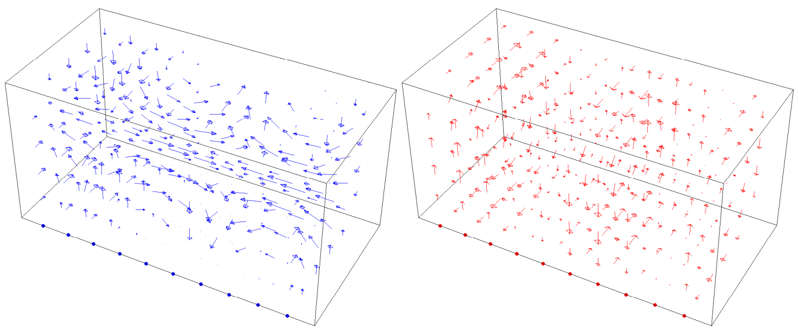
$$E_{y0} = -\frac{k_x}{k_y} E_{x0} \tag{90}$$

Example: TE_{01} Mode in a Rectangular Waveguide



Electric field (left) and magnetic field (right) in the TE_{01} mode in a section of rectangular waveguide.

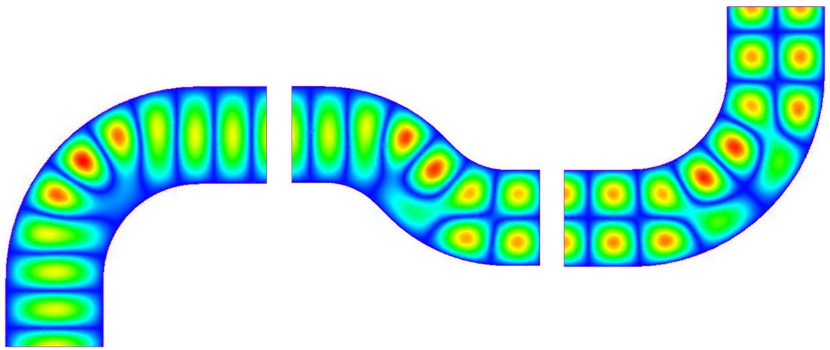
Example: TM_{11} Mode in a Rectangular Waveguide



Electric field (left) and magnetic field (right) in the TM_{11} mode in a section of rectangular waveguide.

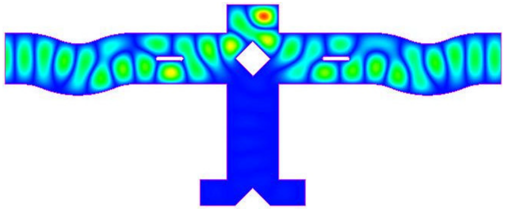
Manipulating Modes

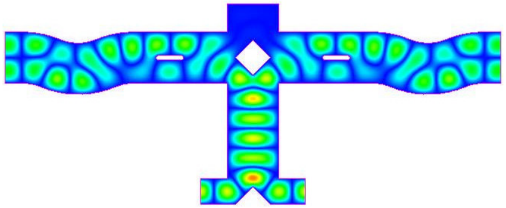
By using an appopriate geometry in a waveguide, it is possible to convert from one mode to another...



Manipulating Modes

The behaviour of a mode depends on the geometry of the waveguide...





Modes of Transmission: TE Modes

As an example, let us consider the energy flow in a waveguide in a TE mode.

In the TE mode of propagation in a waveguide, there is a phase difference of π between the horizontal electric field component E_x and the vertical electric field component E_y.

The electric field is given by:

$$E_x = E_0 \cos k_x x \, \sin k_y y \, e^{j(\omega t - k_z z)} \tag{91}$$

$$E_y = -\frac{k_x}{k_y} E_0 \sin k_x x \, \cos k_y y \, e^{j(\omega t - k_z z)} \tag{92}$$

$$E_z = 0 \tag{93}$$

There is a standing wave pattern in x and y, and the wave is travelling in the z direction.

The lowest mode has $(n_x, n_y) = (0, 1)$ or $(1, 0)$.

Modes of Transmission: TE Modes

The magnetic field associated with the electric field can be obtained from:

$$\nabla \times \vec{E} = -\dot{\vec{B}} = -j\omega \vec{B} \tag{94}$$

We find that:

$$B_x = \frac{k_z k_x}{\omega k_y} E_0 \sin k_x x \, \cos k_y y \, e^{j(\omega t - k_z z)} \tag{95}$$

$$B_y = \frac{k_z}{\omega} E_0 \cos k_x x \, \sin k_y y \, e^{j(\omega t - k_z z)} \tag{96}$$

$$B_z = -j \frac{k_x^2 + k_y^2}{\omega k_y} E_0 \cos k_x x \, \cos k_y y \, e^{j(\omega t - k_3 z)} \tag{97}$$

Notice that the B_z component is 90° out of phase with the other components.

Modes of Transmission: TE Modes

The energy flow within the waveguide is given by the Poynting vector, $\vec{S}$:

$$\vec{S} = \vec{E} \times \vec{H} \tag{98}$$

In a TE mode, $E_z = 0$, and B_z is 90° out of phase with respect to the other field components.

It then follows that the time-average values of the transverse components of the Poynting vector vanish:

$$\langle S_x \rangle_t = \langle S_y \rangle_t = 0 \tag{99}$$

On average, there is no energy flow in the transverse direction in the waveguide.

The same is true for other modes (e.g. TM modes).

Modes of Transmission: TE Modes

The z component of $\vec{E} \times \vec{H}$ has a non-zero time average:

$$\langle S_z \rangle_t = \frac{k_z}{2\omega\mu} E_0^2 \left(\cos^2 k_x x \, \sin^2 k_y y + \frac{k_x^2}{k_y^2} \sin^2 k_x x \, \cos^2 k_y y \right) \quad (100)$$

Hence, there is a net flow of energy along the z axis.

In the lowest frequency mode, TE_{01}:

$$\langle S_z \rangle_t = \frac{k_z}{2\omega\mu} E_0^2 \sin^2 k_y y = \frac{v_g}{2} \varepsilon E_0^2 \sin^2 k_y y \quad (101)$$

The total (time average) power transmitted along the waveguide is:

$$W = \int_0^{a_x} \int_0^{a_y} \langle S_z \rangle_t \, dx \, dy = \frac{1}{4} a_x a_y v_g \varepsilon E_0^2 \quad (102)$$

TE_{01} Mode: Rectangular Waveguide Examples

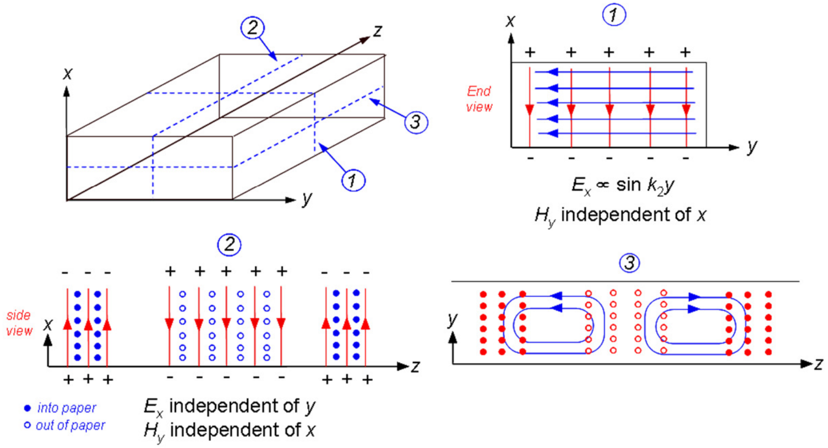
The maximum power for a waveguide is limited by the maximum electric field that can be supported before the dielectric inside the waveguide breaks down and starts to conduct.

For dry air, the breakdown field is approximately 3000 kV/m.

If we assume a maximum electric field of around 1500 kV/m, then we obtain the following estimates for the power that can be transmitted in waveguides of typical dimensions:

Dimensions	f_{min}	Typical f	Power Rating
8 mm × 16 mm	9.5 GHz	12 - 18 GHz	145 kW
34 mm × 72 mm	2 GHz	2.6 - 4 GHz	2.8 MW

Field Plots for TE_{01} Mode



Comments on Waveguides (in All Modes)

- Lines of the magnetic field $\vec{B}$ form closed loops ($\nabla \cdot \vec{B} = 0$).
- On the walls, lines of the electric field $\vec{E}$ start from positive charges, and end on negative charges.
- Lines of $\vec{E}$ and $\vec{B}$ are orthogonal.
- Lines of $\vec{E}$ meet a perfect conductor at 90°.

Comments on Waveguides (in All Modes)

- As time increases, the field pattern moves along the z axis with the group velocity v_g.
- The charge distribution on the walls moves in response to the changing field pattern. This means that there is a current flow, which leads to conversion of the electromagnetic energy in the wave into heat.
- The wall currents flow in a depth of the wall of order of the skin depth. Coating or plating the inside walls of the waveguide with a good conductor (e.g. silver) can help to reduce energy losses.
- We have not considered attenuation in detail. Typical power attenuation lengths (for $1/e$ of the initial power) are of the order of 40 m.

Comments on Waveguides (in All Modes)

- In practice, waveguides are often constructed so that $a_y \approx 2a_x$.
- The terminal devices for transmitting and receiving signals are constructed so that the conductors inside the waveguide coincide with the lines of the electric field $\vec{E}$ in the desired mode. The conductors then need to be "fed" a current of the appropriate frequency for transmitting the wave.
- By suitable design, a particular mode can be transmitted to almost total exclusion of all others.

Dielectric Waveguides

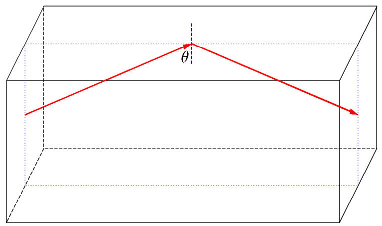
Consider electromagnetic radiation at an interface between two dielectric media.

Total internal reflection occurs if the angle of incidence θ is greater than the critical angle θ_c, i.e. if:

$$\theta > \theta_c = \sin^{-1}\left(\frac{n_2}{n_1}\right) \tag{103}$$

where n_1 is the refractive index of the material in which the wave is travelling, and n_2 is the refractive index of the material on the other side of the interface.

Dielectric Waveguides



Hence, a rectangular block of dielectric can act as a waveguide.

However, there are some important differences that result from the boundary conditions.

We will not solve the general problem, but look only at some specific cases.

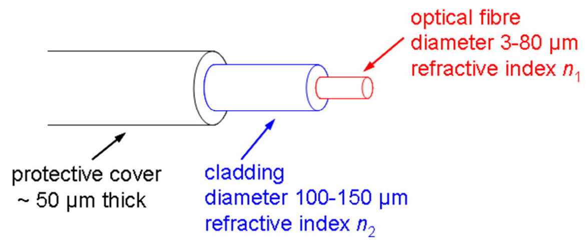
Dielectric Waveguides

Points to note about dielectric waveguides:

- Wave propagation is limited to modes for which $\theta > \theta_c$.
- There will be effects resulting from phase shifts on reflection.
- In practice, dielectric waveguides are constructed with a circular, rather than a rectangular cross section.

Example of a Dielectric Waveguide: Optical Fibre

Step index fibre:

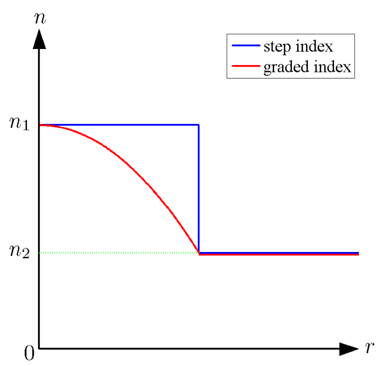


Note that 1 μm = 1/1000 mm. Optical fibres have very small diameters!

To understand fully the wave propagation, we need to solve the wave equation in cylindrical coordinates with boundary conditions for two dielectrics. The wave equation in cylindrical coordinates involves Bessel functions - we will not go into the mathematics.

Example of a Dielectric Waveguide: Optical Fibre

Some fibres have a "graded" refractive index.



Typically, $n_1 - n_2 \approx 0.01$, and the variation of the refractive index is roughly parabolic.

Comments on Optical Fibres

- Optical fibres are packaged using polymer materials, and do not contain any metal. This means that there are no "pick-up" problems: they are insensitive to external electromagnetic noise.
- Optical fibres are waveguides operating at optical frequencies, of the order 10^{15} Hz. This means that there is the possibility of a large bandwidth for carrying large volumes of information.
- By using high quality materials, attenuation can be very small; optical "repeater" stations can be separated by up to 10 km. In the UK, phone and TV links between cities often use optical fibres.

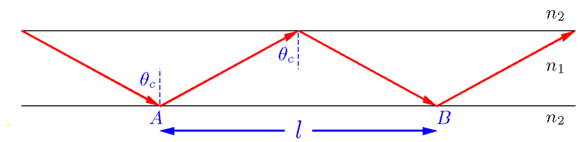
Comments on Optical Fibres

- Optical fibres are cheap and small ($<$ diameter of a human hair). Their packaging is very compact, and they are easier to install than conventional (metal) electric cable.

- Optical fibres are difficult to join if broken; this can be an advantage in providing security (the fibres are hard to "tap").

Spread in Arrival Time of Light Signals

A light pulse reflects of the walls as it travels down an optical fibre. The time taken for the light to travel down the fibre depends on the path taken. Consider the case of a step-index fibre.



The minimum time between points A and B is:

$$t_{min} = \frac{l}{c/n_1} \tag{104}$$

The maximum time is determined by the critical angle:

$$t_{max} = \frac{1}{\sin\theta_c}\frac{l}{c/n_1} \tag{105}$$

Spread in Arrival Time of Light Signals

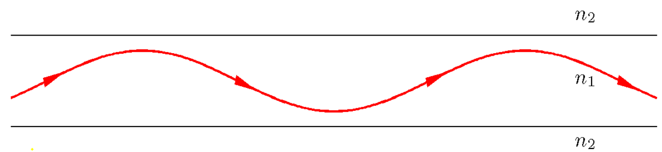
The time difference is:

$$\Delta t = t_{max} - t_{min} = \frac{l}{c/n_1}\left(\frac{1}{\sin\theta_c} - 1\right) = \frac{l}{c/n_1}\frac{n_1 - n_2}{n_2} \tag{106}$$

Assuming a difference in refractive index $(n_1 - n_2)/n_2 \approx 10^{-2}$, and $n_1 = 1.5$, then over a distance $l = 1$ km, the difference in arrival time of the two light pulses is 50 ns.

Spread in Arrival Time of Light Signals

Now consider the case of a parabolic-profile fibre.



The refractive index decreases with distance from the centre of the fibre.

This means that the speed of light increases; so a light pulse taking a "long" trajectory between two points along the fibre makes up some of the lost time by an increase in speed.

In addition, because the trajectory is smooth, rather than having angular corners, the path length is reduced.

Spread in Arrival Time of Light Signals

When these two effects are taken into account, the difference between the maximum and minimum times to travel between two points in a parabolic-profile fibre is given by:

$$\Delta t = \frac{l}{c/n_1}\left(\frac{n_1 - n_2}{n_2}\right)^2 \approx 10^{-4}\frac{l}{c/n_1} \tag{107}$$

where we have again assumed that $(n_1 - n_2)/n_2 \approx 10^{-2}$.

In this case, over a distance $l = 1$ km, the difference in arrival time of the two light pulses is 0.5 ns.

Summary of Part 5: Cavities

You should be able to:

- Explain what is meant by "modes" in a cavity or waveguide.
- Solve Maxwell's equations with the appropriate boundary conditions to find the modes in a rectangular cavity or waveguide with perfectly conducting walls.
- Find an expression for the frequency of a given mode in a rectangular cavity or waveguide, in terms of the dimensions of the cavity or waveguide.
- Show that, in a rectangular cavity with perfectly conducting walls, the average energy flow at any point is zero.

Summary of Part 5 (continued): Waveguides

You should be able to:

- Explain how the boundary conditions on the fields in a rectangular waveguide lead to solutions that represent standing waves in the transverse directions, and travelling waves in the longitudinal direction.
- Derive expressions for the phase velocity and the group velocity in a rectangular waveguide.
- Explain what is meant by the "cut-off frequency" in a waveguide, and sketch a plot showing how the group velocity in a waveguide varies with frequency.
- Explain the principles behind "dielectric waveguides" (optical fibres), and describe the structure of step-index and graded-index optical fibres.
- Describe some of the advantages of optical fibres over conducting wires for carrying signals.
- Explain how graded-index fibres can reduce the spread in arrival times of signals transmitted along the fibre, compared to step-index fibres.