UNIVERSITY OF LIVERPOOL

PHYS370 – Advanced Electromagnetism

Part 4: Waves on Boundaries

When plane waves are incident on a boundary between different media, some energy crosses the boundary, and some is reflected.

We define transmission and reflection coefficients to quantify the transmission and reflection of wave energy. These coefficients are properties of the two media.

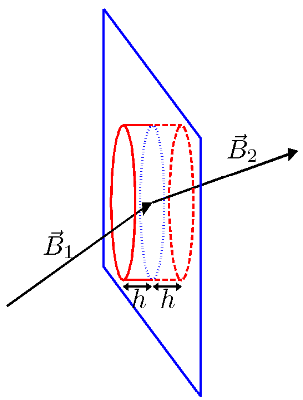
The transmission and reflection coefficients are determined by matching the electric and magnetic fields in the waves at the boundary between the two media.

In this part of the course, we shall consider:

- Boundary conditions on electric and magnetic fields.
- Boundary conditions on fields at the surfaces of conductors.
- Monochromatic plane wave on a boundary:
  - directions of reflected and transmitted waves (laws of reflection and refraction);
  - amplitudes of reflected and transmitted waves (Fresnel's equations);
  - the special case of a boundary between two dielectrics;
  - the special case of the surface of a conductor.
- Monochromatic plane wave on a boundary between two dielectrics:
  - polarisation by reflection;
  - total internal reflection.
- Reflection coefficient for a conducting surface.

Boundary Conditions 1: Normal Component of $\vec{B}$

We can use Maxwell's equations to derive the boundary conditions on the magnetic field across a surface. Consider a "pillbox" across the surface.



Boundary Conditions 1: Normal Component of $\vec{B}$

Take Maxwell's equation:

$$\nabla \cdot \vec{B} = 0 \tag{1}$$

integrate over the volume of the pillbox, and apply Gauss' theorem:

$$\int_V \nabla \cdot \vec{B}\, dV = \oint_S \vec{B} \cdot d\vec{S} = 0 \tag{2}$$

where V is the volume of the pillbox, and S is its surface. We can break the integral over the surface into three parts: over the flat ends (S_1 and S_2) and over the curved wall (S_3):

$$\int_{S_1} \vec{B} \cdot d\vec{S} + \int_{S_2} \vec{B} \cdot d\vec{S} + \int_{S_3} \vec{B} \cdot d\vec{S} = 0 \tag{3}$$

Boundary Conditions 1: Normal Component of $\vec{B}$

In the limit that the length of the pillbox approaches zero, the integral over the curved surface also approaches zero. If each end has a *small* area A, then equation (3) becomes:

$$-B_{1n}A + B_{2n}A = 0 \tag{4}$$

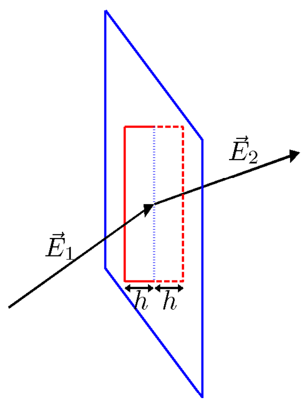
or:

$$B_{1n} = B_{2n} \tag{5}$$

In other words, the normal component of the magnetic field $\vec{B}$ must be continuous across the surface.

Boundary Conditions 2: Tangential Component of $\vec{E}$

Consider a loop spanning the surface.



Boundary Conditions 2: Tangential Component of $\vec{E}$

Take Maxwell's equation:

$$\nabla \times \vec{E} = -\dot{\vec{B}} \tag{6}$$

Integrate over the surface bounded by the loop and apply Stokes' theorem to get:

$$\int_S \nabla \times \vec{E} \cdot d\vec{S} = \oint_C \vec{E} \cdot d\vec{l} = -\frac{\partial}{\partial t} \int_S \vec{B} \cdot d\vec{S} \tag{7}$$

Now take the limit in which the width of the loop becomes zero. The contributions to the integral around the loop C from the narrow ends become zero, as does the integral of the magnetic field across the area bounded by the loop. We are left with:

$$E_{1t}l - E_{2t}l = 0 \tag{8}$$

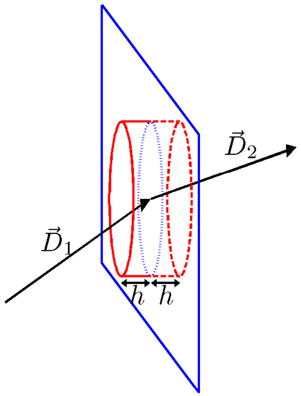
which means that:

$$E_{1t} = E_{2t} \tag{9}$$

Therefore, the tangential component of the electric field is continuous across the boundary.

Boundary Conditions 3: Normal Component of $\vec{D}$

Consider a pillbox crossing the boundary.



surface charge density ρ_s

Boundary Conditions 3: Normal Component of $\vec{D}$

Take Maxwell's equation:

$$\nabla \cdot \vec{D} = \rho \tag{10}$$

Integrate over the volume of the pillbox, and apply Gauss' theorem:

$$\int_V \nabla \cdot \vec{D}\, dV = \oint_S \vec{D} \cdot d\vec{S} = \int_V \rho\, dV \tag{11}$$

Now we take the limit in which the height of the pillbox becomes zero. We assume that there is a surface charge density ρ_s. If the flat ends of the pillbox have (small) area A, then:

$$-D_{1n}A + D_{2n}A = \rho_s A \tag{12}$$

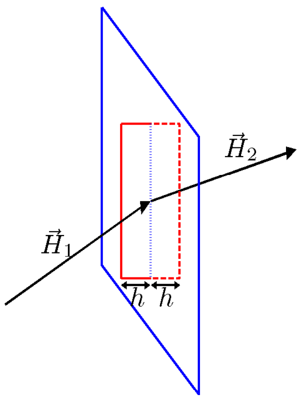
Dividing by the area A, we arrive at:

$$D_{2n} - D_{1n} = \rho_s \tag{13}$$

Note that if the surface charge density is zero, the normal component of $\vec{D}$ is continuous across the surface. However, this is not true for the normal component of $\vec{E}$, unless the two materials have identical permittivities.

Boundary Conditions 4: Tangential Component of $\vec{H}$

Consider a loop across the boundary.



surface current density J_s

Boundary Conditions 4: Tangential Component of $\vec{H}$

Take Maxwell's equation:

$$\nabla \times \vec{H} = \vec{J} + \dot{\vec{D}} \tag{14}$$

Integrate over the surface bounded by the loop, and apply Stokes' theorem to obtain:

$$\int_S \nabla \times \vec{H} \cdot d\vec{S} = \oint_C \vec{H} \cdot d\vec{l} = \int_S \vec{J} \cdot d\vec{S} + \frac{\partial}{\partial t} \int_S \vec{D} \cdot d\vec{S} \tag{15}$$

As before, take the limit where the lengths of the narrow edges of the loop become zero. Then we find that:

$$H_{1t}l - H_{2t}l = J_{s\perp}l \tag{16}$$

or:

$$H_{1t} - H_{2t} = J_{s\perp} \tag{17}$$

where $J_{s\perp}$ represents a surface current density perpendicular to the direction of the tangential component of $\vec{H}$ that is being matched.

Boundary Conditions 4: Tangential Component of $\vec{H}$

The concept of surface current density is analogous to that of surface charge density: it represents a finite current in an infinitesimal layer of the material.

If the material has finite conductivity, then an infinitesimal layer of the material has infinite resistance, and no current can flow (if the electric field is finite).

Therefore, for a material with finite conductivity, we have:

$$H_{1t} = H_{2t} \tag{18}$$

That is, the tangential component of $\vec{H}$ is continuous across the boundary.

Summary of Boundary Conditions

The general conditions on electric and magnetic fields at the boundary between two materials can be summarised as follows:

Boundary condition:	Derived from...	...applied to:
$B_{2n} = B_{1n}$	$\nabla \cdot \vec{B} = 0$	pillbox
$E_{2t} = E_{1t}$	$\nabla \times \vec{E} = -\dot{\vec{B}}$	loop
$D_{2n} - D_{1n} = \rho_s$	$\nabla \cdot \vec{D} = \rho$	pillbox
$H_{2t} - H_{1t} = -J_{s\perp}$	$\nabla \times \vec{H} = \vec{J} + \dot{\vec{D}}$	loop

Boundary Conditions on Surfaces of Conductors

Static electric fields cannot persist inside a conductor. This is simply because the free charges within the conductor will re-arrange themselves to cancel any electric field; this can result in a surface charge density, ρ_s.

We have seen that electromagnetic waves can pass into a conductor, but the field amplitudes fall exponentially with decay length given by the skin depth, δ:

$$\delta \approx \sqrt{\frac{2}{\omega\mu\sigma}} \tag{19}$$

As the conductivity increases, the skin depth gets smaller.

Since both static and oscillating electric fields vanish within a good conductor, we can write the boundary conditions at the surface of such a conductor:

$$\begin{aligned} E_{1t} &\approx 0 & \qquad E_{2t} &\approx 0 \\ D_{1n} &\approx -\rho_s & \qquad D_{2n} &\approx 0 \end{aligned}$$

Boundary Conditions on Surfaces of Conductors

Lenz's law states that a changing magnetic field will induce currents in a conductor that will act to oppose the change.

In other words, currents are induced that will tend to cancel the magnetic field in the conductor.

This means that a good conductor will tend to exclude magnetic fields.

Thus the boundary conditions on *oscillating* magnetic fields at the surface of a good conductor can be written:

$$\begin{aligned} B_{1n} &\approx 0 & \qquad B_{2n} &\approx 0 \\ H_{1t} &\approx J_{s\perp} & \qquad H_{2t} &\approx 0 \end{aligned}$$

We can consider an "ideal" conductor as having infinite conductivity.

In that case, we would expect the boundary conditions to become:

$$\begin{aligned} B_{1n} &= 0 & B_{2n} &= 0 \\ E_{1t} &= 0 & E_{2t} &= 0 \\ D_{1n} &= -\rho_s & D_{2n} &= 0 \\ H_{1t} &= J_{s\perp} & H_{2t} &= 0 \end{aligned}$$

Strictly speaking, the boundary conditions on the magnetic field apply only to oscillating fields, and not to static fields.

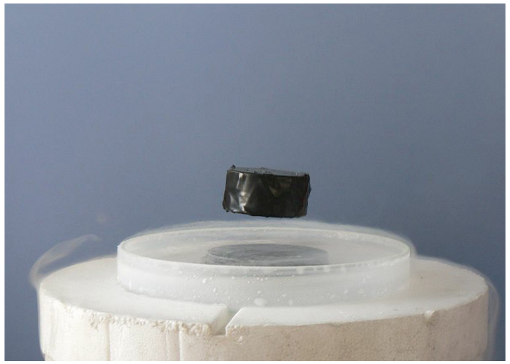
But it turns out that for superconductors, static magnetic fields are excluded as well as oscillating magnetic fields. This is not expected for classical "ideal" conductors.

Although superconductors have infinite conductivity, they cannot be understood in terms of classical theories in the limit $\sigma \to \infty$.

Superconductivity is a quantum phenomenon: one aspect of this is the Meissner effect, which refers to the expulsion of all magnetic fields (static as well as oscillating) from within a superconductor.

In fact, even in a superconductor, the magnetic field is not completely excluded from the material but penetrates a small distance (the London penetration depth, typically around 100 nm) into the material.

As long as the applied magnetic field is not too large, a sample of material cooled below its critical temperature will expel any magnetic field as it undergoes the phase transition to superconductivity: when this happens, a magnet placed on top of the sample will start to levitate.

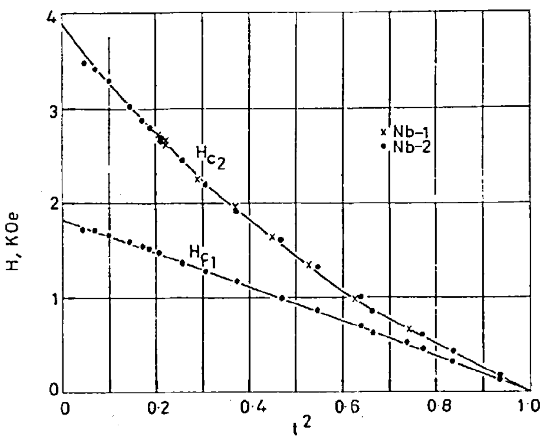


The Meissner effect allows us to classify superconductors into two distinct classes:

- **Type I superconductors:** above a certain critical field H_c (which depends on the temperature), superconductivity is abruptly destroyed.

- **Type II superconductors:** above one critical field value H_{c1}, the magnetic field starts to penetrate, but the electrical resistance remains zero. Above a second, higher critical field value H_{c2}, superconductivity is abruptly destroyed.

Critical Fields in Niobium (Type II Superconductor)



R.A. French, "Intrinsic Type-2 Superconductivity in Pure Niobium," *Cryogenics*, **8**, 301 (1968). Note: $t = T/T_c$. The critical temperature for niobium is $T_c = 9.2$ K.

Waves on Boundaries

We now apply the boundary conditions to an electromagnetic wave incident on a boundary between two different materials.

We shall use the boundary conditions to derive the properties of the reflected and transmitted waves, for a given incident wave.

Consider a monochromatic wave incident at some angle on a boundary. We must consider three waves: the incident wave itself; the reflected wave, and the transmitted wave on the far side of the boundary.

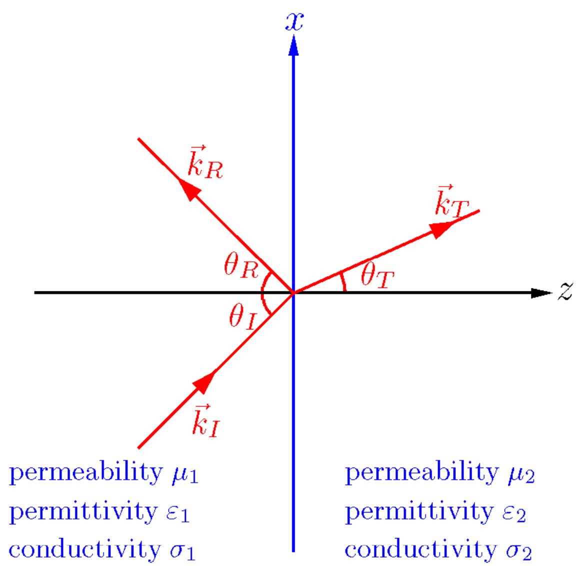
The electric field components for these waves can be written (respectively):

$$\vec{E}_I(\vec{r},t) = \vec{E}_{0I} e^{j(\omega_I t - \vec{k}_I \cdot \vec{r})} \tag{20}$$

$$\vec{E}_R(\vec{r},t) = \vec{E}_{0R} e^{j(\omega_R t - \vec{k}_R \cdot \vec{r})} \tag{21}$$

$$\vec{E}_T(\vec{r},t) = \vec{E}_{0T} e^{j(\omega_T t - \vec{k}_T \cdot \vec{r})} \tag{22}$$

Waves on Boundaries



Waves on Boundaries

Let us first consider the time dependence of the waves. The boundary conditions must apply at all times: for example, the tangential component of the electric field, E_t, must be continuous across the boundary at all points on the boundary at all times.

This means that all waves must have the same time dependence, and therefore:

$$\omega_I = \omega_R = \omega_T = \omega \tag{23}$$

Reflection at a boundary cannot change the frequency of an incident monochromatic wave. Some surfaces reflect some wavelengths better than others, which is why they can appear coloured under white light; but the frequency of the light does not change.

Now let us consider the relationships between the directions in which the waves are moving.

We shall find that these relationships are just the laws of reflection and refraction that we are familiar with from basic optics.

However, our goal is now to derive these laws from Maxwell's equations, by applying the boundary conditions on fields in waves across boundaries.

We start from the fact that the boundary conditions must be satisfied at all points on the boundary.

This means that the waves must all change phase in the same way as we move from one point to another on the boundary.

Since the phase of each of the waves at a position $\vec{r}$ is given by $\vec{k} \cdot \vec{r}$, where $\vec{k}$ is the appropriate wave vector, we must have:

$$\vec{k}_I \cdot \vec{p} = \vec{k}_R \cdot \vec{p} = \vec{k}_T \cdot \vec{p} \tag{24}$$

where $\vec{p}$ is any point on the boundary.

For simplicity, let us choose our coordinates so that the boundary lies in the plane $z = 0$. Then any point $\vec{p}$ on the boundary can be written:

$$\vec{p} = (x, y, 0) \tag{25}$$

Now we can (without loss of generality) further specify the coordinate system so that $\vec{k}_I$ lies in the $x - z$ plane, i.e. the y component of $\vec{k}_I$ is zero:

$$\vec{k}_I = (k_I \sin\theta_I, 0, k_I \cos\theta_I) \tag{26}$$

where θ_I is the angle between the direction of travel of the incident wave and the boundary.

Now let us apply equation (24):

$$\vec{k}_I \cdot \vec{p} = \vec{k}_R \cdot \vec{p} = \vec{k}_T \cdot \vec{p}$$

to points on the boundary with $x = 0$, i.e. $\vec{p} = (0, y, 0)$. We find:

$$k_{Iy} = k_{Ry} = k_{Ty} = 0 \tag{27}$$

Therefore, the directions of the incident, reflected and transmitted waves all lie in the plane $y = 0$.

Now let us consider points on the boundary with $y = 0$, i.e. $\vec{p} = (x, 0, 0)$. This time, using equation (24) gives:

$$k_{Ix} = k_{Rx} = k_{Tx} = k_I \sin\theta_I \tag{28}$$

which (since the vertical components of the wave vectors are all zero) can be written:

$$k_I \sin\theta_I = k_R \sin\theta_R = k_T \sin\theta_T \tag{29}$$

But since the incident and reflected waves are travelling in the same material with the same frequency, the magnitudes of the wave vectors must be the same:

$$k_I = k_R \tag{30}$$

Combining equations (29) and (30) we find:

$$\theta_I = \theta_R \qquad \text{the law of reflection} \tag{31}$$

$$\frac{\sin\theta_I}{\sin\theta_T} = \frac{k_T}{k_I} \qquad \text{the law of refraction (Snell's law)} \tag{32}$$

As an example, consider a monochromatic wave incident on a boundary between two dielectrics (e.g. air and glass).

Since the conductivity is zero on both sides of the boundary, the wave vectors of all waves must be real.

Also, we have:

$$\frac{\omega}{k_I} = v_1, \qquad \frac{\omega}{k_T} = v_2 \tag{33}$$

where v_1 is the phase velocity in medium 1, and v_2 is the phase velocity in medium 2.

Then equation (32) gives us:

$$\frac{\sin\theta_I}{\sin\theta_T} = \frac{v_1}{v_2} \tag{34}$$

We define the refractive index n of a material as the ratio of the speed of light in a vacuum to the speed of light in the material:

$$n = \frac{c}{v} \tag{35}$$

Then equation (34) can be written:

$$\frac{\sin\theta_I}{\sin\theta_T} = \frac{n_2}{n_1} \tag{36}$$

This is the familiar form of Snell's law.

For a wave incident on a conductor, $\vec{k}_T$ will be complex:

$$\vec{k}_T = \vec{\alpha} - j\vec{\beta} \tag{37}$$

For a good conductor (i.e. $\sigma \gg \omega\varepsilon_2$):

$$\alpha \approx \beta \approx \sqrt{\frac{\omega\mu_2\sigma_2}{2}} \tag{38}$$

so:

$$k_T = \sqrt{\vec{\alpha}^2 + \vec{\beta}^2} = \sqrt{\omega\mu_2\sigma_2} \tag{39}$$

Applying the law of refraction (32):

$$\frac{\sin\theta_I}{\sin\theta_T} = \frac{k_T}{k_I} \approx \sqrt{\frac{\sigma_2}{\omega\varepsilon_1}} \gg 1 \tag{40}$$

where we have assumed that $\mu_2 \approx \mu_1$.

Since the largest value of $\sin\theta_I$ is 1, equation (40) tells us that $\sin\theta_T \approx 0$, so the direction of the transmitted wave in a good conductor must be (close to the) normal to the surface.

Having derived the relationships between the directions of the incident, reflected and refracted waves, we turn now to the amplitudes of the waves.

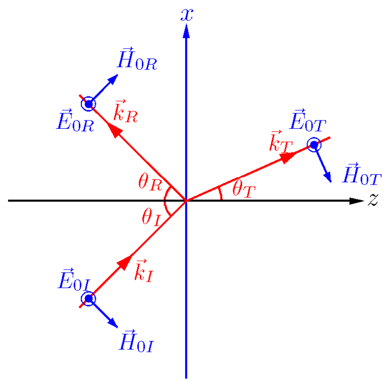
We can again apply the boundary conditions on the electromagnetic fields at a boundary to derive relationships between the wave amplitudes.

It turns out that the relative amplitudes and phases of the waves depend on the electromagnetic impedances of the materials on either side of the boundary.

The results we find are summarised in a set of equations known as Fresnel's equations.

Consider an electromagnetic wave incident on a boundary with the electric field in the wave normal to the plane of incidence.

We call this "N polarisation".



Since the tangential component of the electric field is continuous across the boundary (at any time and any point on the boundary), we have:

$$\vec{E}_{0I} + \vec{E}_{0R} = \vec{E}_{0T} \tag{41}$$

The tangential component of the magnetic field must also be continuous across the boundary:

$$H_{0I}\cos\theta_I - H_{0R}\cos\theta_I = H_{0T}\cos\theta_T \tag{42}$$

The ratio of the electric field to the magnetic intensity in a wave is given by the impedance of the medium:

$$\frac{E_0}{H_0} = Z \tag{43}$$

Substituting from (43) into (42) we get:

$$\frac{(E_{0I} - E_{0R})}{Z_1}\cos\theta_I = \frac{E_{0T}}{Z_2}\cos\theta_T \tag{44}$$

Now we solve equations (41) and (44) for the amplitudes of the reflected and transmitted waves, as fractions of the amplitude of the incident wave:

$$\left(\frac{E_{0R}}{E_{0I}}\right)_N = \frac{Z_2\cos\theta_I - Z_1\cos\theta_T}{Z_2\cos\theta_I + Z_1\cos\theta_T} \tag{45}$$

$$\left(\frac{E_{0T}}{E_{0I}}\right)_N = \frac{2Z_2\cos\theta_I}{Z_2\cos\theta_I + Z_1\cos\theta_T} \tag{46}$$

It is important to remember that equations (45) and (46) apply *only* to the case that the incident wave is polarised with the electric field normal to the plane of incidence, i.e. for N polarisation.

Consider equations (45) and (46) in the special case of a boundary between two dielectrics.

Since the refractive index n of a dielectric is given by:

$$n = \frac{c}{v} = \sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}} \tag{47}$$

we can write:

$$Z = \sqrt{\frac{\mu}{\varepsilon}} = \frac{\mu}{\mu_0}\frac{Z_0}{n} \tag{48}$$

For non-magnetic dielectrics, we have $\mu_1 = \mu_2 = \mu_0$, so that:

$$Z_1 = \frac{Z_0}{n_1}, \qquad Z_2 = \frac{Z_0}{n_2} \tag{49}$$

Then, equations (45) and (46):

$$\left(\frac{E_{0R}}{E_{0I}}\right)_N = \frac{Z_2\cos\theta_I - Z_1\cos\theta_T}{Z_2\cos\theta_I + Z_1\cos\theta_T}$$

$$\left(\frac{E_{0T}}{E_{0I}}\right)_N = \frac{2Z_2\cos\theta_I}{Z_2\cos\theta_I + Z_1\cos\theta_T}$$

become:

$$\left(\frac{E_{0R}}{E_{0I}}\right)_N = \frac{n_1\cos\theta_I - n_2\cos\theta_T}{n_1\cos\theta_I + n_2\cos\theta_T} \tag{50}$$

$$\left(\frac{E_{0T}}{E_{0I}}\right)_N = \frac{2n_1\cos\theta_I}{n_1\cos\theta_I + n_2\cos\theta_T} \tag{51}$$

The Fresnel equations above are frequently written in terms of the refractive index rather than the impedance; but then the equations are valid only for non-magnetic dielectrics.

Now consider the special case of a plane wave incident on the surface of a good conductor, again polarised so that the electric field is parallel to the surface.

Recall that the impedance of a conductor is given in general by:

$$Z = (1+j)\sqrt{\frac{\mu}{\varepsilon}}\sqrt{\frac{\omega\varepsilon}{2\sigma}} \tag{52}$$

If the conductor is non-magnetic, so that $\mu \approx \mu_0$, and if the permittivity is also close to the permittivity of free space, then for a good conductor with $\sigma \gg \omega\varepsilon$:

$$|Z| \approx Z_0\sqrt{\frac{\omega\varepsilon}{\sigma}} \ll Z_0 \tag{53}$$

Reflection and Refraction at the Surface of a Conductor

If we assume that the wave is incident on a good conductor from a dielectric with impedance of the same order of magnitude as the impedance of free space, then we have:

$$Z_1 \approx Z_0, \quad |Z_2| \ll Z_0 \qquad \therefore \quad |Z_2| \ll Z_1 \tag{54}$$

Then, equations (45) and (46) become:

$$\left(\frac{E_{0R}}{E_{0I}}\right)_N = \frac{Z_2 \cos\theta_I - Z_1 \cos\theta_T}{Z_2 \cos\theta_I + Z_1 \cos\theta_T} \approx -1 \tag{55}$$

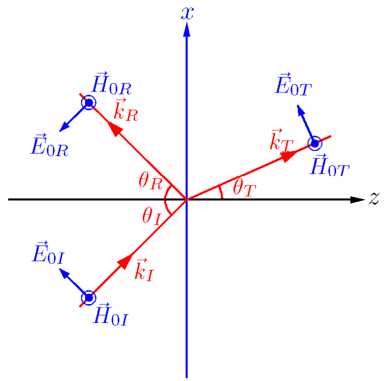
$$\left(\frac{E_{0T}}{E_{0I}}\right)_N = \frac{2Z_2 \cos\theta_I}{Z_2 \cos\theta_I + Z_1 \cos\theta_T} \approx 0 \tag{56}$$

Therefore, there is nearly 100% reflection from a metal surface, with a phase change of 180°. The phase change means that there is a cancellation between the tangential components of the electric field in the incident and reflected waves.

Intensity of Reflected and Refracted Waves

Now consider the case of an monomchromatic wave incident on a boundary, with the electric field parallel to the plane of incidence.

We call this "P polarisation".



Intensity of Reflected and Refracted Waves

We apply the boundary conditions as before.

First, since the component of the electric field tangential to the boundary must be continuous across the boundary, we have:

$$E_{0I} \cos\theta_I - E_{0R} \cos\theta_I = E_{0T} \cos\theta_T \tag{57}$$

Next, since the tangential component of the magnetic field must also be continuous across the boundary:

$$H_{0I} + H_{0R} = H_{0T} \tag{58}$$

We can again express the magnetic intensity in terms of the electric field and the impedance:

$$\frac{E_0}{H_0} = Z \tag{59}$$

Intensity of Reflected and Refracted Waves

Proceeding as before, we find for the ratios of the wave amplitudes:

$$\left(\frac{E_{0R}}{E_{0I}}\right)_P = \frac{Z_1 \cos\theta_I - Z_2 \cos\theta_T}{Z_1 \cos\theta_I + Z_2 \cos\theta_T} \tag{60}$$

$$\left(\frac{E_{0T}}{E_{0I}}\right)_P = \frac{2Z_2 \cos\theta_I}{Z_1 \cos\theta_I + Z_2 \cos\theta_T} \tag{61}$$

Equations (60) and (61) are valid for the electric field in the incident wave parallel (P) to the plane of incidence.

Compare equations (60) and (61) with (45) and (46):

$$\left(\frac{E_{0R}}{E_{0I}}\right)_N = \frac{Z_2 \cos\theta_I - Z_1 \cos\theta_T}{Z_2 \cos\theta_I + Z_1 \cos\theta_T}$$

$$\left(\frac{E_{0T}}{E_{0I}}\right)_N = \frac{2Z_2 \cos\theta_I}{Z_2 \cos\theta_I + Z_1 \cos\theta_T}$$

When the materials on both sides of the boundary are dielectrics, we can write equations (60) and (61) in terms of the refractive indices n_1 and n_2.

$$\left(\frac{E_{0R}}{E_{0I}}\right)_P = \frac{n_2 \cos\theta_I - n_1 \cos\theta_T}{n_2 \cos\theta_I + n_1 \cos\theta_T} \quad (62)$$

$$\left(\frac{E_{0T}}{E_{0I}}\right)_P = \frac{2n_1 \cos\theta_I}{n_2 \cos\theta_I + n_1 \cos\theta_T} \quad (63)$$

When the wave is incident from a dielectric onto the surface of a good conductor ($\sigma \gg \omega\varepsilon$), if we assume non-magnetic materials with permittivity close to that of vacuum, we have:

$$Z_1 \approx Z_0, \quad |Z_2| \ll Z_0 \qquad \therefore \quad |Z_2| \ll Z_1 \quad (64)$$

Then:

$$\left(\frac{E_{0R}}{E_{0I}}\right)_P = \frac{Z_1 \cos\theta_I - Z_2 \cos\theta_T}{Z_1 \cos\theta_I + Z_2 \cos\theta_T} \approx 1 \quad (65)$$

$$\left(\frac{E_{0T}}{E_{0I}}\right)_P = \frac{2Z_2 \cos\theta_I}{Z_1 \cos\theta_I + Z_2 \cos\theta_T} \approx 0 \quad (66)$$

There is nearly 100% reflection from a metal surface: this is the case for both N and P polarisation.

The equations (45), (46), (60) and (61) are known as Fresnel's equations:

$$\left(\frac{E_{0R}}{E_{0I}}\right)_N = \frac{Z_2 \cos\theta_I - Z_1 \cos\theta_T}{Z_2 \cos\theta_I + Z_1 \cos\theta_T}$$

$$\left(\frac{E_{0T}}{E_{0I}}\right)_N = \frac{2Z_2 \cos\theta_I}{Z_2 \cos\theta_I + Z_1 \cos\theta_T}$$

$$\left(\frac{E_{0R}}{E_{0I}}\right)_P = \frac{Z_1 \cos\theta_I - Z_2 \cos\theta_T}{Z_1 \cos\theta_I + Z_2 \cos\theta_T}$$

$$\left(\frac{E_{0T}}{E_{0I}}\right)_P = \frac{2Z_2 \cos\theta_I}{Z_1 \cos\theta_I + Z_2 \cos\theta_T}$$

For a wave incident normal to a boundary, $\theta_I = \theta_T = 0$, and Fresnel's equations (for electric field normal to the plane of incidence) become:

$$\frac{E_{0R}}{E_{0I}} = \frac{Z_2 - Z_1}{Z_2 + Z_1} \quad \text{and} \quad \frac{E_{0T}}{E_{0I}} = \frac{2Z_2}{Z_2 + Z_1} \quad (67)$$

Note that since the energy in a wave is proportional to the square of its amplitude divided by the impedance of the medium, we can easily show that energy is conserved in this case:

$$\frac{E_{0R}^2}{E_{0I}^2} + \frac{Z_1}{Z_2}\frac{E_{0T}^2}{E_{0I}^2} = \frac{(Z_2 - Z_1)^2 + 4Z_1 Z_2}{(Z_2 + Z_1)^2} = 1 \quad (68)$$

and hence:

$$\frac{E_{0R}^2}{Z_1} + \frac{E_{0T}^2}{Z_2} = \frac{E_{0I}^2}{Z_1} \quad (69)$$

Of course, energy is also conserved for general angles of incidence, but the algebra is slightly more complicated.

Wave Incident on a Boundary Between Dielectrics

There are two important phenomena associated with electromagnetic waves incident on a boundary between two dielectrics. These are:

- Polarisation by reflection.
- Total internal reflection.

Both of these effects can be understood from the relationships between the angles and intensities of the incident, reflected and transmitted waves that we have derived. We shall discuss each in turn.

Wave Incident on a Boundary Between Dielectrics

Expressing Fresnel's equations in terms of the refractive indices n_1 and n_2 for the two dielectrics:

$$\left(\frac{E_{0R}}{E_{0I}}\right)_N = \frac{n_1 \cos\theta_I - n_2 \cos\theta_T}{n_1 \cos\theta_I + n_2 \cos\theta_T}$$

$$\left(\frac{E_{0T}}{E_{0I}}\right)_N = \frac{2n_1 \cos\theta_I}{n_1 \cos\theta_I + n_2 \cos\theta_T}$$

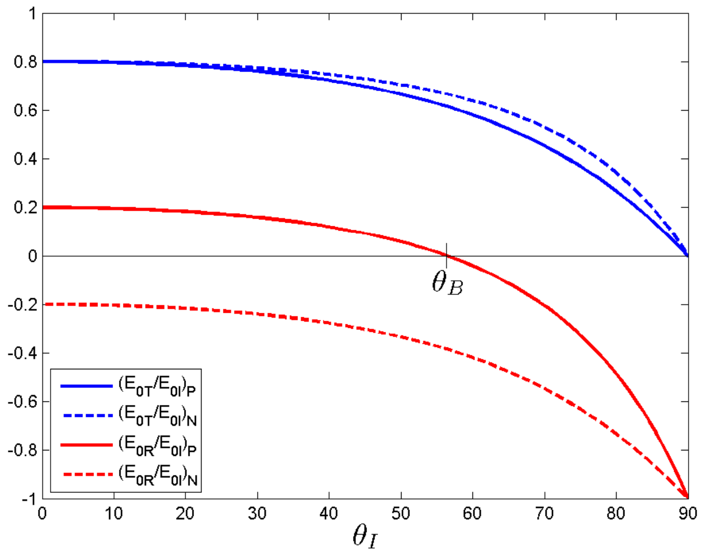
$$\left(\frac{E_{0R}}{E_{0I}}\right)_P = \frac{n_2 \cos\theta_I - n_1 \cos\theta_T}{n_2 \cos\theta_I + n_1 \cos\theta_T}$$

$$\left(\frac{E_{0T}}{E_{0I}}\right)_P = \frac{2n_1 \cos\theta_I}{n_2 \cos\theta_I + n_1 \cos\theta_T}$$

The angles of incidence and transmission are related by the law of refraction (32):

$$\frac{\sin\theta_I}{\sin\theta_T} = \frac{n_2}{n_1}$$

Wave Incident on a Boundary Between Dielectrics



Wave Incident on a Boundary Between Dielectrics

- Since there are no imaginary terms in the ratios of the field amplitudes, E_{0R} and E_{0T} are either in phase with E_{0I} (positive ratio) or π out of phase (negative ratio).
- For all n_1 and n_2, E_{0T} is always in phase with E_{0I}.
- If $n_1 < n_2$ (incident wave in low-density material, and transmitted wave in high-density material):

$$\left(\frac{E_{0R}}{E_{0I}}\right)_N < 0 \qquad (70)$$

so for N polarisation, the reflected and incident waves are always π out of phase.

Depending on the values of θ_I and n_2/n_1, the ratio

$$\left(\frac{E_{0R}}{E_{0I}}\right)_P \qquad (71)$$

can be positive or negative. It is zero when:

$$n_2 \cos\theta_I = n_1 \cos\theta_T \qquad (72)$$

Using the law of refraction (32):

$$\frac{n_1}{n_2} = \frac{\sin\theta_T}{\sin\theta_I} \qquad (73)$$

we find that the condition (72) can be written:

$$\sin\theta_I \cos\theta_I - \sin\theta_T \cos\theta_T = 0 \qquad (74)$$

Using the trigonometric identity:

$$\sin 2A = 2 \sin A \cos A \qquad (75)$$

equation (74) becomes:

$$\sin 2\theta_I - \sin 2\theta_T = 0 \qquad (76)$$

Using a further trigonometric identity:

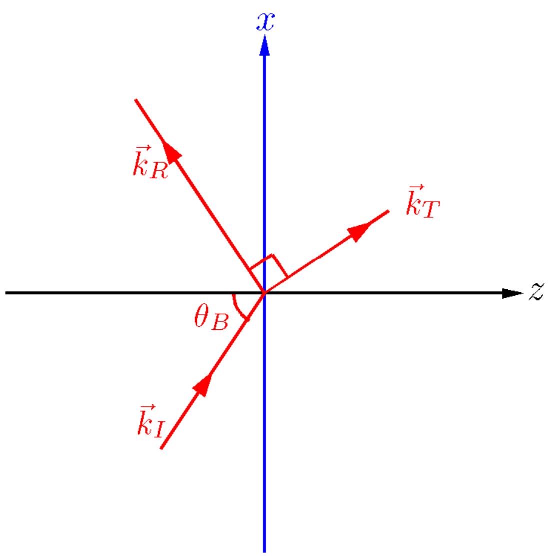
$$\sin A - \sin B = 2 \cos\left(\frac{A+B}{2}\right) \sin\left(\frac{A-B}{2}\right) \qquad (77)$$

equation (76) becomes:

$$\cos(\theta_I + \theta_T) \sin(\theta_I - \theta_T) = 0 \qquad (78)$$

Therefore, the condition for zero reflection (for the electric field parallel to the plane of incidence) becomes:

$$\theta_I + \theta_T = \frac{\pi}{2} \qquad (79)$$



Substituting from equation (79) into the law of refraction (32), we find:

$$\frac{\sin\theta_T}{\sin\theta_I} = \frac{\sin(\frac{\pi}{2} - \theta_I)}{\sin\theta_I} = \frac{\cos\theta_I}{\sin\theta_I} = \frac{n_1}{n_2} \qquad (80)$$

The angle of incidence at which condition (80) is satisfied is called the Brewster angle, θ_B:

$$\tan\theta_B = \frac{n_2}{n_1} \qquad (81)$$

When the angle of incidence is equal to the Brewster angle, a wave with the electric field parallel to the plane of incidence has zero reflected amplitude.

A wave with electric field normal to the plane of incidence is still reflected. As a result, the wave becomes polarised.

Polarisation by Reflection

Polarisation by reflection enables us to use polarising filters to reduce the amount of glare from sunlight reflecting off the surface of a lake, or off a wet road...

Total Internal Reflection

Consider again the law of refraction (32):

$$\sin\theta_T = \frac{n_1}{n_2}\sin\theta_I \tag{82}$$

If $n_1 > n_2$, then for sufficiently large θ_I, we apparently have:

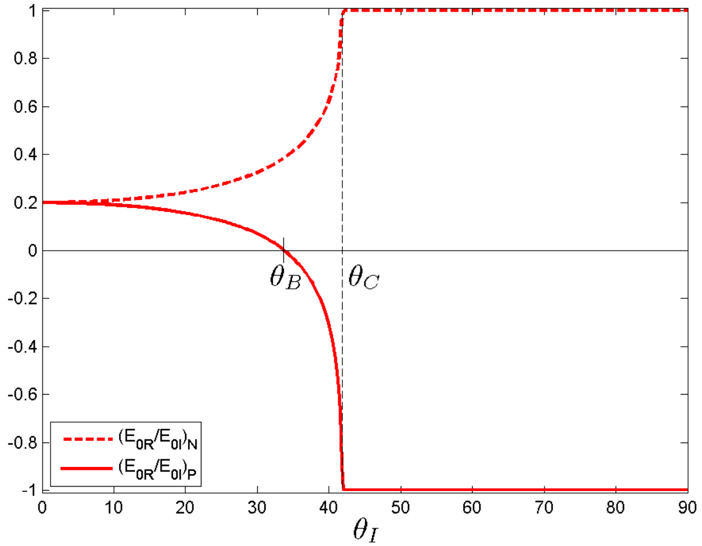
$$\sin\theta_T > 1 \tag{83}$$

We interpret this as meaning that under these conditions, there is *no* refracted wave: all the energy in the incident wave is reflected from the boundary.

The angle of incidence for which $\sin\theta_T = 1$ is called the *critical angle*, θ_c:

$$\sin\theta_c = \frac{n_2}{n_1} \tag{84}$$

Total Internal Reflection



Reflection Coefficient for a Good Conductor

Finally (for this part of the lecture course), we shall consider in a little more detail reflection from the surface of a good conductor.

Recall that we found previously that, for polarisation both normal and parallel to the plane of incidence, the amplitudes of the reflected and transmitted waves were given approximately by:

$$\left(\frac{E_{0R}}{E_{0I}}\right)_N = \frac{Z_2\cos\theta_I - Z_1\cos\theta_T}{Z_2\cos\theta_I + Z_1\cos\theta_T} \approx 1$$

$$\left(\frac{E_{0T}}{E_{0I}}\right)_N = \frac{2Z_2\cos\theta_I}{Z_2\cos\theta_I + Z_1\cos\theta_T} \approx 0$$

We shall try to take our analysis a stage further, so as to arrive at a more accurate estimate for the amount of reflected light. For simplicity, we shall only consider normal incidence.

Consider a wave incident on the surface of a conductor at normal incidence ($\theta_I = 0$). From Fresnel's equations, the amplitude of the reflected wave is given by:

$$\left(\frac{E_{0R}}{E_{0I}}\right)_N = \frac{Z_2 - Z_1}{Z_2 + Z_1} \tag{85}$$

The impedance for the wave in the conductor is given, in general, by:

$$Z_2 = (1+j)\sqrt{\frac{\mu_2}{\varepsilon_2}}\sqrt{\frac{\omega\varepsilon_2}{2\sigma_2}} \tag{86}$$

Assuming non-magnetic materials, and that the conductor satisfies the good conductor condition $\sigma_2 \gg \omega\varepsilon$, we shall have:

$$|Z_2| \ll Z_1 \tag{87}$$

Thus, we can write equation (85) for the reflected wave amplitude in terms of a small parameter x:

$$\left(\frac{E_{0R}}{E_{0I}}\right)_N = -\frac{1-x}{1+x}, \qquad x = \frac{Z_2}{Z_1}, \quad |x| \ll 1 \tag{88}$$

Since x is small, we can make a Taylor series expansion:

$$(1+x)^{-1} = 1 - x + O\left(x^2\right) \tag{89}$$

Using this in our expression for E_{0R}/E_{0I} from equation (88), we find:

$$\begin{aligned}\left(\frac{E_{0R}}{E_{0I}}\right)_N &= -(1-x)(1+x)^{-1} \\ &= -(1-x)(1-x+O(x^2)) \\ &= -(1-2x+O(x^2))\end{aligned} \tag{90}$$

We find:

$$\left(\frac{E_{0R}}{E_{0I}}\right)_N \approx -1 + 2\frac{Z_2}{Z_1} = -1 + (1+j)\sqrt{\frac{\mu_2\varepsilon_1}{\mu_1\varepsilon_2}}\sqrt{\frac{2\omega\varepsilon_2}{\sigma_2}} \tag{91}$$

By a similar calculation, we find:

$$\left(\frac{E_{0T}}{E_{0I}}\right)_N \approx 2\frac{Z_2}{Z_1} = (1+j)\sqrt{\frac{\mu_2\varepsilon_1}{\mu_1\varepsilon_2}}\sqrt{\frac{2\omega\varepsilon_2}{\sigma_2}} \tag{92}$$

Let us define the real parameter η, given by:

$$\eta = \sqrt{\frac{\mu_2\varepsilon_1}{\mu_1\varepsilon_2}}\sqrt{\frac{2\omega\varepsilon_2}{\sigma_2}} \ll 1 \tag{93}$$

In terms of η, the reflected and transmitted wave amplitudes are:

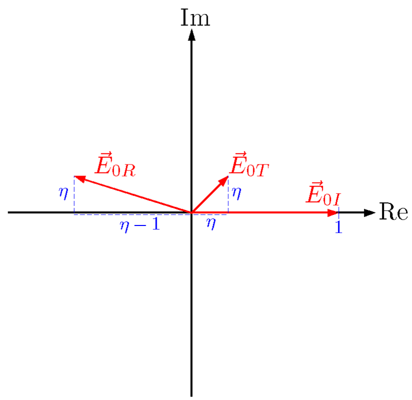
$$\left(\frac{E_{0R}}{E_{0I}}\right)_N \approx -1 + (1+j)\eta \tag{94}$$

$$\left(\frac{E_{0T}}{E_{0I}}\right)_N \approx (1+j)\eta \tag{95}$$

These expressions are convenient for drawing the phases and amplitudes of the reflected and transmitted waves, relative to the incident wave...

$$E_{0R} \approx (\eta - 1 + j\eta)\, E_{0I} \tag{96}$$

$$E_{0T} \approx (\eta + j\eta)\, E_{0I} \tag{97}$$



In the case of a plane wave incident on a good conductor, we find that:

- The phase of E_{0T} leads the phase of E_{0I} by 45°.
- The phase of E_{0R} leads the phase of E_{0I} by:

$$\pi - \tan^{-1}\left(\frac{\eta}{1-\eta}\right) \tag{98}$$

- As $\sigma_2 \to \infty$, $E_{0T} \to 0$ and $E_{0R} \to -E_{0I}$ as expected.
- As $\sigma_2 \to 0$, we do not get the expected dielectric formulae: this situation violates many of the assumptions we made.

We define the reflection coefficient R to be the fraction of energy reflected from a surface. R is the ratio of the time averaged Poynting vector, $\vec{S} = \vec{E} \times \vec{H}$, for the incident and reflected waves.

For a plane wave with normal incidence on a good conductor:

$$R = \frac{\langle \vec{S}_R \rangle}{\langle \vec{S}_I \rangle} = \frac{\langle \vec{E}_R \times \vec{H}_R \rangle}{\langle \vec{E}_I \times \vec{H}_I \rangle} = \left| \frac{E_{0R}}{E_{0I}} \right|^2 \tag{99}$$

Since, from equation (94) the amplitude of the reflected wave is given by:

$$\frac{E_{0R}}{E_{0I}} = -1 + (1+j)\eta \tag{100}$$

we find that:

$$\left| \frac{E_{0R}}{E_{0I}} \right|^2 = (-1 + (1+j)\eta)\,(-1 + (1-j)\eta) \approx 1 - 2\eta \tag{101}$$

where we have used the fact that $|\eta| \ll 1$ to drop a term in η^2.

Therefore:

$$R \approx 1 - 2\eta = 1 - 2\sqrt{\frac{\mu_2 \varepsilon_1}{\mu_1 \varepsilon_2}}\sqrt{\frac{2\omega\varepsilon_2}{\sigma_2}} \tag{102}$$

Consider the example of 3 cm microwave radiation at normal incidence on copper:

frequency of radiation	$\omega = 2\pi \times 10^{10}$ s^{-1}
conductivity of copper	$\sigma_2 = 5.6 \times 10^7\,(\Omega\text{m})^{-1}$
permeability	$\mu_1 = \mu_2 = \mu_0$
permittivity	$\varepsilon_1 = \varepsilon_0$

Using equation (102), we find:

$$R \approx 1 - 2\sqrt{\frac{2\omega\mu_2\varepsilon_1}{\mu_1\sigma_2}} \approx 0.99972 \tag{103}$$

Therefore, 99.97% of the incident power is reflected.

From equation (102),

$$R \approx 1 - 2\sqrt{\frac{2\omega\mu_2\varepsilon_1}{\mu_1\sigma_2}} \tag{104}$$

we observe that:

- R gets closer to 1 at low frequencies.
- Only at very high frequencies does the deviation from 1 become significant for good conductors such as copper, aluminium or silver.

We also note that since the skin depth is small, thin sheets of a good conductor provide excellent shielding for long-wavelength electromagnetic radiation.

Summary of Part 4: Waves on Boundaries

You should be able to:

- Derive (from Maxwell's equations) the boundary conditions on electric and magnetic fields at the interface between two media.
- Apply the boundary conditions on electric and magnetic fields to derive the laws of reflection and refraction.
- Apply the boundary conditions to derive the relative amplitudes of the reflected and refracted waves for different polarisations (Fresnel's equations).
- Apply Fresnel's equations to boundaries between two dielectrics, and to the surfaces of good conductors, to explain the behaviour of waves at such boundaries.
- Plot the variation in reflected and refracted wave amplitudes as functions of angle of incidence, for the boundary between two dielectrics.
- Explain the significance of the Brewster angle and the critical angle, and derive expressions for these angles, in terms of the refractive indices of the media on either side of the boundary.