PHYS370 – Advanced Electromagnetism

Part 2:
Electromagnetic Waves in Dielectric Media

Electromagnetic Wave Equation in "Simple" Materials

Consider a region of space filled by a "simple" material, that is:

- **linear**: μ and ε are constants;

- **isotropic**: there is full rotational symmetry (no preferred or special direction in space);

- **homogeneous**: there is translational symmetry in all directions (no special locations in space);

- **source-free**: the charge density ρ is zero;

- **non-conducting**: the conductivity σ is zero, and hence the current density $\vec{J} = \sigma\vec{E}$ is also zero;

Electromagnetic Wave Equation in "Simple" Materials

In our "simple" material, Maxwell's equations take the form:

$$\nabla \cdot \vec{E} = 0 \tag{1}$$
$$\nabla \cdot \vec{B} = 0 \tag{2}$$
$$\nabla \times \vec{B} = \mu\varepsilon\dot{\vec{E}} \tag{3}$$
$$\nabla \times \vec{E} = -\dot{\vec{B}} \tag{4}$$

If we take the curl of equation (4) we find, using a vector identity:

$$\nabla \times \nabla \times \vec{E} = \nabla(\nabla \cdot \vec{E}) - \nabla^2\vec{E} = -\nabla \times \dot{\vec{B}} \tag{5}$$

Using equation (1), this becomes:

$$\nabla^2\vec{E} = \nabla \times \dot{\vec{B}} \tag{6}$$

Taking the time derivative of equation (3) gives:

$$\nabla \times \dot{\vec{B}} = \mu\varepsilon\ddot{\vec{E}} \tag{7}$$

Then combining equations (6) and (7) we find:

$$\nabla^2\vec{E} - \mu\varepsilon\ddot{\vec{E}} = 0 \tag{8}$$

Electromagnetic Wave Equation in "Simple" Materials

In a "simple" material, the electric field satisfies equation (8):

$$\nabla^2\vec{E} - \mu\varepsilon\ddot{\vec{E}} = 0 \tag{9}$$

Similarly, we find that the magnetic field satisfies:

$$\nabla^2\vec{B} - \mu\varepsilon\ddot{\vec{B}} = 0 \tag{10}$$

These are the equations for waves travelling with speed v, given by:

$$v = \frac{1}{\sqrt{\mu\varepsilon}} \tag{11}$$

Experimentally, we find that v is the speed of light in the material. In a vacuum, the speed of the wave is c, given by:

$$c = \frac{1}{\sqrt{\mu_0\varepsilon_0}} \tag{12}$$

Light is an electromagnetic wave. One of the triumphs of physics in the 19th century was to show the existence of such waves, and derive their properties (including their speed) from Maxwell's equations.

Electromagnetic Wave Equation in "Simple" Materials

The operator:

$$\nabla^2 - \mu\varepsilon\frac{\partial^2}{\partial t^2} \tag{13}$$

is called the *Hertzian* operator. The wave equations (8) and (10) are:

$$\left(\nabla^2 - \mu\varepsilon\frac{\partial^2}{\partial t^2}\right)\left\{\begin{array}{c} \vec{E}(\vec{r},t) \\ \vec{B}(\vec{r},t) \end{array}\right\} = 0 \tag{14}$$

These equations are necessary, but not sufficient, constraints on the possible functions $\vec{E}(\vec{r},t)$ and $\vec{B}(\vec{r},t)$. We must always check that solutions to the wave equations (8) and (10) also satisfy Maxwell's equations.

Monochromatic Plane Wave Solutions to the Wave Equation

We find that the wave equation (8) is solved by:

$$\vec{E}(\vec{r},t) = \vec{E}_0 \cos\left(\omega t - \vec{k}\cdot\vec{r} + \phi_0\right) \tag{15}$$

where $\vec{E}_0$ and $\vec{k}$ are constant vectors, and ω and ϕ_0 are constant scalars. Equation (15) is the equation of a plane wave of frequency ω. The fact that only a single frequency (i.e. a single value of ω) is present in the wave, means that the wave is *monochromatic*.

The field (15) is a valid solution of the wave equation (8) if $\vec{k}$ and ω satisfy:

$$\frac{\omega^2}{\vec{k}^2} = v^2 = \frac{1}{\mu\varepsilon} \tag{16}$$

Equation (16) is known as a *dispersion relation*: it relates the frequency of the wave ω to the wave vector $\vec{k}$.

Maxwell's equations impose further constraints...

Monochromatic Plane Wave Solutions to the Wave Equation

By writing the vectors $\vec{E}_0$ and $\vec{k}$ in terms of components:

$$\begin{aligned} \vec{E}_0 &= (E_{0x}, E_{0y}, E_{0z}) \tag{17} \\ \vec{k} &= (k_x, k_y, k_z) \tag{18} \end{aligned}$$

we find that for the field given by equation (15):

$$\nabla\cdot\vec{E} = \vec{k}\cdot\vec{E}_0 \sin\left(\omega t - \vec{k}\cdot\vec{r} + \phi_0\right) \tag{19}$$

Maxwell's equation (with zero charge density):

$$\nabla\cdot\vec{E} = 0 \tag{20}$$

is only satisfied for all positions $\vec{r}$ and times t, if $\vec{k}$ and $\vec{E}_0$ satisfy:

$$\vec{k}\cdot\vec{E}_0 = 0 \tag{21}$$

Monochromatic Plane Wave Solutions to the Wave Equation

Consider our solution (15) to the wave equation:

$$\vec{E}(\vec{r},t) = \vec{E}_0 \cos\left(\omega t - \vec{k}\cdot\vec{r} + \phi_0\right) \tag{22}$$

At fixed position $\vec{r}$, the field strength $\vec{E}$ varies sinusoidally, with angular frequency ω.

At fixed time t, the field strength varies sinusoidally in the direction of $\vec{k}$, with wavelength $2\pi/k$. Along planes perpendicular to $\vec{k}$, the field is at the same *phase*. (Consider $\vec{r} \to \vec{r} + \vec{r}'$ where $\vec{k}\cdot\vec{r}' = 0$).

Monochromatic Plane Wave Solutions to the Wave Equation

Therefore, the electric field $\vec{E}(\vec{r}, t)$ takes the form of a plane wave advancing in the direction of $\vec{k}$. Since $\vec{k} \cdot \vec{E}_0 = 0$, the field is perpendicular to the direction of motion: it is a *transverse* wave.
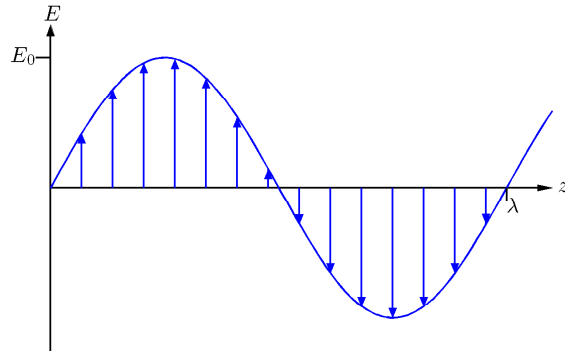
Strengths of Electric and Magnetic Fields in EM Waves

The wave equation (10) for the magnetic field $\vec{B}$ has a similar solution:

$$\vec{B}(\vec{r}, t) = \vec{B}_0 \cos\left(\omega t - \vec{k} \cdot \vec{r} + \phi_0\right) \tag{23}$$

where there are the same constraints on $\vec{B}_0$, $\vec{k}$, and ω.

Despite the fact that we derived independent wave equations for $\vec{E}$ and $\vec{B}$, Maxwell's equations tell us that the electric and magnetic fields are not independent. In particular, we must satisfy:

$$\nabla \times \vec{E} = -\dot{\vec{B}} \tag{24}$$
$$\nabla \times \vec{B} = \mu\varepsilon\dot{\vec{E}} \tag{25}$$

Substituting in the solutions (15) and (23), we find that $\vec{k}$, ω and ϕ_0 must be the same for both $\vec{E}$ and $\vec{B}$, and furthermore we must have:

$$\vec{k} \times \vec{E}_0 = \omega\vec{B}_0 \tag{26}$$
$$\vec{k} \times \vec{B}_0 = -\mu\varepsilon\omega\vec{E}_0 \tag{27}$$
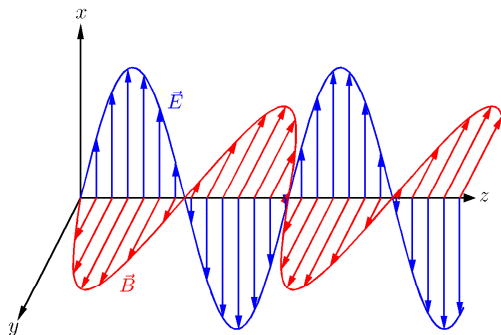
Electric and Magnetic Fields in EM Waves

Maxwell's equations demand that the electric and magnetic fields satisfy (26) and (27):

$$\vec{k} \times \vec{E}_0 = \omega\vec{B}_0$$
$$\vec{k} \times \vec{B}_0 = -\mu\varepsilon\omega\vec{E}_0$$

These constraints can be satisfied if the vectors $\vec{k}$, $\vec{E}_0$ and $\vec{B}_0$ are mutually perpendicular.

Strengths of Electric and Magnetic Fields in EM Waves

The magnitudes of the electric and magnetic fields satisfy:

$$\frac{E_0}{B_0} = \frac{1}{\sqrt{\mu\varepsilon}} = v \tag{28}$$

where v is the phase velocity.

In free space:

$$\frac{E_0}{B_0} = \frac{1}{\sqrt{\mu_0\varepsilon_0}} = c \tag{29}$$

Strengths of Electric and Magnetic Fields in EM Waves

We shall see later that the behaviour of a wave at a boundary between two media depends on the ratio of the electric field $\vec{E}$ to the magnetic intensity $\vec{H}$.

The ratio of the amplitude of the electric field to the amplitude of the magnetic intensity is called the *impedance* Z of the medium.

Using $B_0 = \mu_0 H_0$, we find that in free space:

$$Z_0 = \frac{E_0}{H_0} = \sqrt{\frac{\mu_0}{\varepsilon_0}} \qquad (30)$$

The impedance of free space, Z_0, is a physical constant with value:

$$Z_0 \approx 376.7\,\Omega \qquad (31)$$

Refractive Index

We define the *refractive index* n of a material, to be the ratio of the speed of electromagnetic waves in a vacuum, to the speed of electromagnetic waves in the material:

$$n = \frac{c}{v} = \frac{\sqrt{\mu\varepsilon}}{\sqrt{\mu_0\varepsilon_0}} = \sqrt{\mu_r\varepsilon_r} \qquad (32)$$

For most transparent materials, $\mu_r \approx 1$, in which case:

$$n \approx \sqrt{\varepsilon_r} \qquad (33)$$

Energy in Electromagnetic Fields: Poynting's Theorem

Electromagnetic waves carry energy.

The energy density in an electric field is given by:

$$U_E = \frac{1}{2}\varepsilon\vec{E}^2 \qquad (34)$$

The energy density in a magnetic field is given by:

$$U_H = \frac{1}{2}\mu\vec{H}^2 \qquad (35)$$

The energy flux (energy crossing unit area per unit time) is given by the Poynting vector:

$$\vec{S} = \vec{E} \times \vec{H} \qquad (36)$$

These results follow from Poynting's theorem...

Energy in Electromagnetic Fields: Poynting's Theorem

To derive Poynting's theorem, we start with Maxwell's equations. First, we use:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad (37)$$

Take the scalar product on both sides with the magnetic intensity $\vec{H}$:

$$\vec{H} \cdot \nabla \times \vec{E} = -\vec{H} \cdot \frac{\partial \vec{B}}{\partial t} \qquad (38)$$

Next, we use:

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \qquad (39)$$

Take the scalar product on both sides with the electric field $\vec{E}$:

$$\vec{E} \cdot \nabla \times \vec{H} = \vec{E} \cdot \vec{J} + \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} \qquad (40)$$

Now we take equation (38) minus equation (40):

$$\vec{H} \cdot \nabla \times \vec{E} - \vec{E} \cdot \nabla \times \vec{H} = -\vec{E} \cdot \vec{J} - \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} - \vec{H} \cdot \frac{\partial \vec{B}}{\partial t} \qquad (41)$$

which can be written as:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\varepsilon\vec{E}^2 + \frac{1}{2}\mu\vec{H}^2\right) = -\nabla \cdot \left(\vec{E} \times \vec{H}\right) - \vec{E} \cdot \vec{J} \qquad (42)$$

Equation (42) is Poynting's theorem. Using Gauss' theorem, it may be written in integral form:

$$\frac{\partial}{\partial t}\int_V \left(U_E + U_H\right) dV = -\int_A \vec{S} \cdot d\vec{A} - \int_V \vec{E} \cdot \vec{J}\, dV \qquad (43)$$

where the closed surface A bounds the volume V,

$$U_E = \frac{1}{2}\varepsilon\vec{E}^2 \qquad U_H = \frac{1}{2}\mu\vec{H}^2 \qquad (44)$$

and:

$$\vec{S} = \vec{E} \times \vec{H} \qquad (45)$$

Poynting's theorem in integral form is equation (43):

$$\frac{\partial}{\partial t}\int_V \left(U_E + U_H\right) dV = -\int_A \vec{S} \cdot d\vec{A} - \int_V \vec{E} \cdot \vec{J}\, dV \qquad (46)$$

We note that the last term on the right hand side represents the rate at which the electric field does work on electric charges within the bounded volume V. It is then natural to interpret the first term on the right hand side as the flow of energy in the electromagnetic field across the boundary of the volume V, and the left hand side as the rate of change of the total energy in the electromagnetic field.

With this interpretation, Poynting's theorem (in differential form) expresses the local conservation of energy.

Energy Density

The energy densities in electric and magnetic fields are:

$$U_E = \frac{1}{2}\varepsilon\vec{E}^2, \qquad U_H = \frac{1}{2}\mu\vec{H}^2 \qquad (47)$$

Since the time average of the square of a sine wave is:

$$\langle \sin^2 x \rangle = \frac{1}{2} \qquad (48)$$

it follows that in an electromagnetic wave (15), (23):

$$\langle U_E \rangle = \frac{1}{4}\varepsilon\vec{E}_0^2, \qquad \langle U_H \rangle = \frac{1}{4}\mu\vec{H}_0^2 = \frac{1}{4}\frac{\vec{B}_0^2}{\mu} \qquad (49)$$

So the ratio of the time-averaged energy densities in the electric and magnetic fields in the electromagnetic wave is:

$$\frac{\langle U_E \rangle}{\langle U_H \rangle} = \mu\varepsilon\frac{\vec{E}_0^2}{\vec{B}_0^2} = \mu\varepsilon v^2 = 1 \qquad (50)$$

Thus, the energy in an electromagnetic wave is shared equally between the electric and magnetic fields.

Energy Flux and Impedance

The Poynting vector $\vec{S}$ is defined by:

$$\vec{S} = \vec{E} \times \vec{H} \qquad (51)$$

The Poynting vector gives the instantaneous energy flow crossing unit area normal to the direction of flow, per unit time. Since $\vec{E}_0$ and $\vec{H}_0$ are perpendicular to each other and to $\vec{k}$ (the direction in which the wave is travelling):

$$\vec{S} = \frac{\vec{E}^2}{Z}\hat{k} = Z\vec{H}^2\hat{k} \qquad (52)$$

where $\hat{k}$ is a unit vector in the direction of $\vec{k}$, and Z is the impedance of the medium:

$$Z = \sqrt{\frac{\mu}{\varepsilon}} \qquad (53)$$

Note that the energy flux, like the energy density, depends on the *square* of the field strength.

Energy Flux

The electric field varies sinusoidally with position and with time; and the energy flux is proportional to the square of the field. Therefore, since:

$$\langle \sin^2 x \rangle = \frac{1}{2} \tag{54}$$

the average energy flux (over time or position) is:

$$\langle \vec{S} \rangle = \frac{\vec{E}_0^2}{2Z} \hat{k} \tag{55}$$

Note that this can be written in terms of the average energy density in the wave:

$$\langle \vec{S} \rangle = \langle U \rangle \vec{v} \tag{56}$$

where $\vec{v}$ is the velocity (vector) of the wave.

Complex Notation for Plane Waves

The solution to the wave equation for the electric field is (15):

$$\vec{E}(\vec{r}, t) = \vec{E}_0 \cos\left(\omega t - \vec{k} \cdot \vec{r} + \phi_0\right) \tag{57}$$

Recall that:

$$e^{j\theta} = \cos\theta + j\sin\theta \tag{58}$$

where $j = \sqrt{-1}$. This means that the solution (15) to the wave equation can be written as:

$$\vec{E}(\vec{r}, t) = \text{Re } \vec{E}_0 e^{j\phi_0} e^{j\left(\omega t - \vec{k} \cdot \vec{r}\right)} \tag{59}$$

where "Re" means "the real part of".

Complex Notation for Plane Waves

It turns out that the complex vector field:

$$\vec{E}(\vec{r}, t) = \vec{E}_0 e^{j\left(\omega t - \vec{k} \cdot \vec{r}\right)} \tag{60}$$

(where $\vec{E}_0$ is a complex constant vector) also solves the wave equation. This means that in any *linear* equations, we can use the complex field instead of the real field.

Of course, observable quantities (including the field itself) must be real, so we must remember to take the real part to find the actual physical quantity.

Note that the expressions for the energy density and energy flux (Poynting vector) are *second order* in the fields: we should use the real parts of the fields when applying these expressions.

Superposition

Maxwell's equations are linear. In consequence, if two fields $\vec{E}_1$ and $\vec{E}_2$ are solutions to Maxwell's equations, then the sum $\vec{E}_1 + \vec{E}_2$ is also a solution to Maxwell's equations. Adding two fields together in this way to get a third field is called *superposition*.

Applied to the solution (59) to the wave equation, superposition means that a more general solution can be written as:

$$\vec{E}(\vec{r}, t) = \sum_{\vec{k}} \vec{E}_{0\vec{k}} e^{j(\omega_{\vec{k}} t - \vec{k} \cdot \vec{r})} \tag{61}$$

where $\vec{E}_{\vec{k}}$ is a set of complex coefficients (one for each value of $\vec{k}$ in the summation), and the frequencies $\omega_{\vec{k}}$ satisfy:

$$\frac{\omega_{\vec{k}}^2}{\vec{k}^2} = \frac{1}{\mu\varepsilon} = v^2 \tag{62}$$

By adding together waves with a range of frequencies, phases and directions, we can form very general solutions.

Dispersion

We have seen that plane waves propagate through a simple dielectric with phase velocity v, given by:

$$v = \frac{c}{n} = \frac{1}{\sqrt{\mu\varepsilon}} \tag{63}$$

With our assumption that the permittivity and permeability of the dielectric are constants, the phase velocity is independent of the frequency of the wave.

However, in real dielectrics, we observe that the phase velocity does depend on the frequency of the wave. This effect is known as dispersion. To explain dispersion, we need to develop a more sophisticated model of the dielectric, taking into account the properties of the molecules that constitute the material.
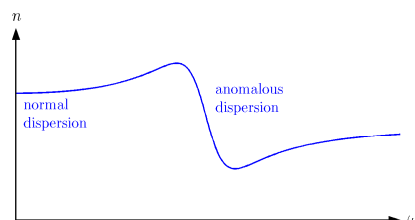
Dispersion

Dispersion is readily observed in familiar experiments. For example, the "splitting" of white light by a prism shows that the refractive index increases with increasing frequency (decreases with increasing wavelength). So blue light travels more slowly than red light through glass, and is refracted more strongly by a prism.
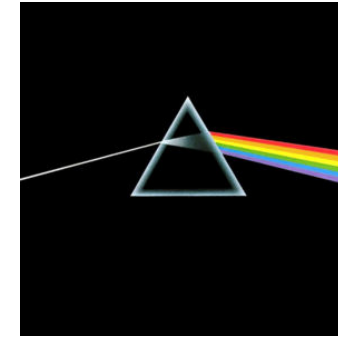
Real Dielectrics

In dielectrics, waves at low frequency propagate with constant velocity $c/\sqrt{\varepsilon_r}$, independent of frequency: our existing theory works well.

As the frequency increases, experimentally we observe that the refractive index rises to a maximum, then drops to a minimum, then rises again.

Real Dielectrics

The drop in refractive index generally happens over a narrow frequency range, and is called *anomalous* dispersion. Experimentally, we find that anomalous dispersion is associated with absorption of the wave energy by the material.
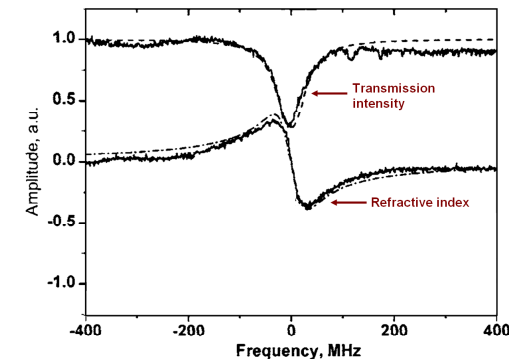
Transmission of infra-red radiation through Teflon vapour.

G. Mouret et al, Applied Physics Letters 88, 181105 (2006)

Real Dielectrics

Most materials have several regions of anomalous dipsersion in different parts of the spectrum. Many materials show absorption in the infra-red region; almost all show absorption in the ultra-violet region. Materials that abosrb in the visible region appear coloured.

By the time we reach X-ray frequencies ($f > 10^{16}$ Hz), anomalous dispersion has stopped, and the refractive index for almost all dielectric materials is almost exactly 1. This means that we cannot make lenses or prisms for X-rays. Absorption has also decreased, meaning that X-rays are a highly penetrating form of radiation.

In the γ-ray region ($f > 10^{19}$ Hz), all dieletric materials become transparent and non-refracting.

Simple Atomic Model of Dielectric Materials

So far, we have ignored the atomic structure of materials. We will now develop a simple atomic model of dielectric materials to explain the variation of the refractive index with frequency of the electromagnetic wave. We will find results in qualitative agreement with experiment.

The effect of an electric field $\vec{E}$ on a dielectric is to polarise the material: on the atomic level, there is a separation of positive charges (nuclei) and negative charges (electrons).
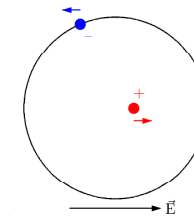
Polarisation

If a positive charge e and a negative charge $-e$ in an atom are separated by a distance Δx, then the polarisation p of the atom is;

$$p = e\Delta x \tag{64}$$

If there are N atoms per unit volume, then the total polarisation per unit volume P is:

$$P = Ne\Delta x \tag{65}$$

Since the separation Δx is proportional to the strength of the electric field, and in the same direction as the field, we can write:

$$\vec{P} = \varepsilon_0 \chi_e \vec{E} \tag{66}$$

where χ_e is the *electric susceptibility*.

If the electric field $\vec{E}$ oscillates (as in an electromagnetic wave), then the polarisation $\vec{P}$ will also oscillate.

Polarisation

At high frequencies, the motion of the electron in an atom cannot immediately follow the variation of the electric field (because of the inertia of the electron). This leads to a *phase difference* between $\vec{P}$ and $\vec{E}$. We can take account of this phase difference by allowing a complex susceptibility:

$$\vec{P} = \varepsilon_0(\chi_{e1} - j\chi_{e2})\vec{E} \tag{67}$$

But since the polarisation contributes to the displacement field $\vec{D}$:

$$\vec{D} = \varepsilon_0\vec{E} + \vec{P} = \varepsilon_0\varepsilon_r\vec{E} \tag{68}$$

it follows that for high frequency waves, we can represent the relative permittivity ε_r as a complex number:

$$\varepsilon_r = \varepsilon_{r1} - j\varepsilon_{r2} \tag{69}$$

The dispersion in a dielectric can be attributed to a complex relative permittivity.

Dispersion in Dielectric Gases

A dielectric gas is the simplest case of a dielectric material: since the material is relatively "dilute", we can neglect interactions between the atoms or molecules. We will consider only non-polar dielectrics, i.e. those without any permanent electric dipole moment.

We will develop a simple model, following Maxwell, Sellmeier and Lorentz. We regard the medium as an assembly of molecular oscillators. If the electrons are displaced, they experience a restoring force $-Kx$; so the equation of motion for an electron is:

$$m\ddot{x} = -Kx \qquad (70)$$

The equation of motion for the electron has the solution:

$$x = Ae^{j\omega_0 t} \qquad (71)$$

where the *natural frequency* ω_0 is given by:

$$\omega_0 = \sqrt{\frac{K}{m}} \qquad (72)$$

Permittivity in a Static Electric Field

Now we apply a static electric field $\vec{E}$, so that the displacement of the electron becomes:

$$x = \frac{eE}{K} \qquad (73)$$

This means that each atom has an induced electric dipole moment:

$$p = \frac{e^2 E}{K} \qquad (74)$$

and the polarisation of the material (with N atoms per unit volume) is given by:

$$P = \frac{Ne^2 E}{K} = \chi_e \varepsilon_0 E \qquad (75)$$

Therefore, we can write for the susceptibility χ_e:

$$\chi_e = \frac{Ne^2}{\varepsilon_0 K} = \frac{Ne^2}{\varepsilon_0 m \omega_0^2} \qquad (76)$$

Permittivity in a Static Electric Field

From equation (76) for the susceptibility, the relative permittivity for a dielectric gas can be written:

$$\varepsilon_r = 1 + \chi_e = 1 + \frac{e^2}{m\varepsilon_0} \frac{N}{\omega_0^2} \qquad (77)$$

In general, because of the different kinds of atoms and molecules that exist in most materials, there can be several different natural frequencies ω_i, so the susceptibility should be written as a sum, so the permittivity becomes:

$$\varepsilon_r = 1 + \frac{e^2}{m\varepsilon_0} \sum_i \frac{N_i}{\omega_i^2} \qquad (78)$$

where N_i is the number of oscillators per unit volume with natural frequency ω_i.

Permittivity in an Oscillating Electric Field

Now consider an oscillating electric field $\vec{E}$:

$$E = E_0 e^{j\omega t} \qquad (79)$$

This may come, for example, from a plane wave passing a fixed point $z = 0$. The electric field will result in a force driving oscillations of the electrons in the material. Note that in general, the natural frequency ω_0 of oscillations of the electron will be different from the frequency ω of the electric field.

The magnetic field in the electromagnetic wave will also contribute to the force on an electron in the material. In general, the total force on a particle of charge e is given by:

$$\vec{F} = e(\vec{E} + \vec{v} \times \vec{B}) \qquad (80)$$

However, for a plane wave, $B \approx E/c$, and for electrons in atoms, $v \ll c$. Thus, the magnetic contribution to the force will be much less than the electric contribution.

Now we assume a damping force, $m\Gamma\dot{x}$, proportional to the velocity of the electron. The full equation of motion is then:

$$m\ddot{x} + m\Gamma\dot{x} + Kx = eE_0 e^{j\omega t} \tag{81}$$

or:

$$\ddot{x} + \Gamma\dot{x} + \omega_0^2 x = \frac{e}{m}E_0 e^{j\omega t} \tag{82}$$

This is the equation of motion for a driven oscillator, which has solution:

$$x = \frac{e/m}{(\omega_0^2 - \omega^2) + j\omega\Gamma}E_0 e^{j\omega t} \tag{83}$$

The electron oscillates with the same frequency ω as the driving force (the electric field); but the amplitude depends on the frequency.

The constants Γ and ω_0 that characterise the medium must be determined by experiment.

We can generalise equation (83) to allow the case of a material containing oscillators with several different ω_i and Γ_i. If the number of oscillators per unit volume is N_i, then the polarisation is:

$$P = \sum_i \frac{N_i e^2/m}{(\omega_i^2 - \omega^2) + j\omega\Gamma_i}E_0 e^{j\omega t} = \chi_e\varepsilon_0 E_0 e^{j\omega t} \tag{84}$$

Since the relative permittivity is:

$$\varepsilon_r = 1 + \chi_e \tag{85}$$

we find:

$$\varepsilon_r = 1 + \frac{e^2}{m\varepsilon_0}\sum_i \frac{N_i}{(\omega_i^2 - \omega^2) + j\omega\Gamma_i} \tag{86}$$

At low frequencies ($\omega \ll \omega_i$), this expression for the relative permittivity reduces to the static field case (78). But near each resonant frequency, $\omega \approx \omega_i$, the magnitude of ε_r varies rapidly with frequency and becomes complex, leading to absorption and anomalous dispersion.

For a complex relative permittivity ε_r we write:

$$\varepsilon_r = \varepsilon_{r1} - j\varepsilon_{r2} \tag{87}$$
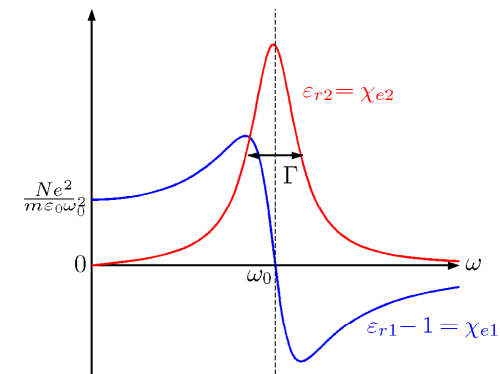
where, using equation (86), we have:

$$\varepsilon_{r1} = 1 + \chi_{e1} = 1 + \frac{e^2}{m\varepsilon_0}\sum_i \frac{N_i(\omega_i^2 - \omega^2)}{(\omega_i^2 - \omega^2)^2 + \omega^2\Gamma_i^2} \tag{88}$$

and:

$$\varepsilon_{r2} = \chi_{e2} = \frac{e^2}{m\varepsilon_0}\sum_i \frac{N_i\omega\Gamma_i}{(\omega_i^2 - \omega^2)^2 + \omega^2\Gamma_i^2} \tag{89}$$

Near one resonant frequency (ω_0) the real and imaginary parts of the susceptibility vary as shown.

The shape of the plot repeats near each resonant frequency ω_i.

The refractive index is given by:

$$n = \sqrt{\varepsilon_r} = \sqrt{1 + \chi_{e1} - j\chi_{e2}} = n_1 - jn_2 \qquad (90)$$

For gases, where the molecules are relatively sparse, the susceptibility is small, i.e. $\chi_{e1} \ll 1$ and $\chi_{e2} \ll 1$. We can then expand the square root in equation (90) as a Taylor series:
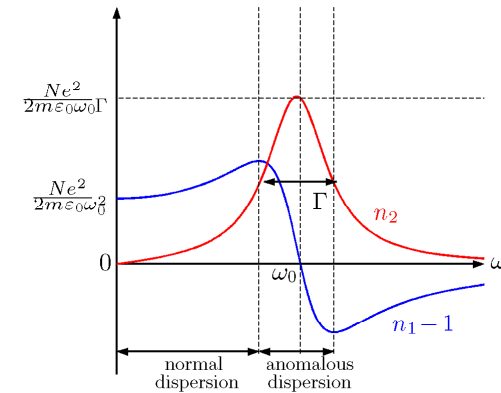
$$n \approx 1 + \frac{1}{2}\chi_{e1} - \frac{1}{2}j\chi_{e2} \qquad (91)$$

Then we find that:

$$n_1 = 1 + \frac{e^2}{2m\varepsilon_0}\sum_i \frac{N_i(\omega_i^2 - \omega^2)}{(\omega_i^2 - \omega^2)^2 + \omega^2\Gamma_i^2} \qquad (92)$$

$$n_2 = \frac{e^2}{2m\varepsilon_0}\sum_i \frac{N_i\omega\Gamma_i}{(\omega_i^2 - \omega^2)^2 + \omega^2\Gamma_i^2} \qquad (93)$$

We can plot the real and imaginary parts of the refractive index:

Note that the imaginary part n_2 peaks at a resonance, ω_0, and is small elsewhere. Write for a wave travelling through a dielectric:

$$E = E_0 e^{j(\omega t - kz)} = E_0 e^{j(\omega t - \frac{\omega}{c}nz)} = E_0 e^{j(\omega t - \frac{\omega}{c}(n_1 - jn_2)z)} \qquad (94)$$

Hence:

$$E = E_0 e^{-n_2\frac{\omega}{c}z} e^{j(\omega t - n_1\frac{\omega}{c}z)} \qquad (95)$$

A non-zero value for n_2 leads to attenuation of the wave in the dielectric - the energy of the wave is transferred to (i.e. is absorbed by) the material. Thus, resonant frequencies in the optical range result in coloured materials.

Now consider the behaviour of n_1. In the limit of low frequency,

$$\omega \to 0, \qquad n_1 \to 1 + \frac{Ne^2}{2m\varepsilon_0\omega_0^2} \qquad (96)$$

At resonance:

$$\omega = \omega_0, \qquad n_1 = 1 \qquad (97)$$

Just above the resonant frequency:

$$\omega > \omega_0, \qquad n_1 < 1 \qquad (98)$$

And in the limit of high frequency:

$$\omega \to \infty, \qquad n_1 \to 1 \qquad (99)$$

Refractive Index in an Oscillating Electric Field

$\omega \ll \omega_0$ normal dispersion $n_1 \uparrow$ as $\omega \uparrow$, $n_2 \approx 0$

$\omega \approx \omega_0$ anomalous dispersion $n_1 \downarrow$ as $\omega \uparrow$, $n_2 > 0$

$\omega \gg \omega_0$ $n_1 < 1$, $n_2 \approx 0$

Note that all substances have more than one resonant frequency. In the quantum theory of atoms, the resonant frequencies are associated with transitions of electrons between energy levels.

Summary of Part 2

You should be able to:

- Derive, from Maxwell's equations, the wave equations for the electric and magnetic fields in dielectrics.

- Derive the relationships (amplitude and direction) between the electric and magnetic fields in a plane wave in dielectric media.

- Derive expressions for the phase velocity and impedance for a plane wave in a dielectric.

- Explain that *dispersion* is the dependence of the propagation of a wave on its frequency.

- Explain the difference between *normal* and *anomalous* dispersion.

- Use a simple atomic model of a dielectric gas to derive an expression for the real and imaginary parts of the refractive index of the gas, and explain how dispersion is related to the motion of electrons in the atoms of the gas.

- Explain that when an electromagnetic wave moves through a conducting medium, the conductivity of the medium can be written as a complex number, with a dependence on the frequency of the wave.