PHYS370 – Advanced Electromagnetism

Part 1: Maxwell's Equations

Contents:

We will learn how a vast range of physical phenomena follow from Maxwell's equations...

Recommended texts:

- I.S. Grant and W.R. Phillips, "Electromagnetism" *Wiley, 2nd Edition, 1990*
- D. Fleisch, "A Student's Guide to Maxwell's Equations" *Cambridge, 2008*
- D. Fleisch, "A Student's Guide to Vectors and Tensors" *Cambridge, 2012*

Free to download:

- B. Thide, "Electromagnetic Field Theory" *http://www.plasma.uu.se/CED/Book/index.html*

Comprehensive texts for the more ambitious:

- J.D. Jackson, "Classical Electrodynamics" *Wiley, 3rd Edition, 1998*
- A. Garg, "Classical Electromagnetism in a Nutshell" *Princeton University Press, 2012*
- A. Zangwill, "Modern Electrodynamics" *Cambridge, 2013*

Notation

Scalar quantities are italicised, like this:

$$x, \quad E_x, \quad \mu_0$$

Vector quantities are written with an arrow:

$$\vec{E}, \quad \vec{B}$$

The components of a vector are written with a subscript, e.g.:

$$\vec{A} = (A_x, A_y, A_z)$$

The magnitude of a vector $\vec{A}$ is written in italics:

$$\left|\vec{A}\right| = A$$

A derivative with respect to time is indicated by a dot:

$$\dot{\vec{B}} = \frac{\partial \vec{B}}{\partial t}$$

Vector Calculus

In electromagnetism, we use vector calculus *all the time*. Make sure you are familiar with the notation, and the algebra!

The four basic operators of vector calculus are (in Cartesian coordinates):

grad (scalar) = vector	$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$
div (vector) = scalar	$\nabla \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$
curl (vector) = vector	$\nabla \times \vec{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}, \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}, \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$
laplacian (scalar) = scalar	$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$
laplacian (vector) = vector	$\nabla^2 \vec{A} = (\nabla^2 A_x, \nabla^2 A_y, \nabla^2 A_z)$

Useful Mathematical Theorems

The mathematical identities (for any vector field $\vec{F}$) are useful:

$$\nabla \times \nabla \times \vec{F} \equiv \nabla(\nabla \cdot \vec{F}) - \nabla^2 \vec{F} \qquad (1)$$

$$\nabla \cdot \nabla \times \vec{F} \equiv 0 \qquad (2)$$

The grad (∇), div ($\nabla\cdot$) and curl ($\nabla\times$) operators acting on sin functions have the following effect:

$$\nabla \sin(\vec{k} \cdot \vec{r}) = \vec{k} \cos(\vec{k} \cdot \vec{r}) \qquad (3)$$

$$\nabla \cdot \vec{A} \sin(\vec{k} \cdot \vec{r}) = \vec{k} \cdot \vec{A} \cos(\vec{k} \cdot \vec{r}) \qquad (4)$$

$$\nabla \times \vec{A} \sin(\vec{k} \cdot \vec{r}) = \vec{k} \times \vec{A} \cos(\vec{k} \cdot \vec{r}) \qquad (5)$$

where $\vec{r} = (x, y, z)$ is a position vector, and $\vec{k}$ and $\vec{A}$ are constant vectors.

In the above equations, we can interchange sin and cos, with a minus sign on the right hand side.

Useful Mathematical Theorems: Gauss and Stokes

Gauss' theorem for any vector field $\vec{A}$:

$$\int_V \nabla \cdot \vec{A}\, dV = \oint_S \vec{A} \cdot d\vec{S} \qquad (6)$$

where S is the closed surface bounding the volume V, and the surface area element $d\vec{S}$ is directed out of the volume V.

Stokes' theorem for any vector field $\vec{A}$:

$$\int_S \nabla \times \vec{A} \cdot d\vec{S} = \oint_C \vec{A} \cdot d\vec{l} \qquad (7)$$

where C is the closed line bounding the area S.

Units:

We use the SI (International System) of units, in which there are seven *base units*:

mass	kilograms	kg
length	meters	m
time	seconds	s
electric current	amperes	A
thermodynamic temperature	kelvin	K
amount of substance	mole	mol
luminous intensity	candela	cd

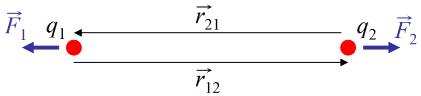
Some useful physical constants:

Speed of light in a vacuum	c	2.998×10^8 ms^{-1}
Impedance of free space	Z_0	376.7 Ω
Permittivity of free space	ε_0	8.854×10^{-12} Fm^{-1}
Permeability of free space	μ_0	$4\pi \times 10^{-7}$ Hm^{-1}
Charge on a positron	e	1.602×10^{-19} C

In 1865, Maxwell published a set of equations that describe completely the behaviour of electromagnetic fields. These equations are used in a huge range of applications, from the properties of materials, to properties of radiation (radio waves to gamma rays).

The theory of electromagnetism has been extensively tested and is hugely successful. It provides a model for a wide variety of *field theories*.

In general, a field theory describes interactions between different objects.

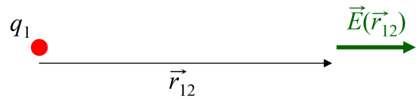


The electrostatic interaction between two point-like objects in a vacuum can be written:

$$\vec{F}_1 = \frac{1}{4\pi\varepsilon_0} q_1 q_2 \frac{\vec{r}_{21}}{|\vec{r}_{21}|^3} \tag{8}$$

$$\vec{F}_2 = \frac{1}{4\pi\varepsilon_0} q_1 q_2 \frac{\vec{r}_{12}}{|\vec{r}_{12}|^3} \tag{9}$$

where $\vec{F}_1$ and $\vec{F}_2$ are the forces on the objects which carry charges q_1 and q_2 respectively; $\vec{r}_{21}$ and $\vec{r}_{12}$ are vectors giving the relative positions of the objects; and ε_0 is a fundamental physical constant that expresses the strength of the interaction per unit charge.



The interaction can be written in terms of an *electric field* that is created by a charged object. For example, we can define the electric field at location $\vec{r}_{12}$ from an isolated point charge q_1 to be:

$$\vec{E}(\vec{r}_{12}) = \frac{1}{4\pi\varepsilon_0} q_1 \frac{\vec{r}_{12}}{|\vec{r}_{12}|^3} \tag{10}$$

In terms of this electric field, the force F_2 on a second point charge, q_2, in the field $\vec{E}(\vec{r}_{12})$ is given by:

$$\vec{F}_2 = q_2 \vec{E}(\vec{r}_{12}) \tag{11}$$

Equation (10) relates the field to its source. Equation (11) tells us the effect of the field on an object in the field. These are the essential ingredients of a field theory.

The full set of equations that relate electric fields ($\vec{E}$ and $\vec{D}$) and magnetic fields ($\vec{B}$ and $\vec{H}$) to their sources (charge density ρ and current density $\vec{J}$) can be written, in differential form, as follows:

$$\nabla \cdot \vec{D} = \rho \qquad \nabla \cdot \vec{B} = 0$$
$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \tag{12}$$

To find explicit expressions for the fields at a given place and time, we have to solve these differential equations with the boundary conditions imposed by the sources of the fields.

The effects of the fields on a point-like object moving in an electromagnetic field can be written (the Lorentz force equation):

$$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right) \tag{13}$$

where $\vec{v}$ is the velocity of the object.

Sometimes, the field equations take a simpler form when they are expressed in terms of *potentials* rather than directly in terms of the fields.

A potential is a mathematical scalar or vector function (of space and time) whose derivative gives the field.

For example, the magnetic field can be expressed in terms of the magnetic vector potential $\vec{A}$:

$$\vec{B} = \nabla \times \vec{A} \tag{14}$$

and the electric field can be expressed in terms of the magnetic vector and electric scalar potential ϕ:

$$\vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t} \tag{15}$$

Note that fields are associated with forces, whereas potentials are associated with energy.

In electrostatics:

- the field is the force per unit charge;
- the potential is the potential energy per unit charge;
- the field is the gradient of the potential (since the force is the gradient of the potential energy).

In terms of the potentials, the electromagnetic field equations become second-order differential equations. But they take a nice, symmetric form:

$$\nabla^2 \vec{A} - \mu\varepsilon \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu \vec{J} \tag{16}$$

$$\nabla^2 \phi - \mu\varepsilon \frac{\partial^2 \phi}{\partial t^2} = -\frac{\rho}{\varepsilon} \tag{17}$$

where μ and ε are quantities that characterise the strength of the electric and magnetic interactions in the material in which the fields exist.

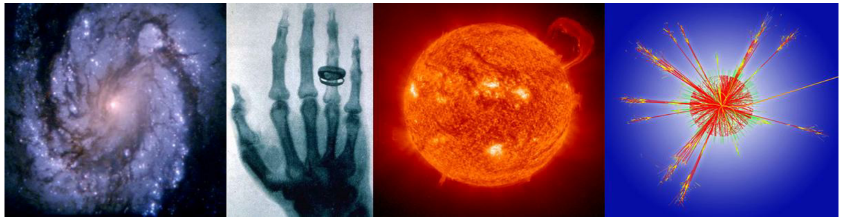
Note that for a static, point-like charge q, Equation (17) has the familiar solution:

$$\phi = \frac{1}{4\pi\varepsilon} \frac{q}{r} \tag{18}$$

where r is the distance from the charge.

How many types of field are there?

We believe that there are only four fundamental types of field (sometimes called simply *forces*) in nature. The familiar ones from everyday experience are gravity (the first field to be described mathematically) and electromagnetism.



The other fundamental forces are the weak nuclear and strong nuclear forces. They differ from gravity and electromagnetism in a number of respects: for example, they act only over very short distances, whereas gravity and electromagnetism are both capable of very long range interactions.

Long-Range and Short-Range Fields

The mathematical description of a short-range interaction is very similar to that of a long-range interaction. Take Equation (17) above, and simply add another term:

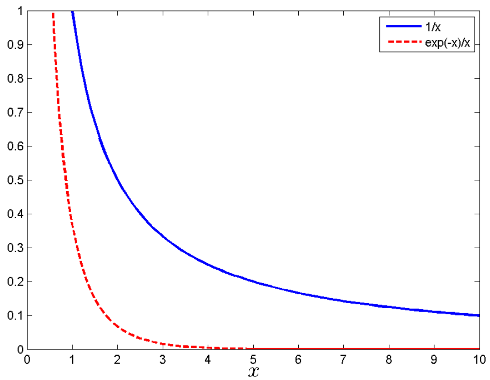
$$\nabla^2\phi - \mu\varepsilon\frac{\partial^2\phi}{\partial t^2} - \frac{\phi}{l^2} = -\frac{\rho}{\varepsilon} \tag{19}$$

where l is a constant. For a static, point-like source, Equation (19) has the solution:

$$\phi = \frac{1}{4\pi\varepsilon}q\frac{e^{-\frac{r}{l}}}{r} \tag{20}$$

For small l, the potential falls off much more quickly than $1/r$...

Long-Range and Short-Range Fields



The constant l characterises the "range" of the force. For gravity and electromagnetism we believe (from experiments) that $1/l = 0$.

Classical and Quantum Fields

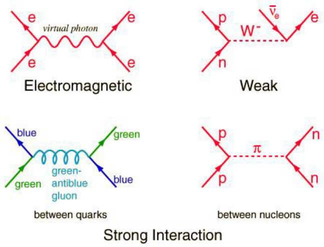
All the equations on the previous slides are classical equations: they take no account of quantum effects, and make no reference to Planck's constant h.

However, we believe that when particles interact, they do so by exchanging discrete amounts of energy, called "quanta". When we develop a classical field theory to include quantum effects, we construct a "quantum field theory".

In general, quantum field theories are much more complicated than classical field theories. But some of the consequences of quantisation can be understood using simplified models.

Classical and Quantum Fields

In quantum field theory, an interaction between two particles is understood in terms of an exchange of a third particle: the type of particle exchanged is determined by the type of interaction.



Using the physical constants $\hbar$ and c, we can define a mass m associated with the length scale l of the interaction:

$$m = \frac{\hbar}{c}\frac{1}{l} \tag{21}$$

In quantum field theory, the mass m is identified with the mass of the exchange particle. Note that as $l \to \infty$, $m \to 0$.

Classical and Quantum Fields

This simple model explains why gravity and electromagnetism are long-range forces (since the graviton and photon have zero mass) while the weak interaction has only a very short-range (since the W and Z bosons have mass of around 90 GeV/c^2).

But it doesn't explain why the strong interaction is also short-range, despite the fact that the gluon has zero mass.

When a field theory is quantised, many new features can appear that are not expected from relatively simple classical field theories. The quantum field theory of the strong interaction is especially complicated.

The electromagnetic field was the first to be understood as a quantum theory. We are still searching for a quantum theory of gravity.

The Electromagnetic Field

The classical electromagnetic field is a good case study for other field theories, including quantum field theories.

Our main goal in this course will be to understand a variety of electromagnetic phenomena in terms of solutions to the field equations (Maxwell's equations). The phenomena we shall study will include:

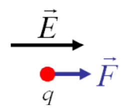
- Electromagnetic waves in various media
- Electromagnetic waves at boundaries between different media
- Propagation of electromagnetic waves in waveguides
- Sources of electromagnetic radiation

We shall begin by reviewing the various quantities associated with electromagnetic fields, and the relationships between them...

The Electric Field

The electric field $\vec{E}$ at a particular point in space is the force per unit static electric charge located at that point:

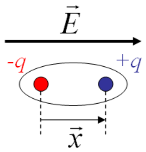
$$\vec{E} = \frac{\vec{F}}{q} \tag{22}$$



In free space, the electric field is very simple. Things get more complicated when we need to describe electric fields within materials...

Electric Polarisation

When a dielectric (non-conducting material) is placed in an electric field, the molecules within the material each acquire an electric dipole moment. The dipole moment measures the displacement of the electric charges within the molecule, in response to the external electric field.



The electric dipole moment $\vec{p}$ is defined as the magnitude of the charge multiplied by the separation:

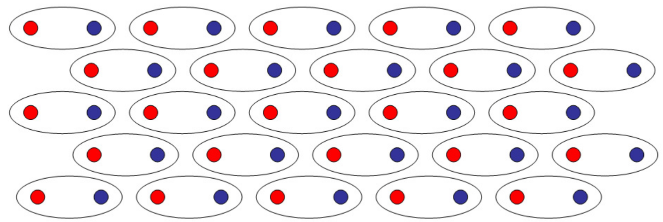
$$\vec{p} = q\vec{x} \tag{23}$$

Note that the dipole moment is a vector quantity.

Electric Polarisation

The electric polarisation $\vec{P}$ of a dielectric is defined as the dipole moment $\vec{p}$ per unit volume. Thus, if there are N molecules per unit volume, each with electric dipole moment $\vec{p}$, the polarisation of the dielectric will be:

$$\vec{P} = N\vec{p} = Nq\vec{x} \tag{24}$$



Electric Polarisation and Susceptibility

In general, the response of a material to an external electric field is very complicated. Even in relatively simple materials, we need a good understanding of the quantum mechanics of the atoms and molecules within the material if we want to calculate the polarisation from first principles.

However, for many materials we can make the approximation that the polarisation is proportional to the external electric field. The constant of proportionality is the product of the permittivity of free space, ε_0, and the electric susceptibility, χ_e:

$$\vec{P} = \chi_e \varepsilon_0 \vec{E} \tag{25}$$

Note that the susceptibility χ_e is a dimensionless quantity. Equation (25) is a good approximation for materials that are homogeneous, isotropic and linear. There are various ways in which the susceptibility of a given material can be measured.

Electric Displacement

The electric displacement $\vec{D}$ is a measure of the electric field within a material, taking into account the polarisation $\vec{P}$:

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P} \tag{26}$$

Note that the polarisation generated by an external electric field will tend to *reduce* the strength of the field.

The electric susceptibility describes how the polarisation depends on the external electric field. Equation (25) tells us that for homogeneous, isotropic, linear dielectrics:

$$\vec{P} = \chi_e \varepsilon_0 \vec{E}$$

Combining equations (26) and (25), we find:

$$\begin{aligned} \vec{D} &= \varepsilon_0 \vec{E} + \chi_e \varepsilon_0 \vec{E} && (27) \\ &= (1 + \chi_e)\varepsilon_0 \vec{E} && (28) \end{aligned}$$

The electric displacement is the measure of the electric field in a material, taking into account the response of the material to the field. The magnitude of the response of a material to an external electric field can be measured by the electric susceptibility χ_e, or by the relative permittivity ε_r.

The relative permittivity ε_r is defined by:

$$\vec{D} = \varepsilon_r \varepsilon_0 \vec{E} \tag{29}$$

Combining equations (28) and (29), we find:

$$\varepsilon_r = 1 + \chi_e \tag{30}$$

Note that, like the susceptibility, the relative permittivity is dimensionless.

A material that has no response to an external electric field (like a vacuum) will have susceptibility zero, and relative permittivity equal to 1.

A material that has a strong response to an external electric field (by acquiring a large polarisation) will have a susceptibiliy much larger than zero, and a relative permittivity much larger than 1.

We sometimes use the permittivity ε, instead of the relative permittivity ε_r. The permittivity is defined by:

$$\varepsilon = \varepsilon_r \varepsilon_0 \tag{31}$$

electric field	$\vec{E}$	newtons/coulomb (NC^{-1})
electric displacement	$\vec{D}$	coulombs/meter^{-2} (Cm^{-2})
permittivity	ε	farads/meter (Fm^{-1})

The electric displacement $\vec{D}$ and the electric field $\vec{E}$ are related by the permittivity ε:

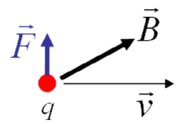
$$\vec{D} = \varepsilon \vec{E} \tag{32}$$

The permittivity is a property of materials. The vacuum also has a permittivity, with value ε_0:

$$\varepsilon_0 = 8.854 \times 10^{-12}\ \mathrm{Fm}^{-1} \tag{33}$$

The ratio of the permittivity of a material to the permittivity of the vacuum is known as the relative permittivity. The relative permittivity measures the electric polarisation induced in a material by an external electric field.

When a charged particle moves through a magnetic field, it experiences a force proportional to the size of the field, the speed of the particle, and the sine of the angle between the field and the velocity. The direction of the force is perpendicular to both the field and the velocity:

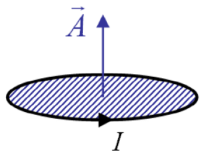


The magnetic force on the particle can be written:

$$\vec{F} = q\vec{v} \times \vec{B} \tag{34}$$

Just as we had to account for the response of materials to external electric fields, we have to account for the response of materials to external magnetic fields.

The magnetic field around an individual atom or molecule can modelled as a current I flowing in a loop enclosing an area $\vec{A}$:



Note that $\vec{A}$ is a vector, with magnitude equal to the area enclosed by the loop, and direction perpendicular to the current loop.

The magnetic dipole moment $\vec{m}$ of the current loop is defined by:

$$\vec{m} = I\vec{A} \tag{35}$$

The magnetisation of a material is defined as the magnetic dipole moment per unit volume. Thus, if there are N magnetic dipoles per unit volume with average dipole moment $\vec{m}$, the magnetisation $\vec{M}$ is given by:

$$\vec{M} = N\vec{m} \tag{36}$$

In a magnetic material, the magnetic moments of atoms and molecules within the material can change in response to an applied external magnetic field. The response of the material to an external magnetic field $\vec{B}$ is measured by the magnetic intensity $\vec{H}$:

$$\vec{B} = \mu_0 \left(\vec{H} + \vec{M} \right) \tag{37}$$

where μ_0 is a fundamental physical constant, the permeability of free space.

In certain kinds of materials (diamagnets and paramagnets), the magnetisation is approximately proportional to the magnetic intensity:

$$\vec{M} = \chi_m \vec{H} \tag{38}$$

Note that this relationship does not hold for ferromagnets, which are more complicated. In particular, ferromagnets display hysteresis, in which the magnetisation depends not only on the magnetic intensity present at a given time, but on the magnetic intensity that was present in the material at earlier times. This means that there cannot be a simple one-to-one relationship between magnetic intensity and magnetisation in ferromagnetic materials.

In cases where equation (38) holds, we can write the relationship (37) between the magnetic field and the magnetic intensity as:

$$\vec{B} = \mu_0 \left(\vec{H} + \chi_m \vec{H} \right) \tag{39}$$

$$= (1 + \chi_m)\, \mu_0 \vec{H} \tag{40}$$

We define the relative permeability μ_r as:

$$\mu_r = 1 + \chi_m \tag{41}$$

so that for diamagnetic and paramagnetic materials:

$$\vec{B} = \mu_r \mu_0 \vec{H} \tag{42}$$

We can also define the magnetic permeability μ as:

$$\mu = \mu_r \mu_0 \tag{43}$$

Summary: The Magnetic Field

magnetic intensity	$\vec{H}$	amperes/meter (Am^{-1})
magnetic field	$\vec{B}$	tesla (T)
permeability	μ	henrys/meter (Hm^{-1})

The magnetic intensity $\vec{H}$ and the magnetic field $\vec{B}$ in a diamagnetic or paramagnetic material are related by the permeability μ:

$$\vec{B} = \mu \vec{H} \tag{44}$$

The permeability is a property of the material. The vacuum also has a permeability, with value μ_0:

$$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{Hm}^{-1} \tag{45}$$

The ratio of the permeability of a material to the permeability of free space is the relative permeability of the material. The relative permeability measures the response of the material to an external magnetic field.

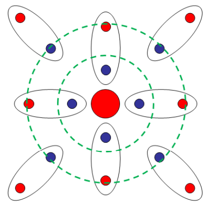
Permittivity and Permeability

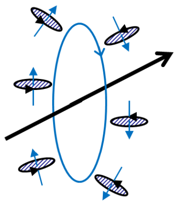
In static cases, sources of the electromagnetic field determine the electric displacement $\vec{D}$ and the magnetic intensity $\vec{H}$:

$$\nabla \cdot \vec{D} = \rho \qquad \nabla \times \vec{H} = \vec{J} \tag{46}$$

The permeability and permittivity describe the magnetisation and polarisation of a material in response to external fields. In most materials, the electric field $\vec{E}$ is *reduced* by the induced electric dipole moment; and the magnetic field $\vec{B}$ is *enhanced* by the induced magnetic moment in the material:

$$\vec{E} = \frac{\vec{D}}{\varepsilon_r \varepsilon_0} \qquad \vec{B} = \mu_r \mu_0 \vec{H} \tag{47}$$





Permittivity and Permeability of the Vacuum

As we shall see later, the speed of light in a vacuum is given by:

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} \tag{48}$$

It turns out that we can choose one of the constants μ_0 and ε_0 for our own convenience; the other is then fixed by the speed of light in a vacuum, from equation (48). Different systems of units make different choices for either μ_0 or ε_0. In SI units, the permeability of free space μ_0 is *defined* to be:

$$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{Hm}^{-1} \tag{49}$$

We then find, using $c = 2.998 \times 10^8\ \mathrm{ms}^{-1}$, that:

$$\varepsilon_0 = 8.854 \times 10^{-12}\ \mathrm{Fm}^{-1} \tag{50}$$

Charge, Charge Density, Current Density and Conductivity

Electric charge is represented by the symbol q, and is measured in coulombs (C). The electric charge density (charge per unit volume) is represented by the symbol ρ, and is measured in $\mathrm{C/m}^3$.

In an electrical conductor, an electric field $\vec{E}$ will cause a flow of electric charge. The flow of charge is given by the current density $\vec{J}$ (which has units of amperes/meter^2, $\mathrm{A/m}^2$).

In an ohmic conductor with conductivity σ, the current density is given by:

$$\vec{J} = \sigma \vec{E} \tag{51}$$

This is equivalent to Ohm's law, $I = V/R$.

Maxwell's equations determine the electric and magnetic fields in the presence of sources (charge and current densities), and in materials of given properties.

$$\nabla \cdot \vec{D} = \rho \quad (52)$$

$$\nabla \cdot \vec{B} = 0 \quad (53)$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad (54)$$

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \quad (55)$$

The physical significance of Maxwell's equations is most easily understood by converting them from differential equations into integral equations...

Differential	Integral
$\nabla \cdot \vec{D} = \rho$	$\oint_S \vec{D} \cdot d\vec{S} = \int_V \rho \, dV$
$\nabla \cdot \vec{B} = 0$	$\oint_S \vec{B} \cdot d\vec{S} = 0$
$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$	$\oint_C \vec{E} \cdot d\vec{l} = -\frac{\partial}{\partial t} \int_S \vec{B} \cdot d\vec{S}$
$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$	$\oint_C \vec{H} \cdot d\vec{l} = \int_S \vec{J} \cdot d\vec{S} + \frac{\partial}{\partial t} \int_S \vec{D} \cdot d\vec{S}$

We can show that the differential forms of Maxwell's equations are equivalent to the integral forms using Gauss' theorem and Stokes' theorem...

Gauss' theorem for any vector field $\vec{A}$:

$$\int_V \nabla \cdot \vec{A} \, dV = \oint_S \vec{A} \cdot d\vec{S} \quad (56)$$

where S is the closed surface bounding the volume V, and the surface area element $d\vec{S}$ is directed out of the volume V.

Stokes' theorem for any vector field $\vec{A}$:

$$\int_S \nabla \times \vec{A} \cdot d\vec{S} = \oint_C \vec{A} \cdot d\vec{l} \quad (57)$$

where C is the closed line bounding the area S.

Gauss' theorem and Stokes' theorem can be used to transform between the differential and integral forms of Maxwell's equations.

For example, consider the first of Maxwell's equations:

$$\nabla \cdot \vec{D} = \rho \quad (58)$$

Apply Gauss' theorem:

$$\int_V \nabla \cdot \vec{D} \, dV = \oint \vec{D} \cdot d\vec{S} \quad (59)$$

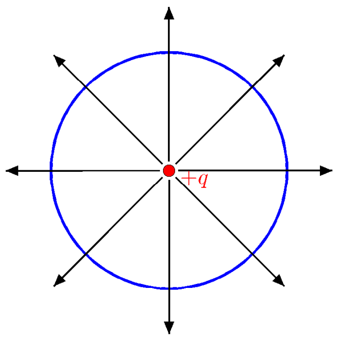
to get the integral form:

$$\oint \vec{D} \cdot d\vec{S} = \int_V \rho \, dV \quad (60)$$

This tells us that the flux of $\vec{D}$ through a closed surface equals the enclosed charge.

Maxwell's Equations: Physical Interpretation (1)

As an example, consider the field around a point charge, q. The field is spherically symmetric, and at a distance r from a point charge, passes through a sphere of surface area $4\pi r^2$.



Maxwell's Equations: Physical Interpretation (1)

Since the system is spherically symmetric, the integral of the electric displacement over the surface of a sphere centered on the point charge is simply equal to the magnitude of the electric diplacement vector multiplied by the surface area of the sphere:

$$\oint_S \vec{D} \cdot d\vec{S} = 4\pi r^2 D \qquad (61)$$

The integral of the charge density over the volume inside the sphere is simply equal to the charge:

$$\int_V \rho \, dV = q \qquad (62)$$

Thus, from equation (60):

$$\oint \vec{D} \cdot d\vec{S} = \int_V \rho \, dV$$

we get:

$$4\pi r^2 D = q \qquad (63)$$

and finally, from $\vec{D} = \varepsilon \vec{E}$, we get Coulomb's law:

$$E = \frac{q}{4\pi \varepsilon r^2} \qquad (64)$$

Maxwell's Equations: Physical Interpretation (2)

Consider the second of Maxwell's equations:

$$\nabla \cdot \vec{B} = 0 \qquad (65)$$

Apply Gauss' theorem:

$$\int_V \nabla \cdot \vec{B} \, dV = \oint \vec{B} \cdot d\vec{S} \qquad (66)$$

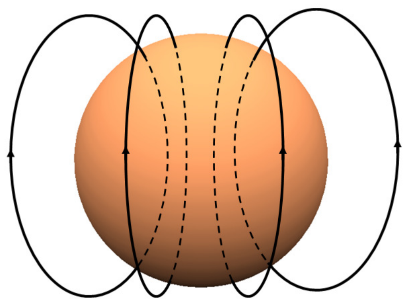
to get the integral form:

$$\oint \vec{B} \cdot d\vec{S} = 0 \qquad (67)$$

This tells us that the flux of $\vec{B}$ through a closed surface equals zero. In other words, as much magnetic field "flows into" a closed surface as "flows out" of the surface.

Maxwell's Equations: Physical Interpretation (2)

The flux of $\vec{B}$ through a closed surface equals zero. As a consequence, there can be no sources or sinks of lines of magnetic flux: the lines must be continuous, and have no beginning or end.



Equation (65) is a statement that there are no magnetic monopoles.

Consider the third of Maxwell's equations:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{68}$$

Apply Stokes' theorem:

$$\int_S \nabla \times \vec{E} \cdot d\vec{S} = \oint_C \vec{E} \cdot d\vec{l} \tag{69}$$

to get the integral form:

$$\oint_C \vec{E} \cdot d\vec{l} = -\frac{\partial}{\partial t} \int_S \vec{B} \cdot d\vec{S} = -\frac{\partial \Phi}{\partial t} \tag{70}$$

This tells us that the circulation of $\vec{E}$ around any closed curve is equal to the rate of change of magnetic flux through any surface spanning the curve.

Electromotive force (emf) is defined by:

$$\text{emf} = \oint_C \vec{E} \cdot d\vec{l} \tag{71}$$

We have seen that applying Stokes' theorem (69) to Maxwell's equation (68):

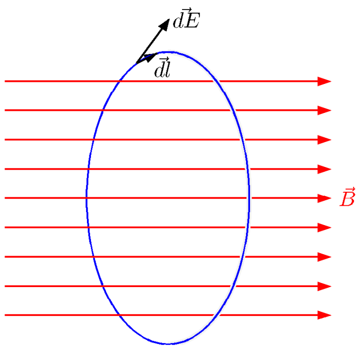
$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

gives:

$$\oint_C \vec{E} \cdot d\vec{l} = -\frac{\partial}{\partial t} \int_S \vec{B} \cdot d\vec{S} = -\frac{\partial \Phi}{\partial t} \tag{72}$$

So Maxwell's third equation is just a statement of Faraday's (and Lenz's) law:

$$\text{emf} = \oint_C \vec{E} \cdot d\vec{l} = -\frac{\partial \Phi}{\partial t} \tag{73}$$



Consider the fourth of Maxwell's equations:

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \tag{74}$$

Apply Stokes' theorem:

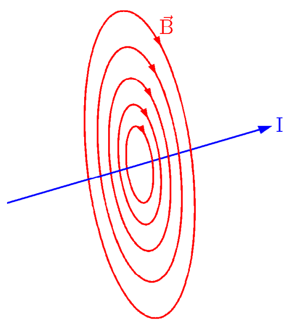
$$\int_S \nabla \times \vec{H} \cdot d\vec{S} = \oint_C \vec{H} \cdot d\vec{l} \tag{75}$$

to get the integral form:

$$\oint_C \vec{H} \cdot d\vec{l} = \int_S \vec{J} \cdot d\vec{S} + \frac{\partial}{\partial t} \int_S \vec{D} \cdot d\vec{S} \tag{76}$$

This tells us that the circulation of $\vec{H}$ around a closed curve is equal to the flux of current density through any surface spanning that curve (Ampere's law) plus the rate of change of electric displacement through any surface spanning the curve (Maxwell's extension to Ampere's law).

Consider the magnetic field around a long, straight wire carrying an electric current I. The magnetic field lines form closed loops perpendicular to, and centered on, the wire.



Consider a disc of radius r perpendicular to, and centered on, the wire. We integrate Maxwell's equation:

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \tag{77}$$

across the surface of the disc (noting that $\dot{\vec{D}} = 0$):

$$\int_S \nabla \times \vec{H} \cdot d\vec{S} = \int_S \vec{J} \cdot d\vec{S} = I \tag{78}$$

Applying Stokes' theorem, we get:

$$\oint_C \vec{H} \cdot d\vec{l} = 2\pi r H = I \tag{79}$$

where H is the magnitude of the magnetic intensity at perpendicular distance r from the wire:

$$H = \frac{I}{2\pi r} \tag{80}$$

Maxwell's extension to Ampere's law introduces the displacement current:

$$\text{displacement current} = \frac{\partial}{\partial t} \int_S \vec{D} \cdot d\vec{S} \tag{81}$$

The presence of the displacement current in Maxwell's equations tells us that a changing electric field gives rise to a magnetic field.

But there is another, very important consequence of this term: it tells us that electric charge is conserved...

Maxwell's fourth equation is:

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \tag{82}$$

Using the vector identity (2) we find:

$$\nabla \cdot \nabla \times \vec{H} = \nabla \cdot \dot{\vec{D}} + \nabla \cdot \vec{J} = 0 \tag{83}$$

Maxwell's first equation is:

$$\nabla \cdot \vec{D} = \rho \tag{84}$$

which (differentiating with respect to time, t) gives:

$$\nabla \cdot \dot{\vec{D}} = \dot{\rho} = \frac{\partial \rho}{\partial t} \tag{85}$$

Combining equations (83) and (85) gives:

$$\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t} \tag{86}$$

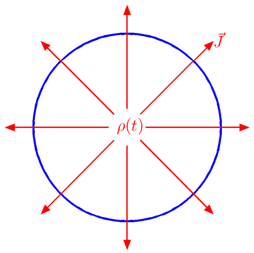
This equation is called the continuity equation for electric charge. It tells us that there is local conservation of electric charge.

The continuity equation (86) follows directly from Maxwell's equations, and tells us that electric charge is conserved locally. In differential form, the continuity equation is:

$$\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t} \tag{87}$$

Using Gauss' theorem, we can convert to integral form, to make the physical interpretation clearer:

$$\oint \vec{J} \cdot d\vec{S} = -\frac{\partial}{\partial t}\int_V \rho \, dV \tag{88}$$



You should be able to:

- Explain the features of a field theory.
- Explain the quantities used to describe electromagnetic fields in free space and in materials (including: electric field; electric displacement; magnetic field; magnetic intensity; electric permittivity; magnetic permeability) and give the relationships between them.
- Write down the field equations for classical electromagnetism (Maxwell's equations), and the Lorentz force equation.
- Write down Gauss' theorem and Stokes' theorem, and use these theorems to convert Maxwell's equations from differential to integral form.
- Derive, from Maxwell's equations, expressions for: the electric field around a point charge; the magnetic field around a straight wire; the emf in a wire loop in a time-dependent magnetic field.
- Derive, from Maxwell's equations, the continuity equation expressing the local conservation of electric charge.