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Damping Rings

Lecture 3

(Some) Collective Effects

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The story so far...

Damping rings are needed in a linear collider to reduce the emittances of the beams produced by the particle sources.

The emittances of a bunch are conserved under symplectic transport, but in a storage ring, the emittances are damped by the effects of synchrotron radiation.

The damping times in a storage ring depend on the beam energy and the energy loss per turn. Wigglers can be used to enhance the synchrotron radiation energy loss, and hence reduce the damping times.

In the horizontal and longitudinal planes, the equilibrium emittances are determined by the balance between radiation damping and quantum excitation.

The equilibrium horizontal emittance and energy spread are determined by the beam energy, the lattice design, and the wiggler parameters. In linear collider damping rings, the damping times and equilibrium emittances are dominated by the wigglers.

Contents

Collective Effects

As particles move along an accelerator beam line, they interact with each other and with their environment (magnets, RF cavities, vacuum chamber…)

So far, we have considered only the effects of the fields of magnets and RF cavities that we use to control the beam and achieve specified bunch dimensions.

We have completely ignored interactions between particles in the beam, and between particles in the beam and the vacuum chamber.

Quantities we have calculated so far (damping times, equilibrium emittances) are independent of the bunch charge.

Interactions between particles within the beam and between particles and the vacuum chamber threaten the quality and stability that are essential to effective operation of a linear collider.

Generally, the impact of these interactions depend on the bunch charge or beam current, and are hence known as “current-dependent” or “collective” effects.

In a storage ring, the collective effects are many and varied. We will look at two examples in detail:

resistive wall instability (a multi-bunch effect);

microwave instability (a single bunch effect).

Lecture 3: (Some) Collective Effects

The objectives of this lecture are:

- to define wake functions and impedances (two important ways to characterise the wake fields in an accelerator);
- to discuss the impact of long-range wake fields on the beam dynamics, and derive an expression for the growth rates of the resistive-wall instability;
- to show how bunch-by-bunch feedback systems can be used to control coupled-bunch instabilities;
- to show how short-range wake fields can lead to distortion of the longitudinal distribution ("potential well distortion");
- to show how short-range wake fields can drive single-bunch instabilities, such as the "microwave instability", and to derive an expression for the microwave instability threshold in a storage ring.

Note: we work in mks units. Wake field and impedance calculations are often done in cgs units. To convert the formulae presented here to cgs units, simply set:

$$\frac{Z_0 c}{4\pi} = 1$$

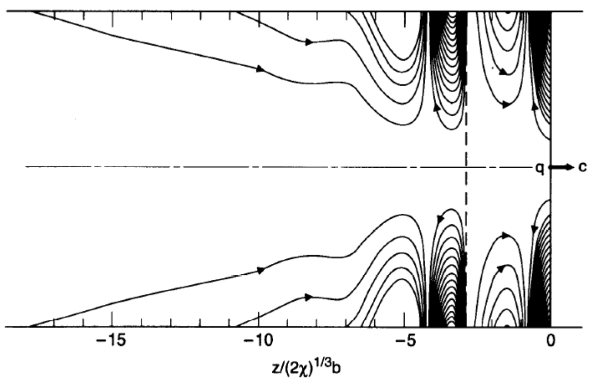
Wake fields

The electromagnetic fields around a bunch of charged particles must satisfy Maxwell's equations.

The presence of a vacuum chamber imposes boundary conditions that modify the fields.

Fields generated by the head of a bunch can act back on particles at the tail, modifying their dynamics and (potentially) driving instabilities.

The electromagnetic fields generated by a particle or a bunch of particles moving through a vacuum chamber are usually described as *wake fields*.



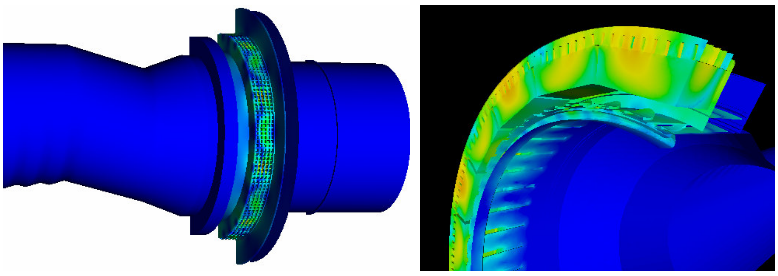
Wake fields following a point charge in a cylindrical beam pipe with resistive walls. (K. Bane)

Wake fields

In simple cases (e.g. resistive wall in a cylindrical vacuum chamber) the fields around a bunch of charge particles can be calculated analytically. However, wake fields are generally calculated numerically, using electromagnetic modelling codes.

For a given accelerator component, this involves defining the geometry and electromagnetic properties of the component, then calculating the electromagnetic fields on a mesh as a charge distribution moves through the component.

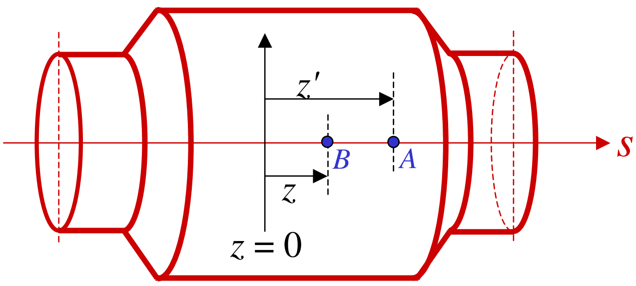
In this lecture, we shall not discuss the codes or the process used to calculate the wake fields, but focus on the dynamical effects of the wake fields.



Calculation of trapped modes in PEP II bellows. Higher-order mode heating is a significant problem for PEP II, and a potential problem for the ILC damping rings.
(Cho Ng, SLAC).

Wake fields and wake functions

The goal of calculating the wake fields is generally to derive a *wake function*. The wake function gives the effect of a leading particle on a following particle, as a function of the longitudinal distance between the two particles.



For example, the change in energy of particle B from the wake field of particle A in the figure, when the particles move through a given accelerator component, can be written:

$$\Delta\delta_B = -\frac{r_e}{\gamma} N_A W_{\parallel}(z - z')$$

where $W_{\parallel}$ is the wake function of the component, eN_A is the charge of particle A, γ is the relativistic factor, r_e is the classical electron radius.

Long-range and short-range wake fields

Depending on the source of the wake field, the wake function can fall off rapidly with distance, on a distance scale comparable to the length of a single bunch. Such "short-range" wake fields are important for single-bunch instabilities, and will be considered later.

In other cases, the wake function extends over the distance from one bunch to the next. Such "long-range" wake fields can drive coupled-bunch instabilities, and these effects are the subject of this lecture.

For long-range wake fields, we can often treat each bunch of particles as a single "macroparticle". This requires the assumptions that:

- all particles in a given bunch see the same wake field, and respond to it in the same way;
- the bunch centroid can move, but the distribution of the bunch around the centroid remains unchanged.

In other words, we assume that the bunch remains coherent. These assumptions may be valid for small effects, but often break down if the coherent motion of a bunch becomes very large.

Longitudinal and transverse wake functions

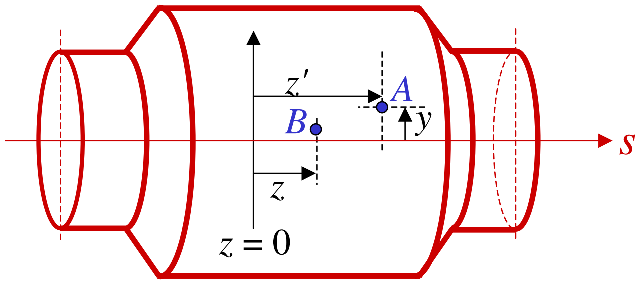
The electromagnetic fields behind a bunch of charged particles exert longitudinal and transverse forces on following particles.

The longitudinal forces are assumed to be independent of the transverse position of the leading bunch; we assume that the energy change of a following bunch can be written simply in terms of a longitudinal wake function and the charge of the leading bunch:

$$\Delta\delta_B = -\frac{r_e}{\gamma} N_A W_{\parallel}(z - z')$$

Longitudinal and transverse wake functions

We assume that the transverse forces depend linearly on the transverse offset of the leading bunch with respect to the reference trajectory. In that case, we can write the transverse deflection of a following bunch in terms of a transverse wake function, the charge of the leading bunch, and the transverse offset of the leading bunch:



$$\Delta p_{y,B} = -\frac{r_e}{\gamma} N_A \; y_A \; W_\perp(z - z')$$

Example: resistive-wall long-range wake functions

Consider the case of a vacuum chamber with conductivity σ, length L, and circular cross-section of radius b. The resistive-wall wake fields have both short-range and a long-range effects.

For the regime:

$$-z >> \sqrt[3]{\frac{b^2}{Z_0 \sigma}}$$

the longitudinal wake function is given by:

$$W_\parallel(z) = \frac{1}{2\pi b}\sqrt{\frac{4\pi}{Z_0 c}\frac{c}{\sigma}}\frac{L}{\sqrt{-z^3}}$$

and the transverse wake function is given by:

$$W_\perp(z) = -\frac{2}{\pi b^3}\sqrt{\frac{4\pi}{Z_0 c}\frac{c}{\sigma}}\frac{L}{\sqrt{-z}}$$

Aluminium has an electrical conductivity of $3.7\times10^7\ \Omega^{-1}\text{m}^{-1}$; so for a beam pipe of radius 1 cm, the range of validity of these expressions is $-z >> 20\ \mu\text{m}$. It might be dangerous to use these wake functions for single-bunch studies, but they should be safe for studies of multi-bunch effects.

Equation of motion for betatron oscillations

In the absence of any wake fields, the equation of motion for the n^{th} bunch moving round a storage ring can be written:

$$\ddot{y}_n + \omega_\beta^2 y_n = 0$$

Note that we use time as the independent variable, and write the "average" betatron frequency as:

$$\omega_\beta = \frac{2\pi\nu_\beta}{T_0}$$

where ν_β is the betatron tune.

We can add the transverse forces from the wake fields as driving terms on the right-hand side of the equation of motion.

Equation of motion with wake fields

If $W_\perp(z)$ represents the wake function over the entire circumference, then the transverse deflection of the n^{th} bunch over one turn can be obtained by summing the wake fields over all bunches over all previous turns:

$$\frac{dp_{y,n}}{dt} = \frac{1}{c}\ddot{y}_n = -\frac{1}{T_0}\frac{r_e}{\gamma}N_0 \sum_k \sum_{m=0}^{M-1} W_\perp\left(-kC - \frac{m-n}{M}C\right) y_m\left(t - kT_0 - \frac{m-n}{M}T_0\right)$$

Here, we assume that the ring is uniformly filled with M equally-spaced bunches, each with a total of N_0 particles.

The sum over k represents a sum over multiple turns; the sum over m represents a sum over all the bunches in the ring.

Note that we use the "retarded time" for each bunch: this gives the position of each bunch at the time that it created the wake field seen by the n^{th} bunch.

Usually, for ultra-relativistic motion, the wake function obeys:

$$W_\perp(z) = 0 \qquad \text{if} \qquad z > 0$$

Equation of motion with wake fields

Including the driving term that comes from the wake fields, the equation of motion for betatron oscillations can be written:

$$\ddot{y}_n + \omega_\beta^2 y_n = -\frac{c}{T_0}\frac{r_e}{\gamma}N_0 \sum_k \sum_{m=0}^{M-1} W_\perp\left(-kC - \frac{m-n}{M}C\right) y_m\left(t - kT_0 - \frac{m-n}{M}T_0\right)$$

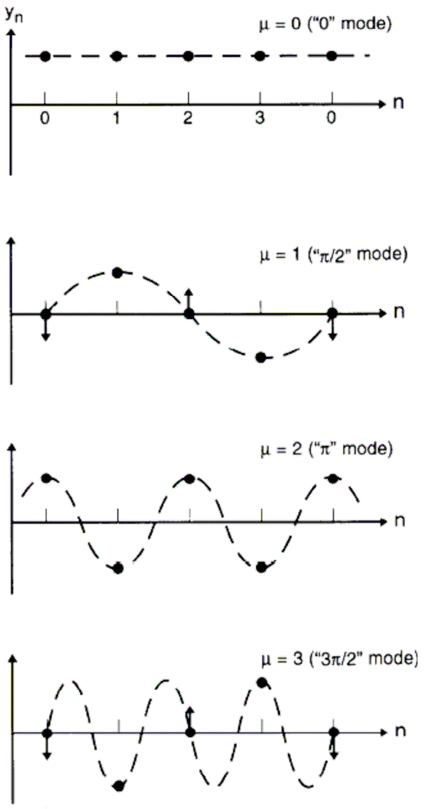
Our task is to solve this equation, to find the behaviour of all the bunches in the ring, in the presence of the long-range wake fields represented by the wake function $W_\perp$.

We shall try a solution of the form:

$$y_n^\mu(t) \propto \exp\left(2\pi i \frac{\mu n}{M}\right)\exp(-i\Omega_\mu t)$$

This solution describes the behaviour of a "mode" consisting of a particular pattern of transverse bunch positions, and oscillating with a particular frequency. The various modes are indexed using the symbol μ. The frequency of a mode is represented by Ω_μ; the imaginary part of Ω_μ gives the growth (or damping) rate of the corresponding mode.

Coupled bunch modes



The mode number μ gives the phase advance between the betatron position of one bunch and the next.

Each bunch performs oscillations with frequency Ω_μ as it moves around the ring.

Because the bunches are coupled by the wake fields, the betatron frequency is shifted from the "nominal" frequency ω_β; the frequency in the presence of the wake fields depends on the mode number.

The real part of $\Omega_\mu - \omega_\beta$ gives the coherent frequency shift; the imaginary part of Ω_μ gives the exponential growth or damping rate for the mode.

Solution of the equation of motion with wake fields

If we substitute our trial solution into the equation of motion, and make the assumption that the mode frequencies are close to the betatron frequency, i.e. that:

$$\Omega_\mu \approx \omega_\beta$$

then we find:

$$\Omega_\mu - \omega_\beta \approx \frac{r_e c}{4\pi\nu_\beta}\frac{N_0}{\gamma}\sum_{m=0}^{M-1}\left[\sum_{k=0}^{\infty} W_\perp\left(-kC - \frac{m-n}{M}C\right)e^{2\pi i\nu_\beta k}\right]e^{2\pi i(\mu+\nu_\beta)\frac{m-n}{M}}$$

Observe that the factor in square brackets in this equation is effectively the Fourier transform of the wake function. We define the *impedance* $Z_\perp$ corresponding to the wake field as the Fourier transform of the wake function:

$$Z_\perp(\omega) = i\frac{Z_0 c}{4\pi}\int_{-\infty}^{\infty} W_\perp(z)\,e^{-i\frac{\omega z}{c}}\frac{dz}{c} \qquad W_\perp(z) = -i\frac{4\pi}{Z_0 c}\frac{1}{2\pi}\int_{-\infty}^{\infty} Z_\perp(\omega)\,e^{i\frac{\omega z}{c}}\,d\omega$$

Defined in this way, the impedance relates the voltage seen by the beam to the beam current in frequency space (Appendix A):

$$\tilde{V}(\omega) = \tilde{I}(\omega)Z_\parallel(\omega)$$

Solution of the equation of motion with wake fields

Using the impedance, we can express the solution of the equation of motion with wake fields in a slightly simpler form than when we used the wake function. The result is (see Appendix B):

$$\Omega_\mu - \omega_\beta \approx -i\frac{4\pi}{Z_0 c}\frac{MN_0 r_e c}{4\pi\gamma\nu_\beta T_0}\sum_{p=-\infty}^{\infty} Z_\perp\left[(pM+\mu)\omega_0 + \omega_\beta\right]$$

Recall that Ω_μ gives the frequency of a bunch in the case that the bunches are arranged in a mode μ:

$$y_n^\mu(t) \propto \exp\left(2\pi i\frac{\mu n}{M}\right)\exp(-i\Omega_\mu t)$$

Thus, we see that associated with the long-range wake field, there are two effects:

- a shift in the frequency of coherent betatron oscillations, given by the imaginary ("reactive") part of the impedance;
- an exponential growth or damping of the betatron oscillations, given by the real ("resistive") part of the impedance.

The size of the effects depends on the "overlap" between the impedance and the betatron frequency.

Example: coupled bunch motion with resistive-wall wake field

As an example, consider the case of the transverse resistive-wall wake fields. The wake function is given by:

$$W_\perp(z) = -\frac{2}{\pi b^3}\sqrt{\frac{4\pi}{Z_0 c}\frac{c}{\sigma}}\frac{L}{\sqrt{-z}} \qquad (z<0)$$

The impedance is found to be:

$$\frac{Z_\perp(\omega)}{L} \approx \frac{c}{\omega}\frac{1-\mathrm{sgn}(\omega)i}{\pi b^3 \delta_{\mathrm{skin}}\sigma}$$

where δ_{skin} is the skin depth:

$$\delta_{\mathrm{skin}} = \sqrt{\frac{4\pi}{Z_0 c}\frac{c^2}{2\pi\sigma|\omega|}}$$

Example: coupled bunch motion with resistive-wall wake field

The frequency shift is given by:

$$\Omega_\mu - \omega_\beta \approx -i\frac{4\pi}{Z_0 c}\frac{M N_0 r_e c}{4\pi\gamma\nu_\beta T_0}\sum_{p=-\infty}^{\infty} Z_\perp\left[(pM+\mu)\omega_0 + \omega_\beta\right]$$

For the resistive-wall impedance, terms with small ω dominate the summation. Therefore, we expect to see the strongest effects in modes for which:

$$(pM+\mu)\omega_0 + \omega_\beta \approx 0 \qquad \Rightarrow \qquad \mu \approx -pM - \nu_\beta$$

Restricting the mode index to $0 \le \mu < M$, the strongest effects of the resistive-wall wake field are for modes:

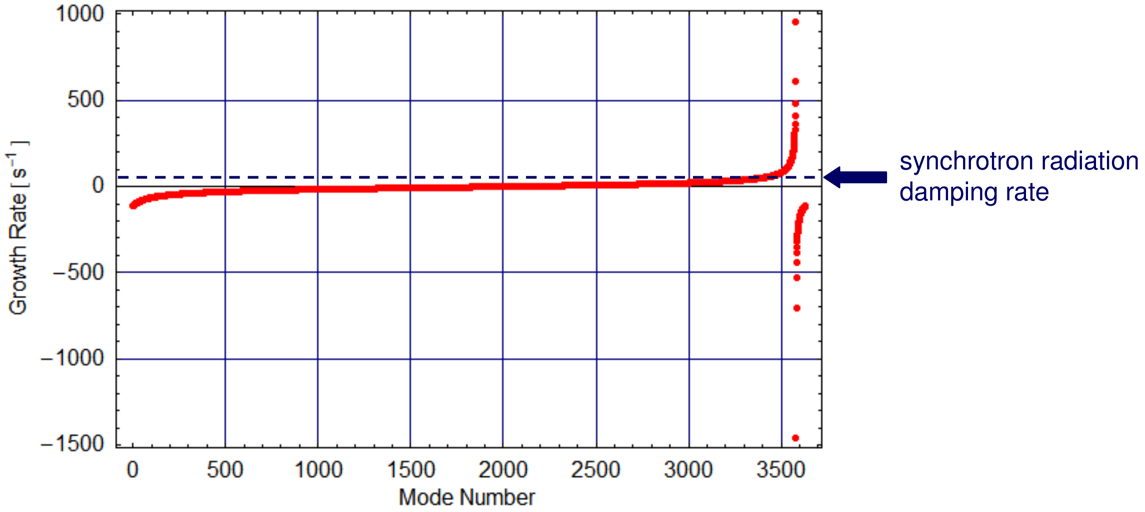
$$\mu \approx M - \nu_\beta$$

These are the modes for which, at fixed points in the ring, the lowest beam oscillation frequencies are observed.

Example: coupled bunch motion with resistive-wall wake field

The plot of growth rates *vs* mode number for the resistive wall wake field has a characteristic shape.

Roughly half the modes grow (are unstable), and half the modes are damped. The strongest effects are for mode numbers around $M - \nu_\beta$, where M is the number of bunches and ν_β is the tune.



Example: coupled bunch motion with resistive-wall wake field

The tune shift is given by:

$$\Omega_\mu - \omega_\beta \approx -i \frac{4\pi}{Z_0 c} \frac{M N_0 r_e c}{4\pi \gamma \nu_\beta T_0} \sum_{p=-\infty}^{\infty} Z_\perp \left[(pM + \mu)\omega_0 + \omega_\beta \right]$$

If we include just the largest term in the summation over the impedance, we find that for the resistive-wall wake field the fastest growth rate of any of the modes is given by:

$$\Gamma \approx \frac{4\pi}{Z_0 c} \frac{C}{2\pi b^3} \frac{c}{\gamma} \frac{\langle I \rangle}{I_A} \frac{1}{2\pi \nu_\beta} \sqrt{\frac{Z_0 c}{4\pi} \frac{c}{\sigma}} \sqrt{\frac{C}{1-\Delta_\beta}}$$

where the tune is written as:

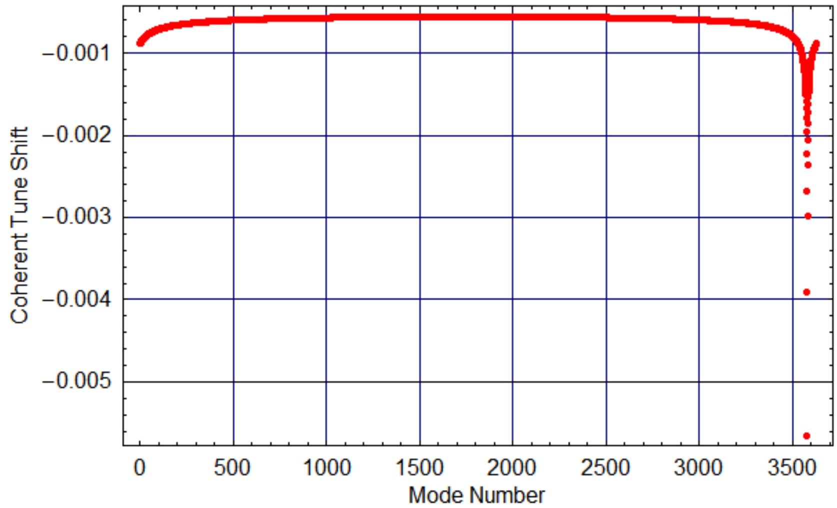
$$\nu_\beta = N_\beta + \Delta_\beta$$

for integer N_β and $0 \le \Delta_\beta < 1$. $\langle I \rangle$ is the average current, and I_A is the Alfven current, given by:

$$I_A = \frac{ec}{r_e} \approx 17.045\,\text{kA}$$

Example: coupled bunch motion with resistive-wall wake field

The real parts of the mode frequencies allows us to calculate the coherent tune shifts:



Approximations in the solution to the equation of motion

Note that in solving the equation of motion, we made a number of assumptions and approximations:

- We assumed that the ring was uniformly filled with equally-spaced bunches, each with the same charge. In general, there will be gaps in the fill, and some variation in charge from bunch to bunch.
- We assumed that the betatron frequency was uniform around the ring. This is equivalent to assuming a constant beta function around the ring. Depending on the lattice design, there may be large variations in beta function around the ring.
- We assumed that the wake function was independent of position; or, equivalently, that we could use an integrated impedance to represent the wake field of the entire ring. In practice, the wake field is a function of position.
- In solving the equation of motion, we assumed that the tune shifts were small. Usually, this is a good approximation.
- The solution we found for the equation of motion is valid if the number of bunches is large.

Instead of finding a solution in the frequency domain, we can write a time-domain simulation to solve the equation of motion for bunches in an accelerator with wake fields. Examples of time-domain simulations are given in Appendix C.

Coupled bunch motion with resistive-wall wake field for the ILC DRs

Assuming an aluminium vacuum chamber with constant beam pipe radius of 3 cm, we find that the shortest growth time for transverse modes in the ILC damping rings from the resistive-wall wake field is approximately 40 turns.

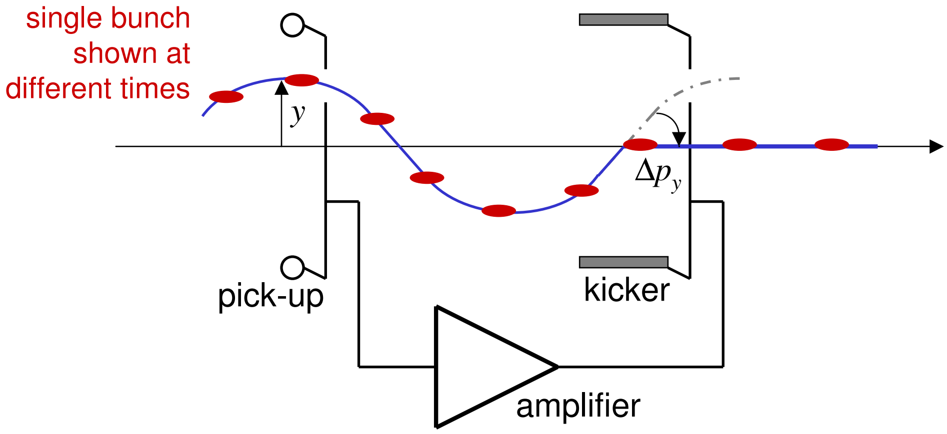
This is very much faster than the damping rate from synchrotron radiation.

- As we inject current into the ring, at some point the beam will become unstable, and the coherent betatron oscillations will increase in amplitude until charge is lost from the ring.
- The transverse resistive-wall wake field scales strongly with the beam pipe radius (~ $1/b^3$) so if we don't achieve 3 cm radius everywhere, the growth rates could be even faster.
- We also need to include the effects of higher-order modes (HOMs) in the RF cavities, which contribute to the impedance and increase the growth rates for modes at the corresponding frequencies.

To suppress coupled-bunch instabilities, we need to use a bunch-by-bunch feedback system.

- Bunch-by-bunch feedback systems use a pickup to detect the position of each bunch in the beam, then applies a corrective "kick" to each bunch, to counteract the effects of long-range wake fields.
- Bunch-by-bunch feedback systems add to the damping provided by natural mechanisms.
- Modern digital feedback systems can achieve damping times of around 20 turns.

Bunch-by-bunch feedback systems



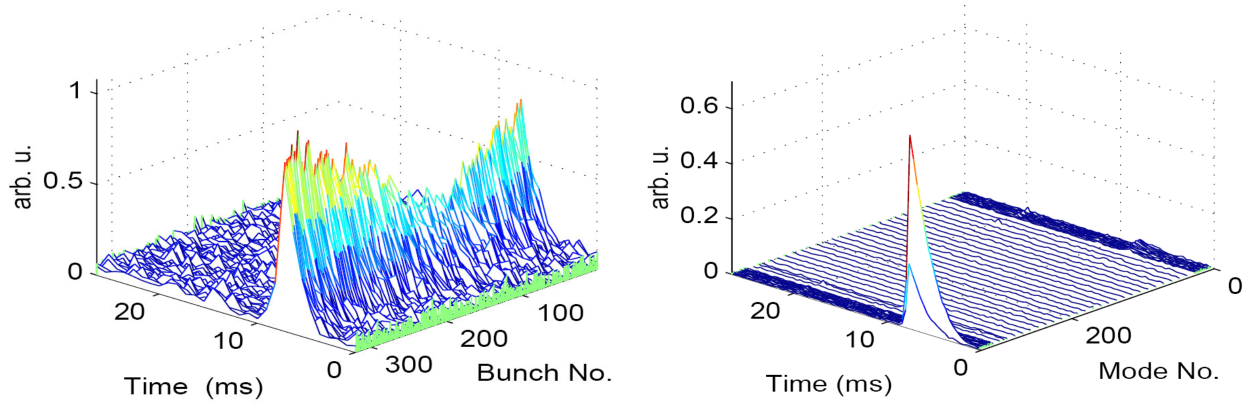
An ideal feedback system will detect the position of a single bunch, and apply a kick that immediately restores the bunch to its correct trajectory.

Because of limitations on the technical components, feedback systems apply a series of small kicks over several turns to correct the trajectories of the bunches in the ring.

Further details, and some analysis, are given in Appendix D.

Example: grow-damp measurement in the ALS

"Grow-damp" measurements are made by storing a beam (with a feedback system to maintain stability at high current), then turning off the feedback kicker for a short time (a few milliseconds), before turning the kicker back on; the beam motion is observed continually on the pick-up.



The plots above show measurements made at the ALS, revealing coupled-bunch modes driven by the resistive-wall wake field.

J. Fox et al, "Multi-bunch instability diagnostics via digital feedback systems at PEP-II, DAΦNE, ALS and SPEAR", Proceedings of the 1999 Particle Accelerator Conference, New York, 1999.

Summary: long-range wake fields

Long-range wake fields couple the motion of different bunches in a storage ring. Depending on a range of factors (including the characteristics of the wake fields, the beam current, beam energy, synchrotron radiation damping rates etc.) this can lead to instabilities, in which the oscillations of the bunches grow exponentially.

Sources of long-range wake fields in storage rings include the finite resistance of the vacuum chamber walls, and higher-order modes in the RF cavities (and, possibly, other components).

The wake fields can be characterised using wake functions (in the time domain), or impedances (in the frequency domain). The wake function and the impedance are related by a Fourier transform.

A simple analytical model of the long-range wake field effects leads to an estimate of the growth rates in terms of the impedance.

Coupled-bunch instabilities can be controlled using bunch-by-bunch feedback systems. For the ILC damping rings, bunch-to-bunch jitter excited by noise in the feedback system (pick-up or amplifier) is a concern.

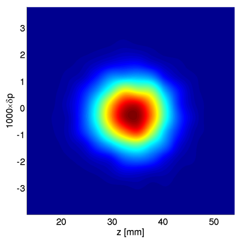
Short-range wake fields

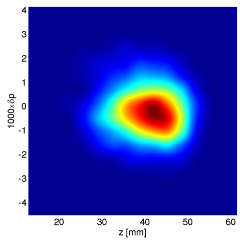
So far, we have looked at the effects of long-range wake fields. We modelled the beam as a set of bunches, with each bunch represented by a single large "point-like" charge. The long-range wake fields coupled the motion of different bunches.

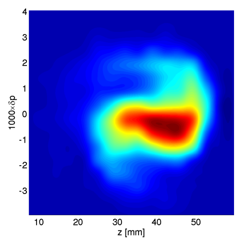
In the rest of this lecture, we will look at the effects of short-range wake fields. A short-range wake field is one that extends only over the length of a single bunch. (In the frequency domain, this corresponds to a very high frequency resonator).

To understand the effects of short-range wake fields, we have to consider the "internal" dynamics of individual bunches.

We will model the bunch as a charge distribution, and try to work out how the distribution function evolves over time, in the presence of a wake field.







The longitudinal bunch distribution

First of all, let us consider the impact of wake fields on the longitudinal bunch distribution, assuming that the distribution can reach a stable equilibrium.

The longitudinal equations of motion for the dynamical variables z and δ are:

$$\frac{dz}{ds} = -\alpha_p \delta$$

$$\frac{d\delta}{ds} = \frac{1}{\alpha_p}\left(\frac{2\pi\nu_s}{C}\right)^2 z + \frac{r_e}{\gamma C}\int_z^\infty \lambda(z') W_\parallel(z-z')\, dz'$$

The first term in the second equation gives the longitudinal "focusing" effect of the RF cavities, which results in synchrotron oscillations with tune ν_s.

The second term in the second equation gives the energy change resulting from the wake fields.

The longitudinal bunch distribution

The longitudinal equations of motion may be derived from a Hamiltonian:

$$H = -\tfrac{1}{2}\alpha_p \delta^2 - \frac{1}{2\alpha_p}\left(\frac{2\pi\nu_s}{C}\right)^2 z^2 - \frac{r_e}{\gamma C}\int_0^z dz' \int_{z'}^{\infty} dz'' \lambda(z'') W_{\parallel}(z' - z'')$$

using Hamilton's equations:

$$\frac{dz}{ds} = \frac{\partial H}{\partial \delta} \qquad \frac{d\delta}{ds} = -\frac{\partial H}{\partial z}$$

It follows from Hamilton's equations that the Hamiltonian itself is a constant of the motion (as long as there is no explicit dependence on the independent variable, s):

$$\frac{dH}{ds} = \frac{\partial H}{\partial z}\frac{dz}{ds} + \frac{\partial H}{\partial \delta}\frac{d\delta}{ds} = \frac{\partial H}{\partial z}\frac{\partial H}{\partial \delta} - \frac{\partial H}{\partial \delta}\frac{\partial H}{\partial z} = 0$$

Hence, any function of the Hamiltonian is a constant of the motion. In particular, if we are looking for an equilibrium distribution that is (by definition) independent of s, we can construct such a distribution as any function of the Hamiltonian.

The longitudinal bunch distribution

In electron storage rings, at low bunch intensity, each bunch arrives (through dissipative radiation processes) at a Gaussian profile in z and δ.

In the absence of the wake fields (i.e. in the limit of low charge), we can write for the Hamiltonian:

$$H = -\tfrac{1}{2}\alpha_p \delta^2 - \frac{1}{2\alpha_p}\left(\frac{2\pi\nu_s}{C}\right)^2 z^2$$

and for the invariant distribution:

$$\lambda(z,\delta) = \lambda_0 \exp\left(\frac{H}{\alpha_p \sigma_\delta^2}\right) = \lambda_0 \exp\left(-\frac{\delta^2}{2\sigma_\delta^2}\right)\exp\left(-\frac{z^2}{2\sigma_z^2}\right)$$

where:

$$\sigma_z = \alpha_p \frac{C}{2\pi\nu_s}\sigma_\delta$$

The longitudinal bunch distribution: potential-well distortion

Generalising this result to the case with wake fields suggests that we can write the equilibrium longitudinal distribution:

$$\lambda(z,\delta) = \lambda_0 \exp\left(-\frac{\delta^2}{2\sigma_\delta^2}\right)\exp\left(-\frac{z^2}{2\sigma_z^2} - \frac{r_e}{\alpha_p \gamma C}\int_0^z dz' \int_{z'}^\infty dz'' \lambda(z'') W_\parallel(z'-z'')\right)$$

Note that the distribution in δ remains Gaussian.

The longitudinal profile (i.e. distribution in z) must obey the equation:

$$\lambda(z) = \lambda_{z0} \exp\left(-\frac{z^2}{2\sigma_z^2} - \frac{r_e}{\alpha_p \gamma C}\int_0^z dz' \int_{z'}^\infty dz'' \lambda(z'') W_\parallel(z'-z'')\right)$$

This equation (or its derivative) is known as the Haissinski equation. It describes the "potential-well distortion", which is the change in shape of the equilibrium longitudinal profile of a bunch in the presence of wake fields in a storage ring.

If we know the wake function, we can solve the Haissinski equation numerically to find the stable longitudinal distribution.

The longitudinal bunch distribution: potential-well distortion

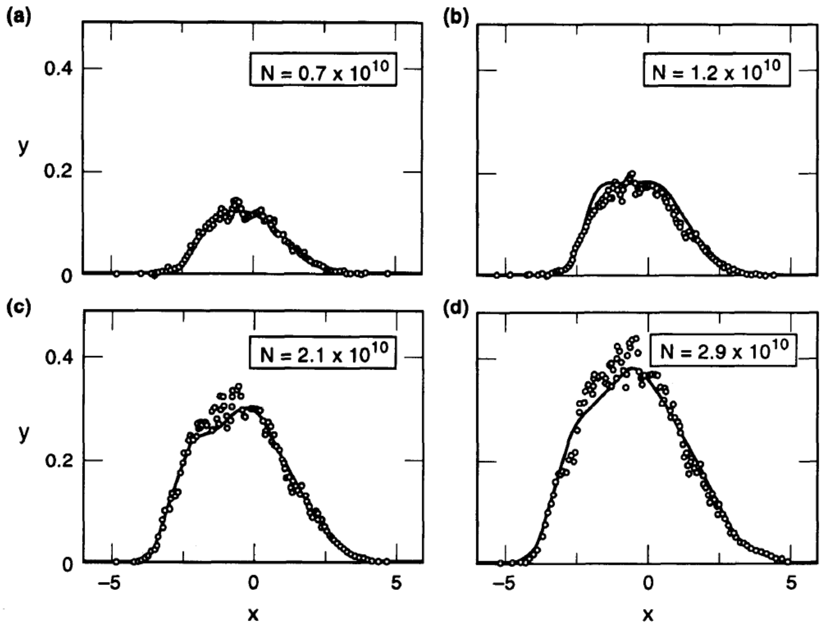


Figure 6.4. Potential-well distortion of bunch shape for various beam intensities for the SLC damping ring. The open circles are the measured results. The horizontal axis is $x = -z/\sigma_{z0}$, where σ_{z0} is the unperturbed rms bunch length. The vertical scale gives $y = 4\pi e\rho(z)/V'_{rf}(0)\sigma_{z0}$. (Courtesy Karl Bane, 1992.)

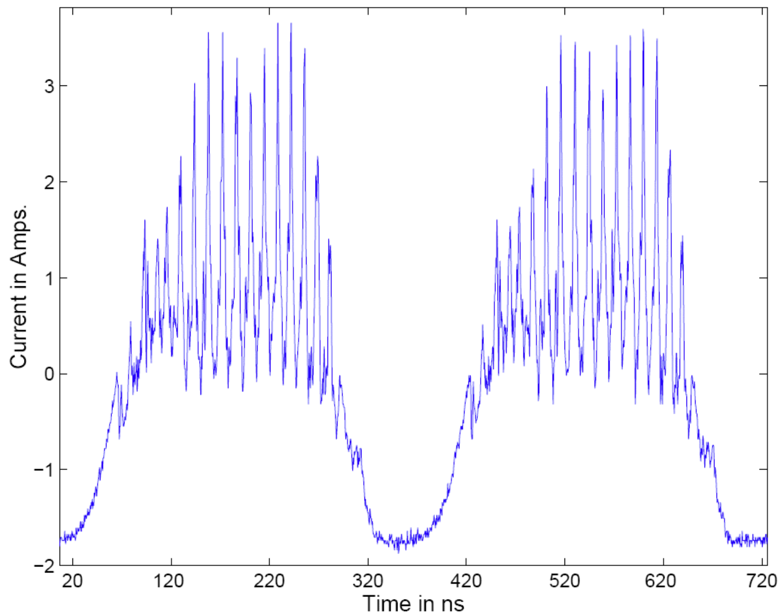
From A. Chao, "Physics of collective beam instabilities in high energy accelerators", Wiley, 1993.

At very low bunch charges, wake fields have little effect on bunches in a storage ring.

As the charge is increased, we start to observe the effects of the wake fields in the distortion of the longitudinal profile of the bunches (potential-well distortion).

As the charge is further increased, the bunch distribution becomes unstable. In this regime, the Haissinski equation is no longer valid, because an equilibrium distribution does not exist.

We need to use different techniques to analyse the dynamics in the longitudinal regime…



Observation of single-bunch longitudinal instability in the Los Alamos PSR, caused by an inductive impedance. From C. Beltran, A.A. Browman and R.J. Macek, "Calculations and observations of the longitudinal instability caused by the ferrite inductors at the Los Alamos Proton Storage Ring", Proceedings of the 2003 Particle Accelerator Conference, Portland, Oregon (2003).

The Vlasov equation

The fundamental equation describing the evolution of a density function in phase space is the Vlasov equation.

We shall work in longitudinal phase space, using dynamical variables θ (the azimuthal angle around the circumference of a storage ring) and δ (the energy deviation). We shall use t (time) as the independent variable.

Consider a distribution of particles in longitudinal phase space, with the local density of particles at time t given by $\Psi(\theta,\delta,t)$. Since the number of particles is conserved, we can write:

$$\frac{d\Psi}{dt} = 0$$

which implies that:

$$\frac{\partial\Psi}{\partial t} + \dot{\theta}\frac{\partial\Psi}{\partial\theta} + \dot{\delta}\frac{\partial\Psi}{\partial\delta} = 0$$

This is the Vlasov equation. Our task is to:

1. find $\dot{\theta}$ and $\dot{\delta}$ by considering the motion of individual particles;
2. solve the Vlasov equation to find the time-evolution of a distribution.

The Vlasov equation and the dispersion relation

Explicit expressions for $\dot{\theta}$ and $\dot{\delta}$ are found by considering the longitudinal dynamics of individual particles. We find (see Appendix E) that:

$$\dot{\theta} = \omega_0\left(1 - \alpha_p\delta\right)$$

$$\dot{\delta} \approx \frac{\Delta\delta}{T_0} = \frac{\omega_0}{2\pi}\frac{I_0}{E/e}\int \Delta\Psi(\delta)\,d\delta \cdot Z_{\parallel}(\omega_n)\,e^{i(n\theta-\omega_n t)}$$

Substituting these expressions into the Vlasov equation, we find (see Appendix E) that the Vlasov equation can be cast into integral form:

$$1 = iZ_{\parallel}(\omega_n)\frac{\omega_0}{2\pi}\frac{I_0}{E/e}\int\frac{\partial\Psi_0(\delta)/\partial\delta}{(n\omega-\omega_n)}\,\mathrm{d}\delta$$

This is an integral equation, which we need to solve to find the mode frequency ω_n for a given impedance $Z_{\parallel}(\omega_n)$, and a given mode (specified by the mode number n, which gives the number of periods of the density perturbation over the entire circumference of the ring).

This equation relates the mode frequency ω_n to the mode number n; it is therefore usually called the "dispersion relation".

The Vlasov equation for longitudinal phase space

Solving the Vlasov equation (or dispersion relation) is generally no easy task, and various numerical and analytical techniques have been devised to provide assistance.

Numerical techniques are often more satisfactory, since they allow one to study the dynamics including a detailed description of the impedance (obtained, for example, by modelling the vacuum chamber). The way in which the beam behaves can be sensitive to details of the impedance.

Sometimes, a detailed description of the impedance is not available, or is not reliable, but a rough estimate of the beam dynamics is still desired. Then we can make some crude approximations, and obtain order-of-magnitude analytical estimates for such quantities as the instability threshold.

Numerical techniques may or may not use the linearised Vlasov equation (the equation including the perturbation terms only to first order). Analytical techniques always use the linearised equation. By solving the linearised equation, we can only hope to identify instability thresholds: we cannot properly describe the behaviour of a mode that grows exponentially.

Dispersion relation for a beam with zero energy spread

As an example, let us consider the case of a beam with zero energy spread:

$$\Psi_0(\delta) = \delta(\delta)$$

(The notation is somewhat unfortunate, but this means that the distribution is a delta function). Such a beam is sometimes called a "cold" beam. Integrating by parts gives:

$$\int \frac{\partial \Psi_0 / \partial \delta}{(n\omega - \omega_n)} d\delta = \int \frac{\Psi_0}{(n\omega - \omega_n)^2} n \frac{\partial \omega}{\partial \delta} d\delta = \frac{n\omega_0 \alpha_p}{(n\omega_0 - \omega_n)^2}$$

The dispersion relation then gives:

$$(n\omega_0 - \omega_n)^2 = iZ_{\parallel}(\omega_n) \frac{I_0}{E/e} \frac{n\omega_0^2 \alpha_p}{2\pi}$$

and hence:

$$\omega_n = n\omega_0 \pm \sqrt{iZ_{\parallel}(\omega_n) \frac{I_0}{E/e} \frac{n\omega_0^2 \alpha_p}{2\pi}}$$

Dispersion relation for a beam with zero energy spread

With zero energy spread, the dispersion relation gives the frequency for mode n:

$$\omega_n = n\omega_0 \pm \sqrt{iZ_{\|}(\omega_n)\frac{I_0}{E/e}\frac{n\omega_0^2\alpha_p}{2\pi}}$$

We see that there are two solutions for the frequency. In general, a mode will consist of a superposition of the two frequencies.

Significantly, we observe that, except for the case that the impedance $Z_{\|}(\omega_n)$ has complex phase $3\pi/2$, there is always a solution for the frequency that has a positive imaginary part. Consequently, when we put this solution into the equation for the distribution:

$$\Psi(\theta,\delta;t) = \Psi_0(\delta) + \Delta\Psi(\delta)e^{i(n\theta-\omega_n t)}$$

we see that the distribution is *always* unstable. Physically, this is because there is no process in our model that will damp a mode that is driven by the impedance. In the absence of any impedance, a density perturbation in the beam will persist indefinitely; and if an impedance is introduced with which the perturbation resonates, the perturbation will start to grow.

Energy spread, Landau damping and beam stability

Real beams have some energy spread, and this leads (in combination with the momentum compaction factor) to a variation in the revolution frequency of the particles.

The range of revolution frequencies results in any initial density perturbation becoming "smeared out" or decohering, at a rate dependent on the energy spread and the momentum compaction factor.

If there is an impedance in the ring with which a density perturbation can resonate, then the damping from the decoherence competes with the antidamping from the impedance.

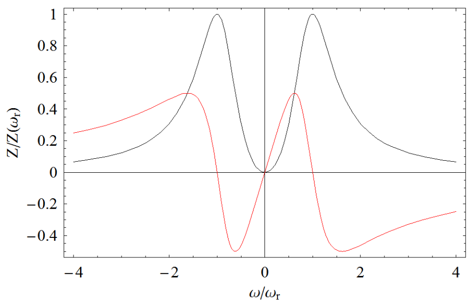
With a low impedance or at low beam intensities, or with a large energy spread or momentum compaction factor, the antidamping from the decoherence dominates, and the beam remains stable. This is a manifestation of Landau damping.

If the beam intensity is increased, then at some point the resonance becomes strong enough that the antidamping dominates: the beam becomes unstable.

Dispersion relation for a beam with Gaussian energy spread

Analysis of a beam with a Gaussian energy spread, and using a broad-band resonator model for the impedance:

$$\frac{Z_{\parallel}(\omega)}{n} \approx \frac{Z_{\parallel}(\omega_r)}{n} \frac{1 - i\frac{\omega^2-\omega_r^2}{\omega_r \omega}}{1 + \frac{(\omega^2-\omega_r^2)^2}{\omega_r^2\omega^2}}$$



yields the *approximate* stability condition (see Appendix F):

$$\left|\frac{Z_{\parallel}(n\omega_0)}{n}\right| < \sqrt{\frac{\pi}{2}} Z_0 \frac{\gamma \alpha_p \sigma_\delta^2 \sigma_z}{r_e N_0}$$

This is known as the Keil-Schnell-Boussard criterion. It gives a rather crude estimate of the stability condition, but can be useful to gauge the regime, if a detailed impedance model for the accelerator is not available.

Characteristics of the microwave instability

The single-bunch instability model we have developed here, is generally known as the "microwave instability", because it leads to density fluctuations in a bunch on a length scale of ~ 1 mm, and generates detectable microwave radiation.

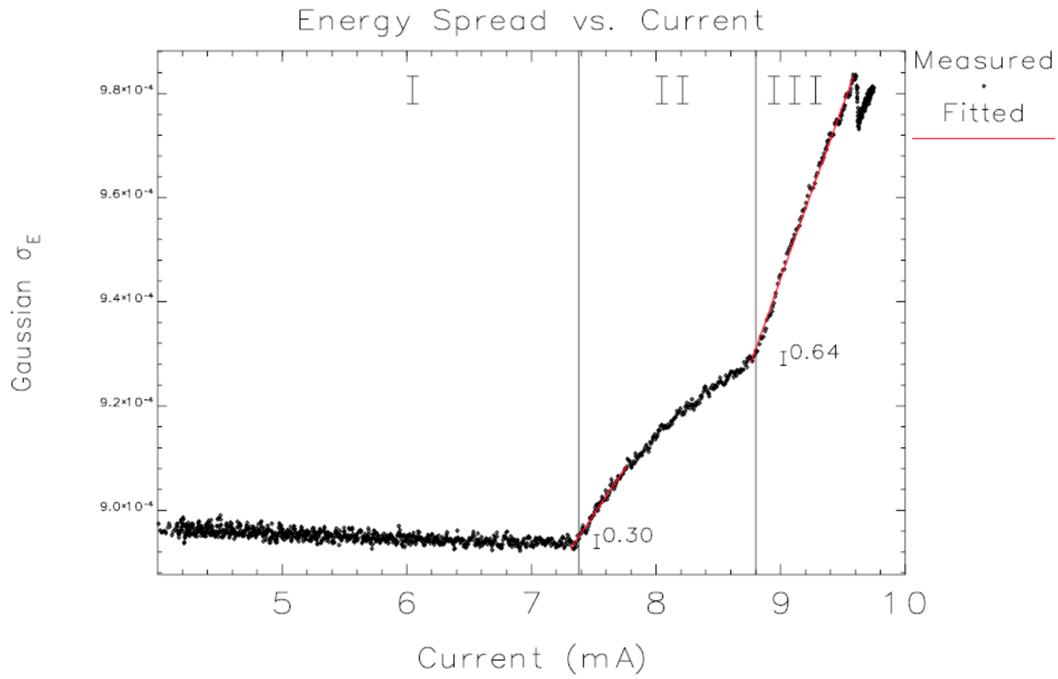
Since we analysed the problem by making a linear approximation to the Vlasov equation, all we can hope to do is estimate the instability threshold (which is the point at which Landau damping is insufficient to keep the beam stable).

Observations (which are understood in terms of further development of the theory, and by simulations) suggest that above threshold, the bunch undergoes a steady increase in energy spread, which varies according to a 1/3 power law with the bunch current:

$$\sigma_\delta = \sigma_{\delta 0} + k(N - N_{th})^{1/3}$$

Associated with the increase in energy spread is a proportionate increase in bunch length. The microwave instability is sometimes known as "turbulent bunch lengthening".

Characteristics of the microwave instability

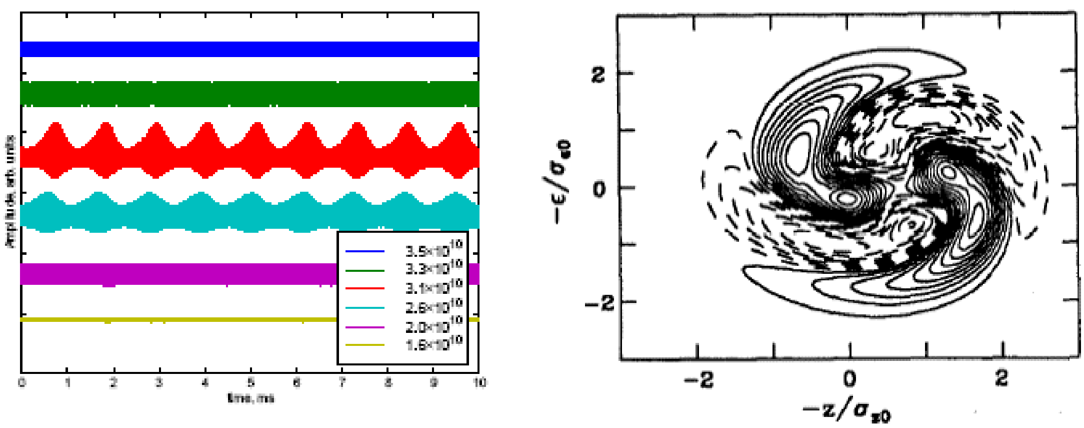


Y.-C. Chae et al, "Measurement of the longitudinal microwave instability in the APS storage ring", Proceedings of the 2001 Particle Accelerator Conference, Chicago (2001).

Observations of single-bunch instabilities in the SLC damping rings

The dynamics of single-bunch instabilities, depending on the beam conditions and the wake fields, can become very complex.

A significant operational problem for the SLC damping rings was associated with a "bursting" mode of instability, in which the bunch distribution never reached a steady equilibrium.



Single-bunch instability in the SLC damping rings.
Left: Experimental observation (B. Podobedov, BNL). Right: Simulation (K. Oide, KEK).

Microwave instability threshold for the ILC damping rings

The important parameters for the single-bunch instability threshold are:

- the bunch length;

  Longer is better, to reduce the peak current; but there is an upper limit set by what the bunch compressors can deal with.
- the energy spread;

  Larger is better, but again there are limits from the bunch compressors. In the ILC damping rings, the energy spread is essentially determined by the beam energy and the field of the damping wigglers.
- the beam energy;

  A higher energy is better, but increases costs, and the equilibrium emittances.
- the bunch charge;

  Lower is better, but the bunch charge is set by the luminosity requirements.
- the momentum compaction factor.

  Which we do have some control over in the lattice design. Larger is better, but if the momentum compaction factor is too large, a very high RF voltage is needed to achieve the specified bunch length. Also, the synchrotron tune becomes large, which can cause problems with synchro-betatron resonances.

Microwave instability threshold for the ILC damping rings

Work to determine the single-bunch instability thresholds in the ILC damping rings is planned; but at present, we do not have an impedance model.

The only parameter we can really apply to control the microwave instability threshold is the momentum compaction factor. The question is, what is the appropriate value to aim for in the lattice design?

We resort to the Keil-Schnell-Boussard criterion to make a rough estimate for some lattice designs.

Lattice	Energy	α_p	σ_δ	σ_z	N_0	Impedance threshold
OCS	5 GeV	1.62×10^{-4}	1.29×10^{-3}	6 mm	2×10^{10}	134 mΩ
BRU	3.74 GeV	11.9×10^{-4}	0.97×10^{-3}	9 mm	2×10^{10}	622 mΩ
MCH	5 GeV	4.09×10^{-4}	1.30×10^{-3}	9 mm	2×10^{10}	510 mΩ
TESLA	5 GeV	1.22×10^{-4}	1.29×10^{-3}	6 mm	2×10^{10}	100 mΩ

Achieving an impedance ~ 100 mΩ could be possible, but challenging.

Single-bunch instabilities: summary

Beam instabilities show complicated dynamics. The basic equation describing the evolution of a distribution is the Vlasov equation, which is difficult to solve in practical cases.

A range of techniques have been developed to find solutions to the Vlasov equation in situations of interest. These include simplifying approximations (perturbation theory) and numerical methods. Generally, it is advisable to cross-check results from different techniques.

Perturbations in the bunch distribution tend to be smoothed out by the natural motions of particles in the bunch (Landau damping). This is most effective if there is a large spread in dynamical behaviour across particles within the bunch (e.g. variation in revolution frequency arising from the energy spread).

If perturbations in the bunch distribution resonate with the impedance, then an instability can develop. The linearised Vlasov equation can be used to estimate the intensity threshold at which the instability occurs (if the ring impedance is known), but cannot describe the behaviour above threshold.

If the ring impedance is not known, then we can make some crude estimates for the general properties of the impedance, and estimate the impedance threshold using, for example, the Keil-Schnell-Boussard criterion.

Other collective effects

The resistive wall and microwave instabilities are just two effects that need to be addressed in the design and operation of linear collider damping rings.

There are many other effects of real or potential concern. These include:

- Electron cloud instability (in the positron damping ring)
- Ion effects (in the electron damping ring)
- Intrabeam scattering
- Touschek scattering
- Space charge effects
- Coherent synchrotron radiation

Electron cloud instability is a major R&D issue for the damping rings.

The instability occurs when electrons accumulate to high density in the vacuum chamber, and interact with the beam.

Electron cloud effects have been observed in many proton and positron machines, and were a limitation on operational performance of the B-factories before installation of solenoids for suppressing build-up of electron cloud.

A lot of effort is being invested in developing ways to suppress the build-up of electron cloud; such as coating with low-SEY materials, clearing electrodes…

Appendices

Appendix A: Physical interpretation of impedance

The longitudinal wake function is defined so that:

$$\Delta\delta(z) = -\frac{r_e}{\gamma} N(z') W_{\parallel}(z - z')$$

For the case of a charge distribution $\lambda(z')$ (number of particles per unit length):

$$\Delta\delta(z) = -\frac{r_e}{\gamma} \int \lambda(z') W_{\parallel}(z - z')\, dz'$$

We write the longitudinal charge distribution as a mode decomposition:

$$\lambda(z') = \frac{1}{C} \int \tilde{\lambda}(\omega) e^{i\frac{\omega z'}{c}}\, d\omega$$

Substitute this into the equation for the energy change, and make the change of variables $z' \to z - z'$:

$$\Delta\delta(z) = \frac{1}{C}\frac{r_e}{\gamma} \iint \tilde{\lambda}(\omega)\, e^{i\frac{\omega z}{c}}\, W_{\parallel}(z')\, e^{-i\frac{\omega z'}{c}}\, dz'\, d\omega$$

Appendix A: Physical interpretation of impedance

We define the longitudinal impedance:

$$Z_{\parallel}(\omega) = \frac{Z_0 c}{4\pi} \int W_{\parallel}(z) e^{-i\frac{\omega z}{c}} \frac{dz}{c}$$

In terms of the impedance, the energy loss as function of longitudinal position z becomes:

$$\Delta\delta(z) = \frac{c}{C} \frac{r_e}{\gamma} \frac{4\pi}{Z_0 c} \int \tilde{\lambda}(\omega) Z_{\parallel}(\omega) e^{i\frac{\omega z}{c}} d\omega$$

and hence:

$$\int \Delta\delta(z) e^{-i\frac{\omega z}{c}} \frac{dz}{c} = \frac{c}{C} \frac{r_e}{\gamma} \frac{4\pi}{Z_0 c} \tilde{\lambda}(\omega) Z_{\parallel}(\omega)$$

which can be written:

$$\int \frac{\Delta E(z)}{e} e^{-i\frac{\omega z}{c}} \frac{dz}{c} = \frac{1}{C} ec\tilde{\lambda}(\omega) Z_{\parallel}(\omega)$$

The left hand side is the Fourier transform of a voltage (the energy change of a particle over one turn of the accelerator). The right hand side is the product of the current spectrum and the impedance.

Appendix A: Physical interpretation of impedance

Finally, we can write:

$$\tilde{V}(\omega) = \tilde{I}(\omega) Z_{\parallel}(\omega)$$

In other words, the impedance -- defined as the Fourier transform of the wake function -- relates the voltage seen by the beam (resulting from the interaction of the beam with its surroundings) to the beam current in frequency space.

Appendix B: Multibunch motion with wake fields

From the definition of the impedance we can write:

$$\sum_{k=0}^{\infty} W_\perp\left(-kC - \frac{m-n}{M}C\right)e^{2\pi i\nu_\beta k} = -i\frac{4\pi}{Z_0 c}\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \sum_{k=0}^{\infty} Z_\perp(\omega)\, e^{-i\frac{\omega}{c}\left(kC+\frac{m-n}{M}C\right)}\; e^{2\pi i\nu_\beta k}$$

$$= -i\frac{4\pi}{Z_0 c}\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \sum_{k=0}^{\infty} Z_\perp(\omega)\, e^{-i(\omega-\omega_\beta)T_0 k}\; e^{-i\frac{\omega}{c}\left(\frac{m-n}{M}\right)C}$$

$$= -i\frac{4\pi}{Z_0 c}\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \sum_{k=0}^{\infty} Z_\perp(\omega+\omega_\beta)\, e^{-i\omega T_0 k}\; e^{-i(\omega+\omega_\beta)\left(\frac{m-n}{M}\right)T_0}$$

Note that we can write the summation over k in terms of a Dirac delta function:

$$\sum_{k=0}^{\infty} e^{-i\omega T_0 k} = \sum_{p'=-\infty}^{\infty} \delta\left(\frac{\omega T_0}{2\pi} - p'\right)$$

Appendix B: Multibunch motion with wake fields

Then we can perform the integral over ω, which gives:

$$\sum_{k=0}^{\infty} W_\perp\left(-kC - \frac{m-n}{M}C\right)e^{2\pi i\nu_\beta k} = -i\frac{4\pi}{Z_0 c}\frac{1}{T_0}\sum_{p'=-\infty}^{\infty} Z_\perp(p'\omega_0+\omega_\beta)\, e^{-i(p'\omega_0+\omega_\beta)\left(\frac{m-n}{M}\right)T_0}$$

Hence we find:

$$\Omega_\mu - \omega_\beta \approx -i\frac{r_e c}{4\pi\nu_\beta}\frac{N_0}{\gamma}\sum_{m=0}^{M-1}\left[\frac{4\pi}{Z_0 c}\frac{1}{T_0}\sum_{p'=-\infty}^{\infty} Z_\perp(p'\omega_0+\omega_\beta)\, e^{-i(p'\omega_0+\omega_\beta)\left(\frac{m-n}{M}\right)T_0}\right] e^{2\pi i(\mu+\nu_\beta)\frac{m-n}{M}}$$

$$\approx -i\frac{4\pi}{Z_0 c}\frac{r_e c}{4\pi\nu_\beta}\frac{N_0}{\gamma}\frac{1}{T_0}\sum_{m=0}^{M-1}\;\sum_{p'=-\infty}^{\infty} Z_\perp(p'\omega_0+\omega_\beta)\, e^{-2\pi i(p'-\mu)\left(\frac{m-n}{M}\right)}$$

Finally, we observe that, for large M, the summation over m vanishes, unless:

$$p' - \mu = pM$$

where p is an integer.

Appendix B: Multibunch motion with wake fields

This gives the final result:

$$\Omega_\mu - \omega_\beta \approx -i \frac{4\pi}{Z_0 c} \frac{MN_0 r_e c}{4\pi\gamma\nu_\beta T_0} \sum_{p=-\infty}^{\infty} Z_\perp\left[(pM + \mu)\omega_0 + \omega_\beta\right]$$

Recall that Ω_μ gives the frequency of a bunch in the case that the bunches are arranged in a mode μ:

$$y_n^\mu(t) \propto \exp\left(2\pi i \frac{\mu n}{M}\right)\exp(-i\Omega_\mu t)$$

Thus, we see that associated with the long-range wake field, there are two effects:

- a shift in the frequency of coherent betatron oscillations, given by the imaginary (“reactive”) part of the impedance;
- an exponential growth or damping of the betatron oscillations, given by the real (“resistive”) part of the impedance.

The size of the effects depends on the “overlap” between the impedance and the betatron frequency.

Appendix B: Multibunch motion with wake fields

Note that we evaluate the impedance at frequencies $(pM+\mu)\omega_0+\omega_\beta$. This can be understood in terms of the beam spectrum. At a fixed point in the ring, the beam signal looks like:

$$\text{signal} \propto \sum_{k=-\infty}^{\infty} \sum_{n=0}^{M-1} y_n^{(\mu)}(t)\,\delta\left(t - kT_0 + \frac{n}{M}T_0\right)$$

$$\propto \sum_{k=-\infty}^{\infty} \sum_{n=0}^{M-1} e^{i\left(2\pi\mu\frac{n}{M} - \omega_\beta t\right)}\,\delta\left(t - kT_0 + \frac{n}{M}T_0\right)$$

The beam spectrum is the Fourier transform of the signal:

$$\text{spectrum} \propto \sum_{k=-\infty}^{\infty} \sum_{n=0}^{M-1} e^{i2\pi\mu\frac{n}{M}} \exp\left(i(\omega-\omega_\beta)\left(k - \frac{n}{M}\right)T_0\right)$$

$$\propto M\omega_0 \sum_{p=-\infty}^{\infty} \delta\left(\omega - \left(\omega_\beta + pM\omega_0 + \mu\omega_0\right)\right)$$

To find the effect of the wake field, we have to evaluate the impedance at frequencies corresponding to frequencies present in the beam spectrum.

Appendix C: Time-domain simulations of resistive wall instability

Instead of finding a solution in the frequency domain, we can write a time-domain simulation to solve the equation of motion for bunches in an accelerator with wake fields.

A time domain simulation must track individual bunches, applying the betatron motion and the wake fields in discrete steps.

Solving the equations of motion in this way allows us to consider:

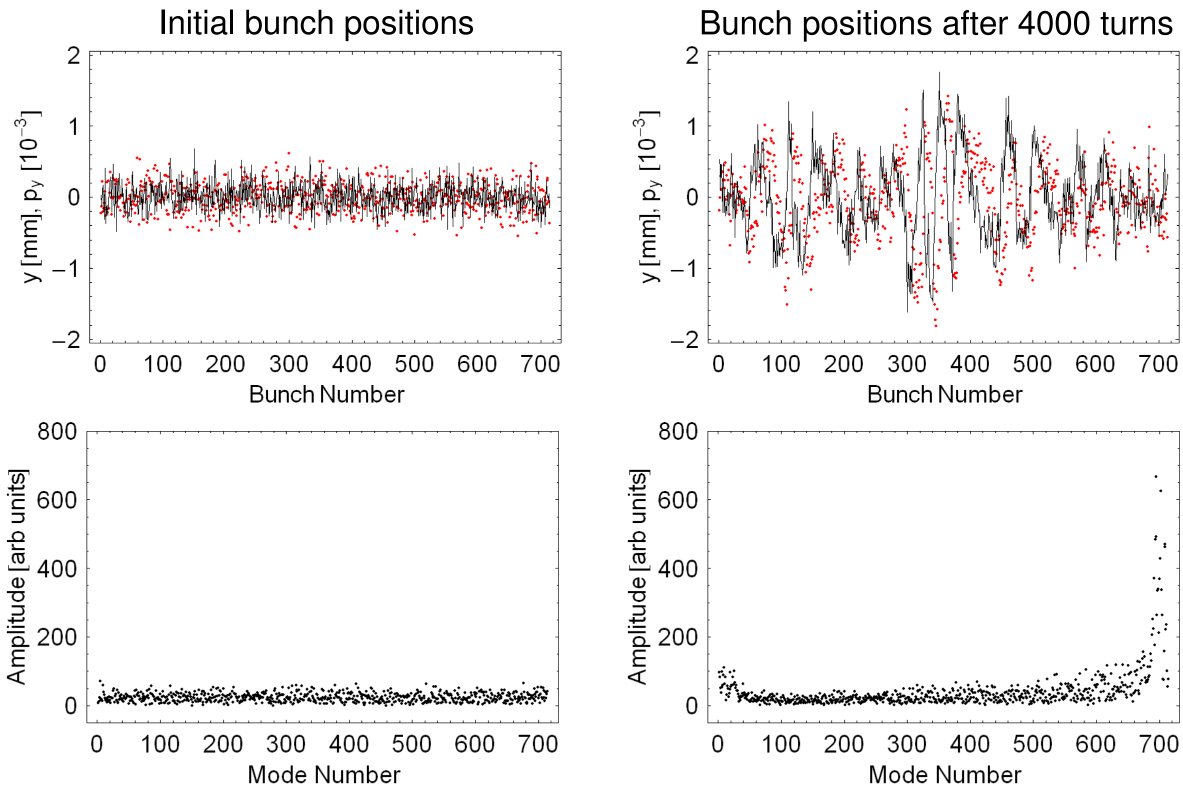
- variations in bunch patterns and bunch charges;
- variations in beta functions around the ring;
- variations in wake fields around the ring.

Time domain simulations allow us to include effects such as synchrotron radiation damping, and bunch-by-bunch feedback systems.

To estimate the growth rates, we can perform a Fourier analysis of the bunch positions after each complete turn, then plot the amplitudes of the Fourier modes as a function of turn number.

The following examples are from studies for the damping rings for the NLC (the Next Linear Collider)…

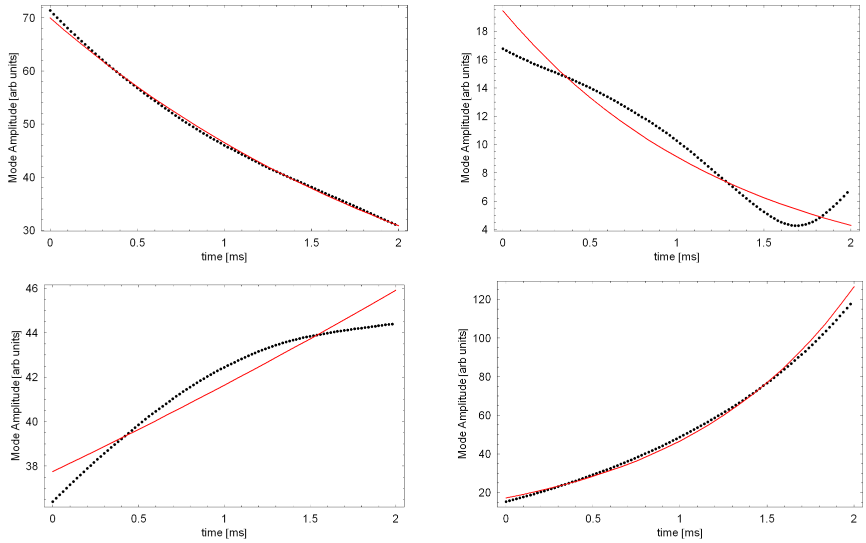
Appendix C: Time-domain simulations of resistive wall instability



Appendix C: Time-domain simulations of resistive wall instability

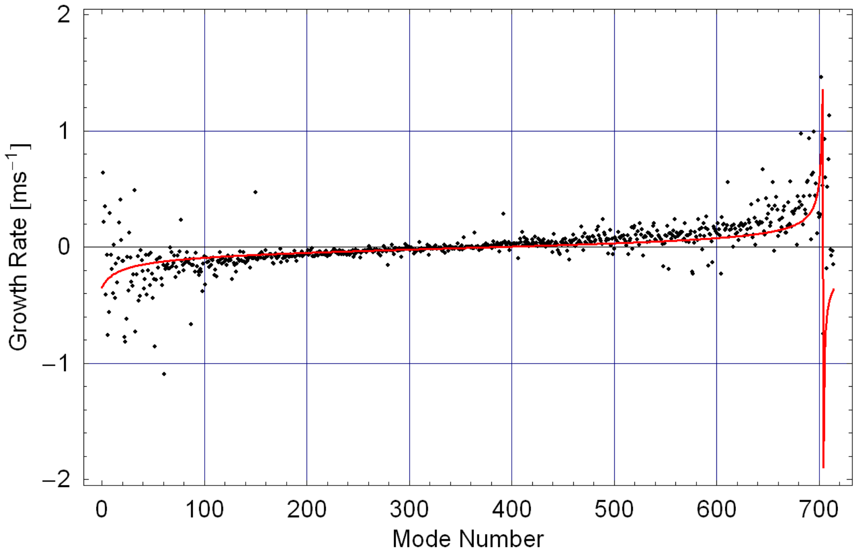
Determining the growth rates can be problematic, because the mode amplitudes do not exactly follow exponential behaviour.

We can estimate the growth rates by fitting an exponential curve to plots of the mode amplitude as a function of bunch number.



Appendix C: Time-domain simulations of resistive wall instability

The growth rates estimated from time-domain simulations fit the analytical estimates reasonably well.



Comparison between simulation (black points) and analytical estimate (red line) of resistive-wall growth rates in the NLC main damping rings.

The parameters that determine the damping rate from the feedback system are:

- the beta functions at the pick-up and the kicker;
- the betatron phase advance between the pick-up and the kicker;
- the amplifier gain, g defined by:

$$\Delta p_y(s_2) = g \cdot y(s_1)$$

where $y(s_1)$ is the bunch position at the pick-up (at location s_1), and $\Delta p_y(s_2)$ is the kick applied to the bunch by the kicker (at location s_2).

Let us calculate the damping rate of the feedback system in terms of these parameters…

In terms of the action J and angle ϕ variables, the transverse coordinate and momentum of a particle at the pick-up can be written as:

$$y_1 = \sqrt{2\beta_1 J_1}\cos\phi_1$$

$$p_{y1} = -\sqrt{\frac{2J_1}{\beta_1}}(\sin\phi_1 + \alpha_1\cos\phi_1)$$

Following the kicker, the coordinate and momentum are:

$$y_2 = \sqrt{2\beta_2 J_1}\cos(\phi_1 + \Delta\phi_{21})$$

$$p_{y2} = -\sqrt{\frac{2J_1}{\beta_2}}[\sin(\phi_1 + \Delta\phi_{21}) + \alpha_2\cos(\phi_1 + \Delta\phi_{21})] + g\ y_1$$

which we can write in terms of a new action J_2 and angle ϕ_2:

$$y_2 = \sqrt{2\beta_2 J_2}\cos\phi_2$$

$$p_{y2} = -\sqrt{\frac{2J_2}{\beta_2}}(\sin\phi_2 + \alpha_2\cos\phi_2)$$

Appendix D: Bunch-by-bunch feedback systems

After some algebra, we find that the new action is given by:

$$J_2 = J_1\left[1 - 2g\sqrt{\beta_1\beta_2}\cos\phi_1 \sin(\phi_1 + \Delta\phi_{21}) + g^2\beta_1\beta_2\cos^2\phi_1\right]$$

Averaging over the initial phase angle ϕ_1 we find:

$$J_2 = J_1\left[1 - g\sqrt{\beta_1\beta_2}\sin\Delta\phi_{21} + \frac{1}{2}g^2\beta_1\beta_2\right]$$

If the phase advance $\Delta\phi_{21}$ is close to the (optimal) value of 90°, the new action can be written:

$$J_2 \approx J_1 \exp\left(-\frac{2t}{\tau_{FB}}\right)$$

where τ_{FB} is the damping time of the feedback system:

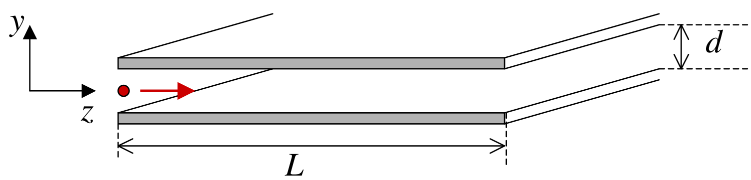
$$\frac{1}{\tau_{FB}} = \frac{g\sqrt{\beta_1\beta_2}\sin\Delta\phi_{21}}{2T_0}$$

Appendix D: Bunch-by-bunch feedback systems

From the required damping rate (set by the fastest growth rate of any of the coupled-bunch modes driven by the wake fields), we can calculate the required gain for the feedback system.

The gain of the feedback system determines the voltage applied to the kicker. We can calculate the voltage as follows.

Consider a kicker consisting of two infinitely wide parallel plates of length L, separated by a distance d and with a voltage V between them.



The deflection of the bunch from passing between the plates is:

$$\Delta p_y = 2\frac{V}{E/e}\frac{L}{d}$$

Appendix D: Bunch-by-bunch feedback systems

The kicker voltage per unit bunch offset is given by:

$$\frac{dV}{dy} = \frac{1}{2}\frac{E}{e}\frac{d}{L}g$$

For example, consider a feedback system used to damp the resistive wall instability in the ILC damping rings. If we assume a maximum growth time of 40 turns, beta functions of 10 m at the pick-up and kicker, and a phase advance of 90° between them, the required gain for the feedback system is:

$$g = 2\frac{T_0}{\tau_{FB}}\frac{1}{\sqrt{\beta_1\beta_2}} = 0.005$$

If we assume kickers of length 20 cm and separated by 2 cm, and a 5 GeV beam, the kicker voltage per unit bunch offset at the pick-up is:

$$\frac{dV}{dy} = \frac{1}{2}\frac{E}{e}\frac{d}{L}g = 1.25\ \text{kV/mm}$$

Appendix D: Bunch-by-bunch feedback systems

Finally, we consider the effect of noise on the pick-up, or in the amplifier. This will lead to some variation in the applied kick from the “correct” value; which will result in some excitation of betatron motion.

Let us represent the noise in the feedback system by the addition of a quantity δ_y to the bunch position measured by the pick-up:

$$y_1 \to y_1 + \delta y$$

This will modify the change in the action resulting from the voltage applied to the kicker:

$$J_2 \approx J_1 \exp\left(-\frac{2t}{\tau_{tot}}\right) + \frac{1}{2}g^2\beta_2\delta y^2$$

Including the effect of noise in the feedback system, we can write the equation of motion for the action as:

$$\frac{dJ}{dt} \approx \frac{g^2\beta_2\langle\delta y^2\rangle}{2T_0} - \frac{2}{\tau_{tot}}J$$

We see that the action reaches an equilibrium:

$$J_{equ} \approx \frac{\tau_{tot}}{4T_0} g^2 \beta_2 \langle \delta y^2 \rangle$$

τ_{tot} is the total damping time, including all damping and antidamping effects (synchrotron radiation, wake fields, feedback system…)

Let us assume that we double the gain of the feedback system, compared to that required to exactly balance the resistive-wall instability, so that:

$$g = 4\frac{T_0}{\tau_{RW}}\frac{1}{\sqrt{\beta_1\beta_2}} \qquad J_{equ} \approx 4\frac{T_0}{\tau_{RW}}\frac{\langle \delta y^2 \rangle}{\beta_1}$$

Let us also assume that the specification on the bunch-to-bunch beam jitter is a fraction f of the beam size:

$$2J_{equ} < f^2 \varepsilon_y$$

This sets an upper limit on the feedback system noise:

$$\langle \delta y^2 \rangle < \frac{f^2}{8}\frac{\tau_{RW}}{T_0}\beta_1 \varepsilon_y$$

As an example, consider the ILC damping rings. Let us assume that f = 10%, the beta function at the pick-up is 10 m, that the resistive-wall growth time is 40 turns, and that the equilibrium vertical emittance is 2 pm.

$$\sqrt{\langle \delta y^2 \rangle} < 1\ \mu\text{m}$$

In other words, the pick-up needs a resolution of better than 1 μm (neglecting any additional noise from the amplifier). This is a challenging, but not unrealistic specification.

The resolution of the feedback system pick-up is an issue for the ILC damping rings because of:

- the high gain required to damp fast coupled-bunch growth rates;
- the very small vertical emittance specification;
- the specification for very low levels of bunch-to-bunch jitter.

The specification can be relaxed by increasing the beta function at the pick-up. However, the resolution specification varies only as the square root of the beta function; so an increase in the beta function by a factor of 10 (to 100 m) would only relax the specificaiton on the pick-up resolution to 3 μm.

Appendix E: Derivation of the dispersion relation from the Vlasov Equation

In general, the revolution frequency depends on the energy deviation. To first order in the energy deviation, we can write:

$$\dot{\theta} = \omega_0 (1 - \alpha_p \delta)$$

where ω_0 is the angular revolution frequency ($\approx 2\pi c/C$) for a particle with the reference energy, and α_p is the momentum compaction factor.

Appendix E: Derivation of the dispersion relation from the Vlasov Equation

We now need to find $\dot{\delta}$.

Let us consider a "coasting beam" model, in which the beam is continuously distributed around the ring, and particles only change their energy deviation as a result of the longitudinal wake fields in the ring. We will justify this apparently crude model of an electron beam in a storage ring later.

Consider further a distribution that can be written as:

$$\Psi(\theta, \delta; t) = \Psi_0(\delta) + \Delta\Psi(\delta) e^{i(n\theta - \omega_n t)}$$

This represents a stationary distribution Ψ_0 which is uniform around the ring but with some arbitrary dependence on the energy deviation; with some perturbation $\Delta\Psi$ with sinusoidal dependence on position around the ring (with n periods in one circumference), and again arbitrary dependence on the energy deviation.

The density perturbation has a time dependence given by the frequency ω_n, which in general can be a function of the spatial dependence of the distribution. The imaginary part of ω_n will determine the stability of the beam.

Appendix E: Derivation of the dispersion relation from the Vlasov Equation

The beam current in our model has two frequency components:

- a DC component, which, if we assume that the impedance $Z_{\|}(\omega)$ vanishes for $\omega \to 0$, makes no contribution to any energy change of the particles in the beam;
- a component at frequency ω_n which, if the impedance has a non-zero component $Z_{\|}(\omega_n)$ leads to an energy change in each revolution.

If the beam distribution is normalised so that:

$$\int_{-\infty}^{\infty} \Psi_0(\delta)\, d\delta = 1$$

and the total beam current is I_0, then the perturbation to the current from the density perturbation $\Delta\Psi$ is:

$$\Delta I(\theta,t) = I_0 \int \Delta\Psi(\delta)\, d\delta\, e^{i(n\theta - \omega_n t)}$$

Appendix E: Derivation of the dispersion relation from the Vlasov Equation

In terms of the variables θ and t, the longitudinal coordinate z of a particle in a "bunch" (for zero energy deviation and assuming velocity $\approx c$) is:

$$z = \frac{\theta}{2\pi} C - ct$$

Hence:

$$\frac{\omega_n z}{c} = \frac{\omega_n}{\omega_0}\theta - \omega_n t \approx n\theta - \omega_n t$$

where we assume that $\omega_n \approx n\omega_0$. The perturbation in the current as a function of z is then:

$$\Delta I(\theta,t) = I_0 \int \Delta\Psi(\delta)\, d\delta\, e^{i\frac{\omega_n z}{c}}$$

The frequency spectrum of the perturbation in the current is:

$$\Delta\tilde{I}(\omega) = \int \Delta I(z)\, e^{-i\frac{\omega z}{c}} \frac{dz}{c} = 2\pi I_0 \int \Delta\Psi(\delta)\, d\delta \cdot \delta(\omega - \omega_n)$$

As expected, the current spectrum contains the single frequency ω_n.

Appendix E: Derivation of the dispersion relation from the Vlasov Equation

We can now calculate the energy loss for a particle in one turn through the accelerator, using the total impedance $Z_{\|}(\omega)$:

$$\frac{\Delta E(z)}{e} = \frac{1}{2\pi}\int \Delta\tilde{I}(\omega) Z_{\|}(\omega) e^{i\frac{\omega z}{c}} d\omega$$

Since the current spectrum contains just a single frequency, this becomes:

$$\frac{\Delta E(z)}{e} = I_0 \int \Delta\Psi(\delta)\, d\delta \cdot Z_{\|}(\omega_n) e^{i\frac{\omega_n z}{c}}$$

$$\Delta\delta(z) = \frac{I_0}{E/e}\int \Delta\Psi(\delta)\, d\delta \cdot Z_{\|}(\omega_n) e^{i\frac{\omega_n z}{c}}$$

The *rate of change* of the energy deviation is then:

$$\dot{\delta} \approx \frac{\Delta\delta}{T_0} = \frac{\omega_0}{2\pi}\frac{I_0}{E/e}\int \Delta\Psi(\delta)\, d\delta \cdot Z_{\|}(\omega_n)\, e^{i(n\theta - \omega_n t)}$$

Appendix E: Derivation of the dispersion relation from the Vlasov Equation

Now we have expressions for the rate of change of the longitudinal variables:

$$\dot{\theta} = \omega_0(1 - \alpha_p \delta)$$

$$\dot{\delta} \approx \frac{\Delta\delta}{T_0} = \frac{\omega_0}{2\pi}\frac{I_0}{E/e}\int \Delta\Psi(\delta)\, d\delta \cdot Z_{\|}(\omega_n)\, e^{i(n\theta - \omega_n t)}$$

which we can substitute into the Vlasov equation:

$$\frac{\partial\Psi}{\partial t} + \dot{\theta}\frac{\partial\Psi}{\partial\theta} + \dot{\delta}\frac{\partial\Psi}{\partial\delta} = 0$$

Keeping terms to first order in the perturbation $\Delta\Psi$, we find that the Vlasov equation becomes:

$$(n\omega - \omega_n)\Delta\Psi(\delta) = i\frac{\omega_0}{2\pi}\frac{I_0}{E/e}\int \Delta\Psi(\delta)\, d\delta \cdot \frac{\partial\Psi_0(\delta)}{\partial\delta} Z_{\|}(\omega_n)$$

where

$$\omega = \omega_0(1 - \alpha_p \delta)$$

Appendix E: Derivation of the dispersion relation from the Vlasov Equation

If we write the Vlasov equation in the form:

$$\Delta\Psi(\delta) = iZ_{\parallel}(\omega_n)\frac{\omega_0}{2\pi}\frac{I_0}{E/e}\int \Delta\Psi(\delta)\,d\delta \cdot \frac{\partial\Psi_0(\delta)/\partial\delta}{(n\omega - \omega_n)}$$

then we observe that by integrating both sides over δ we obtain:

$$1 = iZ_{\parallel}(\omega_n)\frac{\omega_0}{2\pi}\frac{I_0}{E/e}\int \frac{\partial\Psi_0(\delta)/\partial\delta}{(n\omega - \omega_n)}\,\mathrm{d}\delta$$

This is an integral equation, which we need to solve to find the mode frequency ω_n for a given impedance $Z_{\parallel}(\omega_n)$, and a given mode (specified by the mode number n, which gives the number of periods of the density perturbation over the entire circumference of the ring).

This equation relates the mode frequency ω_n to the mode number n; it is therefore usually called the "dispersion relation".

Appendix F: The Keil-Schnell-Boussard criterion

Typically, we expect to find that a beam in an electron storage ring has a Gaussian energy spread:

$$\Psi_0(\delta) = \frac{e^{-\frac{\delta^2}{2\sigma_\delta^2}}}{\sqrt{2\pi}\sigma_\delta}$$

Substituting this into the dispersion relation gives:

$$1 = i\frac{Z_{\parallel}(\omega_n)}{n}\frac{1}{(2\pi)^{3/2}}\frac{I_0}{E/e}\frac{1}{\alpha_p\sigma_\delta^2}\int \frac{\zeta e^{-\frac{1}{2}\zeta^2}}{\zeta + \Delta_n}d\zeta$$

where

$$\Delta_n = \frac{\omega_n - n\omega_0}{n\omega_0\alpha_p\sigma_\delta}$$

Appendix F: The Keil-Schnell-Boussard criterion

Let us write the dispersion relation in the form:

$$\frac{1}{(2\pi)^{3/2}}\frac{I_0}{E/e}\frac{1}{\alpha_p\sigma_\delta^2}\frac{Z_\parallel(\omega_n)}{n} = U + iV$$

where:

$$U + iV = \left[i\int \frac{\zeta e^{-\frac{1}{2}\zeta^2}}{\zeta + \Delta_n} d\zeta \right]^{-1}$$

If the impedance is known, we can plot the real and imaginary parts of:

$$\frac{1}{(2\pi)^{3/2}}\frac{I_0}{E/e}\frac{1}{\alpha_p\sigma_\delta^2}\frac{Z_\parallel(n\omega_0)}{n}$$

for a range of values of n. The modes for any part of the curve lying in the region:

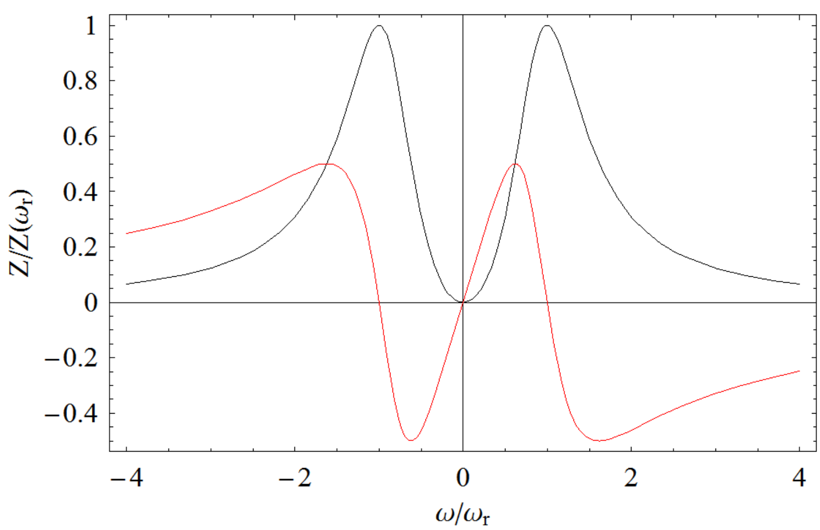
$$\mathrm{Im}\,\Delta_n > 0 \qquad \Rightarrow \qquad \mathrm{Im}\,\omega_n > 0$$

are unstable.

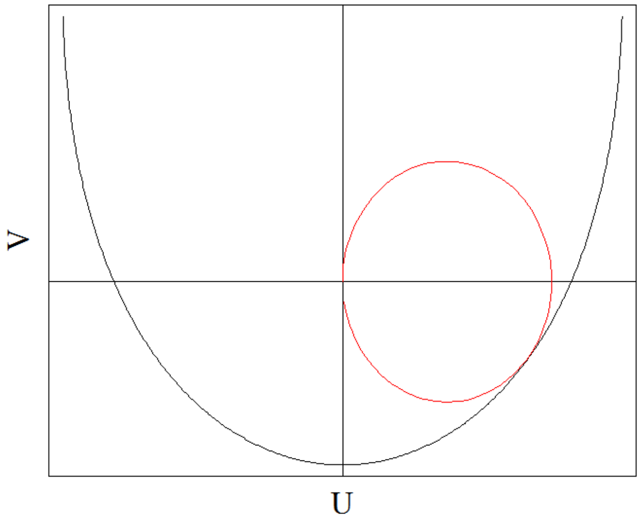
Appendix F: The Keil-Schnell-Boussard criterion

As an example, consider an impedance represented by a broad-band resonator:

$$\frac{Z_\parallel(\omega)}{n} \approx \frac{Z_\parallel(\omega_r)}{n}\frac{1 - i\frac{\omega^2-\omega_r^2}{\omega_r\omega}}{1 + \frac{(\omega^2-\omega_r^2)^2}{\omega_r^2\omega^2}}$$



Appendix F: The Keil-Schnell-Boussard criterion



Let us plot (red curve) the real and imaginary parts of:

$$\frac{1}{(2\pi)^{3/2}} \frac{I_0}{E/e} \frac{1}{\alpha_p \sigma_\delta^2} \frac{Z_\parallel(n\omega_0)}{n}$$

and (black curve) the boundary given by:

$$\mathrm{Im}\,\Delta_n = 0$$

we find that the curves touch for:

$$\frac{1}{(2\pi)^{3/2}} \frac{I_0}{E/e} \frac{1}{\alpha_p \sigma_\delta^2} \frac{Z_\parallel(\omega_r)}{n} \approx \frac{\pi}{6}$$

This gives the stability condition:

$$I_0 < \frac{\sqrt{2\pi^5}}{3} \alpha_p \sigma_\delta^2 \frac{E/e}{Z_\parallel(\omega_r)/n}$$

Appendix F: The Keil-Schnell-Boussard criterion

In the stability diagram shown on the previous slide, the first mode to become unstable has a frequency given by:

$$\omega \approx 1.2\omega_r$$

We expect the resonant frequency of the broad-band impedance to satisfy:

$$\omega_r >> \frac{c}{b}$$

where b is the beam pipe radius. Since, in an electron storage ring, the beam pipe radius is typically of the same order of magnitude as the bunch length, this means that, for the first mode to become unstable, we can expect:

$$\frac{C}{n} << \sigma_z$$

In other words, the period of the unstable mode is likely to be shorter than the bunch length. In this situation, if the timescale of the instability is short compared to the synchrotron period, a bunched beam can behave like a coasting beam, from point of view of the instability.

Appendix F: The Keil-Schnell-Boussard criterion

The stability condition that we derived for the broad-band impedance was:

$$I_0 < \frac{\sqrt{2\pi^5}}{3} \alpha_p \sigma_\delta^2 \frac{E/e}{Z_\parallel(\omega_r)/n}$$

where I_0 is the average current. We assume (following Boussard) that for a bunched beam, we can use the same stability condition, but simply replace the average current by the peak current:

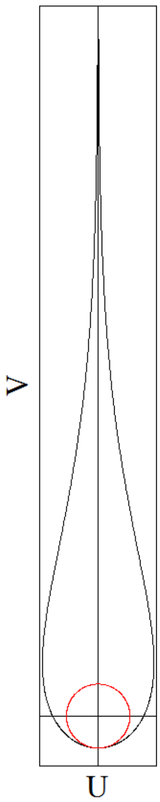
$$I_0 \to \hat{I} = \frac{ecN_0}{\sqrt{2\pi}\sigma_z}$$

where N_0 is the number of particles in a bunch, and σ_z is the rms bunch length. We then find that the stability condition can be written as:

$$\frac{Z_\parallel(\omega_r)}{n} < \frac{\pi^2}{6} Z_0 \frac{\gamma \alpha_p \sigma_\delta^2 \sigma_z}{r_e N_0}$$

Note that this expression tells us that the beam is always unstable if $\sigma_\delta = 0$, or if $\alpha_p = 0$; which agrees with our earlier results for a "cold" beam.

Appendix F: The Keil-Schnell-Boussard criterion



The broad-band resonator model is not usually very good for storage rings. In the design of an accelerator, a significant amount of effort goes into modelling the impedance, so as to be able to determine the instability thresholds. But at an early stage, it is difficult to know what the impedance is likely to look like.

In this situation, an approximation that is sometimes made, is simply to replace the boundary obtained from $\mathrm{Im}\Delta_n = 0$ with a circle of radius $1/\sqrt{2\pi}$. In that case, the stability condition becomes:

$$\frac{1}{(2\pi)^{3/2}} \frac{I_0}{E/e} \frac{1}{\alpha_p \sigma_\delta^2} \left| \frac{Z_\parallel(n\omega_0)}{n} \right| < \frac{1}{\sqrt{2\pi}}$$

or:

$$\left| \frac{Z_\parallel}{n} \right| < 2\pi \frac{E/e}{I_0} \alpha_p \sigma_\delta^2$$

This is known as the Keil-Schnell criterion.

Note that the exact shape of the instability boundary depends on the shape of the energy distribution (in this case, Gaussian).

Appendix F: The Keil-Schnell-Boussard criterion

We can apply the Keil-Schnell criterion to bunched beams, by making the same assumption as before; namely, that we can simply replace the average current (of a coasting beam) by the peak current (of a bunched beam).

The result is known as the Keil-Schnell-Boussard criterion:

$$\left|\frac{Z_{\parallel}}{n}\right| < \sqrt{\frac{\pi}{2}} Z_0 \frac{\gamma \alpha_p \sigma_\delta^2 \sigma_z}{r_e N_0}$$

Note that the result is very close to that we obtained for the broad-band impedance model; there just some variation in the numerical constant.

As should be apparent, the Keil-Schnell-Boussard criterion cannot give anything other than a very crude estimate of the instability threshold in a storage ring. However, it may be good enough for a rough order-of-magnitude estimate in cases where an impedance model is not available.

Wherever possible, several different methods (including, for example, tracking) should be used to obtain reliable estimates of instability thresholds.