2nd International Accelerator School for Linear Colliders, Erice, Sicily
1 – 10 October, 2007

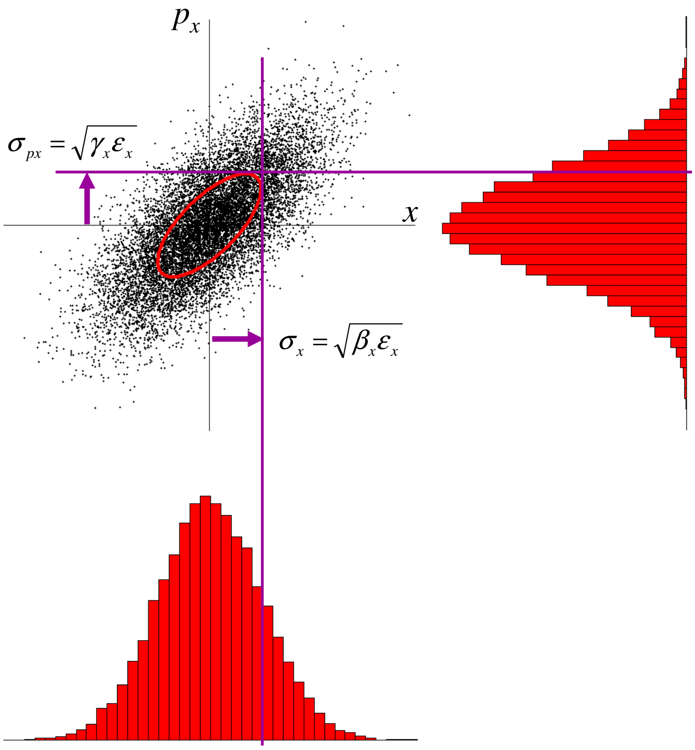
Damping Rings

Lecture 2
Low Emittance Storage Rings

Andy Wolski
University of Liverpool and the Cockcroft Institute

Contents

Summary of Lecture 1: Emittance



$$\varepsilon_x = \sqrt{\langle x^2\rangle\langle p_x^2\rangle - \langle x p_x\rangle^2}$$

Area of ellipse = $\pi\varepsilon_x$

$$\langle x^2\rangle = \beta_x \varepsilon_x$$

$$\langle p_x^2\rangle = \gamma_x \varepsilon_x$$

$$\langle x p_x\rangle = -\alpha_x \varepsilon_x$$

$$\beta_x \gamma_x - \alpha_x^2 = 1$$

Summary of Lecture 1: synchrotron radiation damping

The energy loss per turn is given by:

$$U_0 = \frac{C_\gamma}{2\pi} E_0^4 I_2 \qquad C_\gamma = 8.846\times 10^{-5} \text{ m/GeV}^3$$

The emittances damp as:

$$\varepsilon(t) = \varepsilon(0)\exp\left(-\frac{2t}{\tau}\right)$$

The radiation damping times are given by:

$$\tau_x = \frac{2}{j_x}\frac{E_0}{U_0}T_0 \qquad \tau_y = \frac{2}{j_y}\frac{E_0}{U_0}T_0 \qquad \tau_z = \frac{2}{j_z}\frac{E_0}{U_0}T_0$$

The damping partition numbers are:

$$j_x = 1 - \frac{I_4}{I_2} \qquad j_y = 1 \qquad j_z = 2 + \frac{I_4}{I_2}$$

Summary of Lecture 1: synchrotron radiation integrals

The first, second and fourth synchrotron radiation integrals are:

$$I_1 = \oint \frac{\eta_x}{\rho} ds$$

$$I_2 = \oint \frac{1}{\rho^2} ds$$

$$I_4 = \oint \frac{\eta_x}{\rho} \left(\frac{1}{\rho^2} + 2k_1 \right) ds \qquad k_1 = \frac{e}{P_0} \frac{\partial B_y}{\partial x}$$

Lecture 2: Low Emittance Storage Rings

The objectives of this lecture are:

- to explain how quantum effects lead to excitation of horizontal and longitudinal emittance;
- to derive expressions for the equilibrium horizontal emittance and energy spread, by considering the balance between radiation damping and quantum excitation;
- to derive expressions for the natural emittance in different types of lattice;
- to discuss how wigglers can be used to reduce the natural emittance in a storage ring (as well as reducing the damping times);
- to derive an expression for the natural emittance in a wiggler-dominated storage ring.

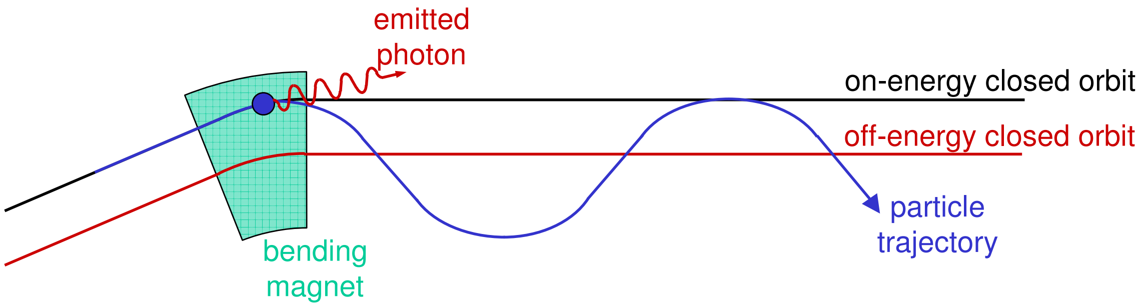
Quantum excitation

If radiation were a purely classical process, the emittances would damp to zero.

However radiation is emitted in discrete units (photons), which induces some "noise" on the beam.

The effect of the noise is to increase the emittance.

The beam eventually reaches an equilibrium determined by a balance between the radiation damping and the quantum excitation.



Quantum excitation of horizontal emittance

By considering the change in the phase-space variables when a particle emits radiation carrying momentum dp, we find that the associated change in the betatron action is:

$$dJ_x = -w_1 \frac{dp}{P_0} + w_2 \left(\frac{dp}{P_0} \right)^2$$

where w_1 and w_2 are functions of the Twiss parameters, the dispersion, and the phase-space variables (see Lecture 1 Appendix A).

The time evolution of the action can then be written:

$$\frac{dJ_x}{dt} = -w_1 \frac{1}{P_0} \frac{dp}{dt} + w_2 \frac{dp}{P_0^2} \frac{dp}{dt}$$

In the classical approximation, we can take $dp \to 0$ in the limit of small time interval, $dt \to 0$. In this approximation, the second term on the right hand side in the above equation vanishes, and we are left only with damping. But since radiation is quantized, it makes no real sense to take $dp \to 0$…

Quantum excitation of horizontal emittance

To take account of the quantization of synchrotron radiation, we write the time-evolution of the action as:

$$\frac{dJ_x}{dt} = -w_1 \frac{1}{P_0}\frac{dp}{dt} + w_2 \frac{dp}{P_0^2}\frac{dp}{dt} \qquad \therefore \qquad \frac{dJ_x}{dt} = -w_1 \dot{N}\frac{\langle u \rangle}{P_0 c} + w_2 \dot{N}\frac{\langle u^2 \rangle}{P_0^2 c^2}$$

where u is the photon energy, and $\dot{N}$ is the number of photons emitted per unit time.

In Appendix A (of this Lecture), we show that this leads to the equation for the evolution of the emittance, including both damping and excitation:

$$\frac{d\varepsilon_x}{dt} = -\frac{2}{\tau_x}\varepsilon_x + \frac{2}{j_x \tau_x} C_q \gamma^2 \frac{I_5}{I_2}$$

where the fifth synchrotron radiation integral I_5 is given by:

$$I_5 = \oint \frac{\mathcal{H}_x}{|\rho^3|} ds$$

and the "quantum constant" C_q is given by: $C_q = \frac{55}{32\sqrt{3}}\frac{\hbar}{mc} \approx 3.832 \times 10^{-13}$ m

Quantum excitation of horizontal emittance

The equilibrium horizontal emittance is found from:

$$\left.\frac{d\varepsilon_x}{dt}\right|_{\varepsilon_x = \varepsilon_0} = 0 \qquad \therefore \qquad \frac{2}{\tau_x}\varepsilon_0 = \frac{2}{j_x \tau_x} C_q \gamma^2 \frac{I_5}{I_2}$$

The equilibrium horizontal emittance is given by:

$$\varepsilon_0 = C_q \gamma^2 \frac{I_5}{j_x I_2}$$

Note that ε_0 is determined by the beam energy, the lattice functions (Twiss parameters and dispersion) in the dipoles, and the bending radius in the dipoles.

ε_0 is sometimes called the “natural emittance” of the lattice, since it is the horizontal emittance that will be achieved in the limit of zero bunch charge: as the current is increased, interactions betweens particles in a bunch can increase the emittance above the equilibrium determined by radiation effects.

Quantum excitation of vertical emittance

In many storage rings, the vertical dispersion in the absence of alignment, steering and coupling errors is zero, so $\mathcal{H}_y = 0$. However, the equilibrium vertical emittance is larger than zero, because the vertical opening angle of the radiation excites some vertical betatron oscillations.

The fundamental lower limit on the vertical emittance, from the opening angle of the synchrotron radiation, is given by(1):

$$\varepsilon_y = \frac{13}{55}\frac{C_q}{j_y I_2}\oint \frac{\beta_y}{|\rho^3|}ds$$

In most storage rings, this is an extremely small value, typically four orders of magnitude smaller than the natural (horizontal) emittance.

In practice, the vertical emittance is dominated by magnet alignment errors. Storage rings typically operate with a vertical emittance that is of order 1% of the horizontal emittance, but many can achieve emittance ratios somewhat smaller than this.

(1) T. Raubenheimer, SLAC Report 387, p.19 (1991).

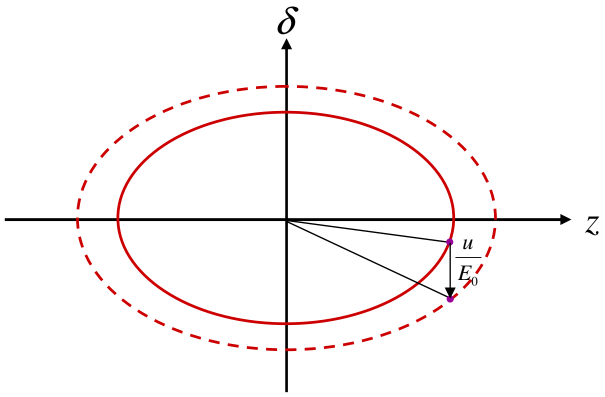
Quantum excitation of synchrotron oscillations

Quantum effects excite longitudinal emittance as well as transverse emittance. Consider a particle with longitudinal coordinate z and energy deviation δ, which emits a photon of energy u.

$$\delta' = \hat{\delta}'\sin\theta' = \hat{\delta}\sin\theta - \frac{u}{E_0}$$

$$z' = \frac{\alpha_p c}{\omega_s}\hat{\delta}'\cos\theta' = \frac{\alpha_p c}{\omega_s}\hat{\delta}\cos\theta$$

$$\therefore \quad \hat{\delta}'^2 = \hat{\delta}^2 - 2\hat{\delta}\frac{u}{E_0}\sin\theta + \frac{u^2}{E_0^2}$$



Averaging over the bunch gives:

$$\Delta\sigma_\delta^2 = \frac{\langle u^2\rangle}{2E_0^2} \qquad \text{where} \qquad \sigma_\delta^2 = \frac{1}{2}\langle\hat{\delta}^2\rangle$$

Quantum excitation of synchrotron oscillations

Including the effects of radiation damping, the evolution of the energy spread is:

$$\frac{d\sigma_\delta^2}{dt} = \frac{1}{2E_0^2 C_0} \oint \dot{N}\langle u^2 \rangle\, ds - \frac{2}{\tau_z}\sigma_\delta^2$$

Using equation (A3) from Appendix A for $\dot{N}\langle u^2\rangle$, we find:

$$\frac{d\sigma_\delta^2}{dt} = C_q \gamma^2 \frac{2}{j_z \tau_z} \frac{I_3}{I_2} - \frac{2}{\tau_z}\sigma_\delta^2$$

We find the equilibrium energy spread from $d\sigma_\delta^2/dt = 0$:

$$\sigma_{\delta 0}^2 = C_q \gamma^2 \frac{I_3}{j_z I_2}$$

The third synchrotron radiation integral I_3 is given by:

$$I_3 = \oint \frac{1}{|\rho^3|}\, ds$$

Natural energy spread

The equilibrium energy spread determined by radiation effects is:

$$\sigma_{\delta 0}^2 = C_q \gamma^2 \frac{I_3}{j_z I_2}$$

This is often referred to as the "natural" energy spread, since collective effects can often lead to an increase in the energy spread with increasing bunch charge.

The natural energy spread is determined essentially by the beam energy and by the bending radii of the dipoles. Note that the natural energy spread *does not depend on the RF parameters (either voltage or frequency)*.

The corresponding equilibrium bunch length is:

$$\sigma_z = \frac{\alpha_p c}{\omega_s}\sigma_\delta$$

We can increase the synchrotron frequency ω_s, and hence reduce the bunch length, by increasing the RF voltage, or by increasing the RF frequency.

Summary: radiation damping and quantum excitation

Including the effects of radiation damping and quantum excitation, the emittances vary as:

$$\varepsilon(t) = \varepsilon(0)\exp\left(-\frac{2t}{\tau}\right) + \varepsilon(\infty)\left[1-\exp\left(-\frac{2t}{\tau}\right)\right]$$

The damping times are given by:

$$j_x\tau_x = j_y\tau_y = j_z\tau_z = 2\frac{E_0}{U_0}T_0$$

The damping partition numbers are given by:

$$j_x = 1-\frac{I_4}{I_2} \qquad j_y = 1 \qquad j_z = 2+\frac{I_4}{I_2}$$

The energy loss per turn is given by:

$$U_0 = \frac{C_\gamma}{2\pi}E_0^4 I_2 \qquad C_\gamma = 8.846\times10^{-5}\ \text{m/GeV}^3$$

Summary: radiation damping and quantum excitation

The natural emittance is:

$$\varepsilon_0 = C_q\gamma^2\frac{I_5}{j_x I_2} \qquad C_q = 3.832\times10^{-13}\ \text{m}$$

The natural energy spread and bunch length are given by:

$$\sigma_\delta^2 = C_q\gamma^2\frac{I_3}{j_z I_2} \qquad \sigma_z = \frac{\alpha_p c}{\omega_s}\sigma_\delta$$

The momentum compaction factor is:

$$\alpha_p = \frac{I_1}{C_0}$$

The synchrotron frequency and synchronous phase are given by:

$$\omega_s^2 = -\frac{eV_{RF}}{E_0}\frac{\omega_{RF}}{T_0}\alpha_p\cos\varphi_s \qquad \sin\varphi_s = \frac{U_0}{eV_{RF}}$$

Summary: synchrotron radiation integrals

The synchrotron radiation integrals are:

$$I_1 = \oint \frac{\eta_x}{\rho} ds$$

$$I_2 = \oint \frac{1}{\rho^2} ds$$

$$I_3 = \oint \frac{1}{|\rho|^3} ds$$

$$I_4 = \oint \frac{\eta_x}{\rho}\left(\frac{1}{\rho^2} + 2k_1\right) ds \qquad k_1 = \frac{e}{P_0}\frac{\partial B_y}{\partial x}$$

$$I_5 = \oint \frac{\mathcal{H}_x}{|\rho|^3} ds \qquad \mathcal{H}_x = \gamma_x \eta_x^2 + 2\alpha_x \eta_x \eta_{px} + \beta_x \eta_{px}^2$$

Application to the ILC damping rings

Many of the ILC damping ring parameters are constrained by luminosity requirements and technology limitations in other systems.

The main parameter constraints are:

Injected normalised positron emittance	0.01 m
Extracted normalised horizontal emittance	8 µm
Extracted normalised vertical emittance	20 nm
Beam store time	200 ms

We also have a number of parameter choices:

Damping ring circumference	6.7 km
Beam energy	5 GeV
Equilibrium normalised vertical emittance	14 nm
Main dipole field strength	0.146 T

Given these parameters, we can calculate (a homework problem!):

1. The damping times that we need to achieve in the damping rings.
2. The energy loss per turn needed in the damping rings.
3. The integrated wiggler field strength needed to achieve the energy loss.

Application to the ILC damping rings

With strong enough dipoles, or long enough wiggler to provide synchrotron radiation energy loss, we can achieve the necessary damping times.

But what about the extracted emittances?

The upper limit on the longitudinal emittance is not too demanding: but the transverse emittances must be very small, compared to those achieved in most existing storage rings.

The horizontal emittance is a function of the lattice design. How do we design a lattice to achieve the specified horizontal emittance?

Lecture 2 objectives: lattices for low-emittance electron storage rings

In the rest of this lecture, we shall:

- derive expressions for the natural emittance in two types of lattice:
  - FODO
  - TME (theoretical minimum emittance)
- discuss how wigglers can be used to reduce the natural emittance in a storage ring (as well as reducing the damping times);
- derive an expression for the natural emittance in a wiggler-dominated storage ring.

In Appendices B and C, we give some material in which we:

- discuss the double-bend achromat (DBA) lattice, and derive an expression for the natural emittance in a DBA lattice.
- consider how the emittance of an achromat may be reduced by "detuning" from the zero-dispersion conditions.
- derive an expression for the natural emittance in a multi-bend achromat, including the triple-bend achromat (TBA).

Calculating the natural emittance in a lattice

In the first part of this lecture, we showed that the natural emittance is given by:

$$\varepsilon_0 = C_q \gamma^2 \frac{I_5}{j_x I_2}$$

where C_q is a physical constant, γ is the relativistic factor, j_x is the horizontal damping partition number, and I_5 and I_2 are synchrotron radiation integrals

j_x, I_5 and I_2 are all functions of the lattice, and independent of the beam energy.

In most storage rings, if the bends have no quadrupole component, the damping partition number $j_x \approx 1$. In this case, we just need to evaluate the two synchrotron radiation integrals:

$$I_5 = \int \frac{\mathcal{H}_x}{\rho^3} ds \qquad\qquad I_2 = \int \frac{1}{\rho^2} ds$$

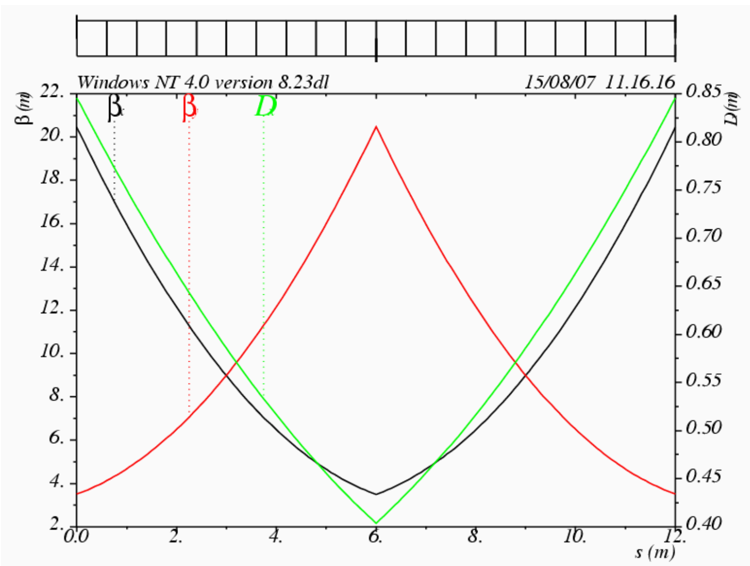
If we know the strength and length of all the dipoles in the lattice, it is straightforward to evaluate I_2.

Evaluating I_5 is more complicated: it depends on the lattice functions…

Case 1: natural emittance in a FODO lattice

Let us consider the case of a simple FODO lattice. To simplify this case, we will use the following approximations:

- the quadrupoles are represented as thin lenses;
- the space between the quadrupoles is completely filled by the dipoles.



Case 1: natural emittance in a FODO lattice

With the approximations in the previous slide, the lattice functions (Twiss parameters and dispersion) are completely determined by the following parameters:

- the focal length f of a quadrupole;
- the bending radius ρ of a dipole;
- the length L of a dipole.

The bending angle θ of a dipole is given by: $\theta = \frac{L}{\rho}$

In terms of these parameters, the horizontal beta function and dispersion at the centre of the horizontally-focusing quadrupole are given by:

$$\beta_x = \frac{4f\rho\sin\theta(2f\cos\theta + \rho\sin\theta)}{\sqrt{16f^4 - \left[\rho^2 - (4f^2+\rho^2)\cos 2\theta\right]^2}} \qquad \eta_x = \frac{2f\rho(2f + \rho\tan\frac{\theta}{2})}{4f^2+\rho^2}$$

By symmetry, at the centre of a quadrupole, $\alpha_x = \eta_{px} = 0$.

Case 1: natural emittance in a FODO lattice

We also know how to evolve the lattice functions through the lattice, using the transfer matrices, M.

For the Twiss parameters, we use: $A(s) = M \cdot A(0) \cdot M^{\mathrm{T}}$

where: $A = \begin{pmatrix} \beta_x & -\alpha_x \\ -\alpha_x & \gamma_x \end{pmatrix}$

The dispersion can be evolved using: $\begin{pmatrix} \eta_x \\ \eta_{px} \end{pmatrix}_s = M \cdot \begin{pmatrix} \eta_x \\ \eta_{px} \end{pmatrix}_{s=0} + \begin{pmatrix} \rho(1-\cos\frac{s}{\rho}) \\ \sin\frac{s}{\rho} \end{pmatrix}$

For a thin quadrupole, the transfer matrix is given by: $M = \begin{pmatrix} 1 & 0 \\ -1/f & 0 \end{pmatrix}$

For a dipole, the transfer matrix is given by: $M = \begin{pmatrix} \cos\frac{s}{\rho} & \rho\sin\frac{s}{\rho} \\ -\frac{1}{\rho}\sin\frac{s}{\rho} & \cos\frac{s}{\rho} \end{pmatrix}$

Case 1: natural emittance in a FODO lattice

With the expressions for the Twiss parameters and dispersion from the previous two slides, we can evaluate the synchrotron radiation integral I_5.

Note: by symmetry, we need to evaluate the integral in only one of the two dipoles in the FODO cell.

The algebra is rather formidable. The result is most easily expressed as a power series in the dipole bending angle θ. We find that:

$$\frac{I_5}{I_2} = \left(4 + \frac{\rho^2}{f^2}\right)^{-\frac{3}{2}} \left[8 - \frac{\rho^2}{2f^2}\theta^2 + O\left(\theta^4\right)\right]$$

Case 1: natural emittance in a FODO lattice

To find a simple expression for the natural emittance, we make two further approximations:

- We assume that the dipole bending angle θ is small, i.e. $\theta << 1$, in which case only the first term in the series expression for I_5/I_2 survives.
- We assume that the bending radius ρ is large compared to the quadrupole focal length f, i.e. $\rho >> f$.

With these two assumptions, we have:

$$\frac{I_5}{I_2} \approx 8\frac{f^3}{\rho^3}$$

Making the approximation $j_x \approx 1$ (since we have no quadrupole component in the dipole), and writing $\rho = L/\theta$, we have:

$$\varepsilon_0 \approx C_q \gamma^2 \left(\frac{2f}{L}\right)^3 \theta^3$$

Case 1: natural emittance in a FODO lattice

We have derived an approximate expression for the natural emittance of a lattice consisting entirely of FODO cells:

$$\varepsilon_0 \approx C_q \gamma^2 \left(\frac{2f}{L} \right)^3 \theta^3$$

Notice how the emittance scales with the beam and lattice parameters:

- The emittance is proportional to the *square* of the energy.
- The emittance is proportional to the *cube* of the bending angle. Increasing the number of cells in a complete circular lattice reduces the bending angle of each dipole, and reduces the emittance.
- The emittance is proportional to the *cube* of the quadrupole focal length. Stronger quadrupoles have shorter focal lengths, and reduce the emittance.
- The emittance is inversely proportional to the *cube* of the cell (or dipole) length. Shortening the cell reduces the lattice functions, and reduces the emittance.

Case 1: natural emittance in a FODO lattice

The phase advance in a FODO cell is given by:

$$\cos\mu_x = 1 - \frac{L^2}{2f^2}$$

This means that a stable lattice must have:

$$\frac{f}{L} \geq \frac{1}{2}$$

In the limiting case, $\mu_x = 180°$, and we have the *minimum emittance in a FODO lattice*:

$$\frac{f}{L} = \frac{1}{2} \qquad \varepsilon_{0,FODO,\min} \approx C_q \gamma^2 \theta^3$$

More typically, a FODO lattice might have a phase advance per cell $\mu_x = 90°$, in which case:

$$\frac{f}{L} = \frac{1}{\sqrt{2}} \qquad \varepsilon_0 \approx C_q \gamma^2 2\sqrt{2}\theta^3$$

Case 1: natural emittance in a FODO lattice

Using the above formulae, we estimate that to achieve a natural emittance of 0.5 nm in a 5 GeV storage ring constructed from FODO cells with 90° phase advance per cell, then neglecting any benefits from a wiggler, we would need 372 dipoles.

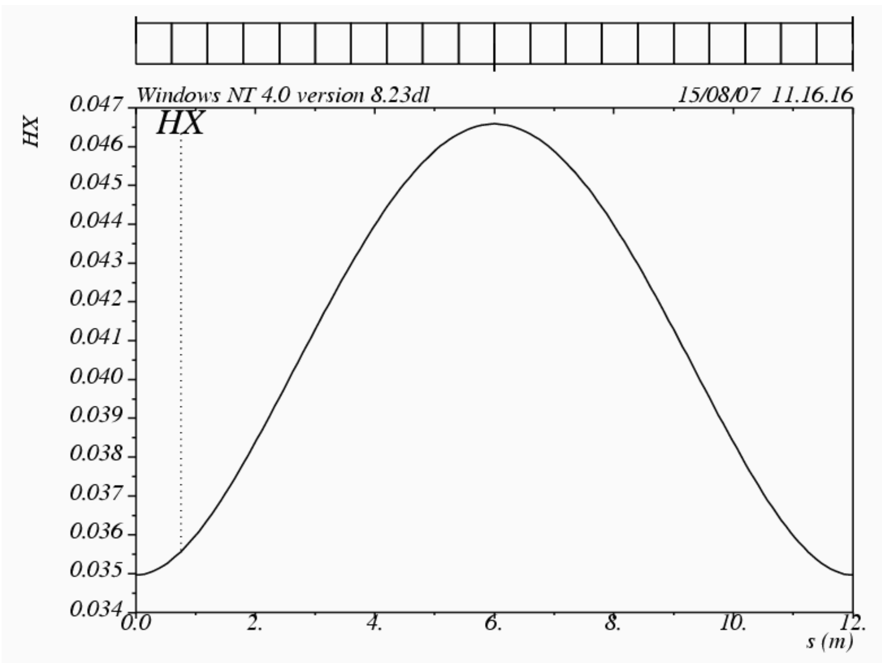
> In fact, we should not neglect benefits from a wiggler… but we haven't quite got that far, yet.

That's a lot of magnets. Is it possible to design the lattice more efficiently, to achieve the same natural emittance with fewer magnets?

A clue is provided if we look at the curly-H function in a FODO lattice…

Case 1: natural emittance in a FODO lattice

The curly-H function remains at a relatively large, roughly constant value throughout the lattice. Perhaps we can reduce it…



Case 2: natural emittance in a TME lattice

The FODO lattice is a very simple structure, but one which is designed without any real attempt to minimise the natural emittance.

We have seen that increasing the phase advance per cell reduces the natural emittance - but perhaps we can do better by considering more carefully the conditions for minimising the natural emittance.

To derive the conditions for the "theoretical minimum emittance" (TME), we write down an expression for:

$$I_5 = \int \frac{\mathcal{H}_x}{\rho^3} ds$$

with arbitrary initial dispersion η_0, η_{p0}, and Twiss parameters α_0 and β_0 in a dipole with given bending radius ρ and angle θ.

Then we minimise I_5 with respect to variations in η_0, η_{p0}, α_0 and β_0…

…the principles are straightforward, but the algebra is rather complicated.

Case 2: natural emittance in a TME lattice

The result is:

$$\varepsilon_{0,TME,\min} \approx \frac{1}{12\sqrt{15}} C_q \gamma^2 \theta^3$$

The minimum emittance is obtained with dispersion at the entrance to a dipole:

$$\eta_0 = \frac{1}{6} L\theta + O(\theta^3) \qquad \eta_{p0} = -\frac{\theta}{2} + O(\theta^3)$$

and with Twiss functions at the entrance:

$$\beta_0 = \frac{8}{\sqrt{15}} L + O(\theta^3) \qquad \alpha_0 = \sqrt{15} + O(\theta^2)$$

Case 2: natural emittance in a TME lattice

Note that with the conditions for minimum emittance:

$$\eta_0 = \frac{1}{6} L\theta + O(\theta^3) \qquad \eta_{p0} = -\frac{\theta}{2} + O(\theta^3)$$

$$\beta_0 = \frac{8}{\sqrt{15}} L + O(\theta^3) \qquad \alpha_0 = \sqrt{15} + O(\theta^2)$$

the dispersion and the beta function reach minima in the centre of the dipole. The values at the centre of the dipole are:

$$\eta_{\min} = \rho\left(1 - 2\frac{\sin\frac{\theta}{2}}{\theta}\right) = \frac{L\theta}{24} + O(\theta^4)$$

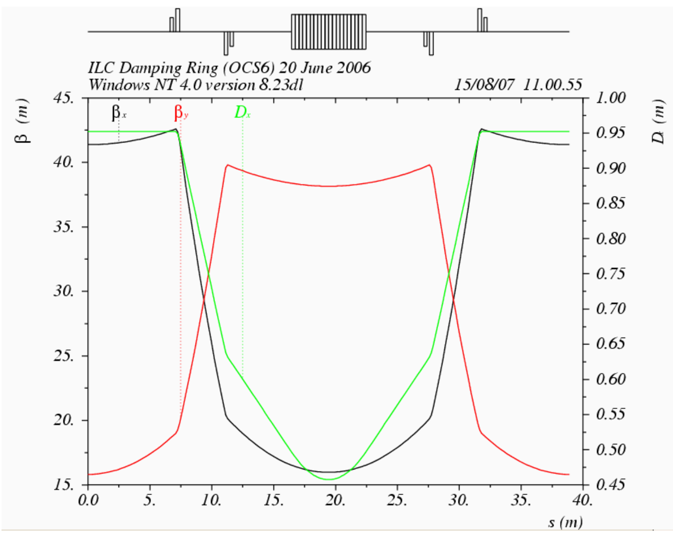
$$\beta_{\min} = \frac{L}{2\sqrt{15}} + O(\theta^3)$$

What do the lattice functions look like in a single cell of a TME lattice?

Because of symmetry in the dipole, we can consider a TME lattice cell as containing a single dipole (as opposed to two dipoles, which we had in the cases of the FODO and DBA lattices)…

Case 2: natural emittance in a TME lattice

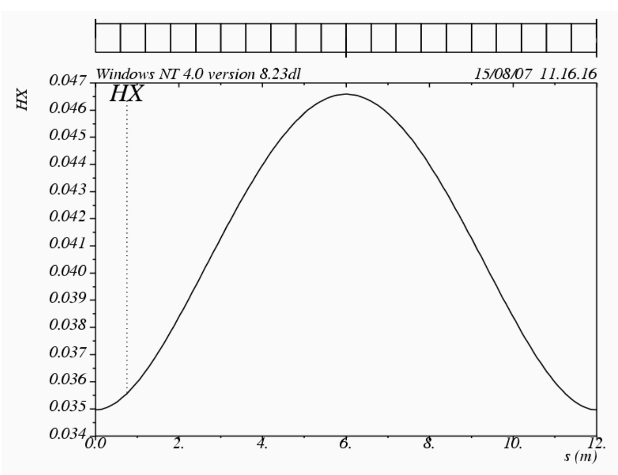
Example of lattice functions in a single cell of a "TME" lattice (different cases may look somewhat different):

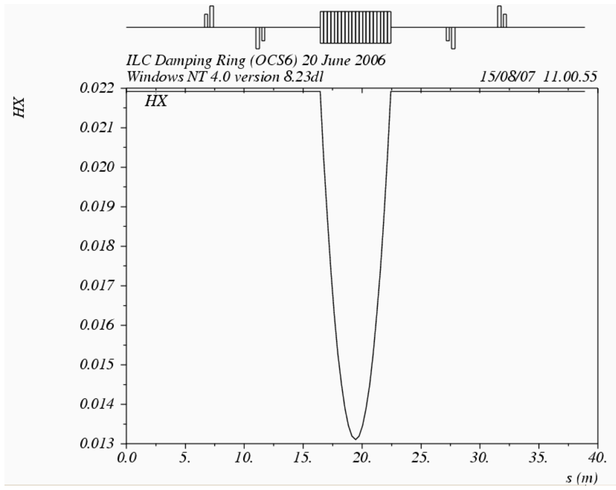


Note: *the lattice functions shown in this example are for a single arc cell in the present ILC damping rings baseline lattice. This does not actually achieve the exact conditions needed for absolute minimum emittance. A more complicated lattice would be needed for this…*

Case 2: natural emittance in a TME lattice

Compare the curly-H function in a FODO cell (left) with those in a TME cell (right), with the same dipole parameters in each case:





The curly-H function is significantly smaller in the TME lattice: this was our design objective.

Summary: natural emittance in FODO, DBA and TME lattices

Lattice Style	Minimum Emittance	Conditions
90° FODO	$\varepsilon_0 \approx 2\sqrt{2} C_q \gamma^2 \theta^3$	$\frac{f}{L} = \frac{1}{\sqrt{2}}$
180° FODO	$\varepsilon_0 \approx C_q \gamma^2 \theta^3$	$\frac{f}{L} = \frac{1}{2}$
DBA (see Appendix B)	$\varepsilon_0 \approx \frac{1}{4\sqrt{15}} C_q \gamma^2 \theta^3$	$\eta_0 = \eta_{p0} = 0$ $\beta_0 \approx \sqrt{12/5} L \qquad \alpha_0 \approx \sqrt{15}$
TME	$\varepsilon_0 \approx \frac{1}{12\sqrt{15}} C_q \gamma^2 \theta^3$	$\eta_{\min} \approx \frac{L\theta}{24} \qquad \beta_{\min} \approx \frac{L}{2\sqrt{15}}$

Note: the approximations are valid for small dipole bending angle, θ.

Comments on lattice design for low emittance lattices

The results we have derived have been for "ideal" lattices that perfectly achieve the stated conditions in each case.

In practice, lattices rarely, if ever, achieve the ideal conditions. In particular, the beta function in an achromat is usually not optimal for low emittance; and the dispersion and beta function in a TME lattice are not optimal.

The main reasons for this are:

- It is difficult to control the beta function and dispersion to achieve the ideal low-emittance conditions with a small number of quadrupoles.
- There are other strong dynamical constraints on the design that we have not considered: in particular, the lattice needs a large dynamic aperture to achieve a good beam lifetime.

The dynamic aperture issue is particularly difficult for low emittance lattices. The dispersion in low emittance lattices is generally low, while the strong focusing leads to high chromaticity. Therefore, very strong sextupoles are often needed to correct the natural chromaticity. This limits the dynamic aperture.

The consequence of all these issues is that in practice, the natural emittance of a lattice of a given type is usually somewhat larger than might be expected using the formulae given here.

Further Options and Issues

There are (of course) many other options besides FODO and TME for the lattice "style".

In Appendices B and C, we discuss:

- The DBA lattice (popular in third-generation synchrotron light sources).
- Detuning the DBA to reduce the emittance.
- Use of multi-bend achromats to reduce the emittance.

In the remainder of the main part of this lecture, we will discuss:

- Effects of insertion devices on the natural emittance in a storage ring.
- Natural emittance in wiggler-dominated storage rings.

Effect of insertion devices on the natural emittance

Insertion devices such as wigglers and undulators are commonly used in third generation light sources to generate radiation with particular properties.

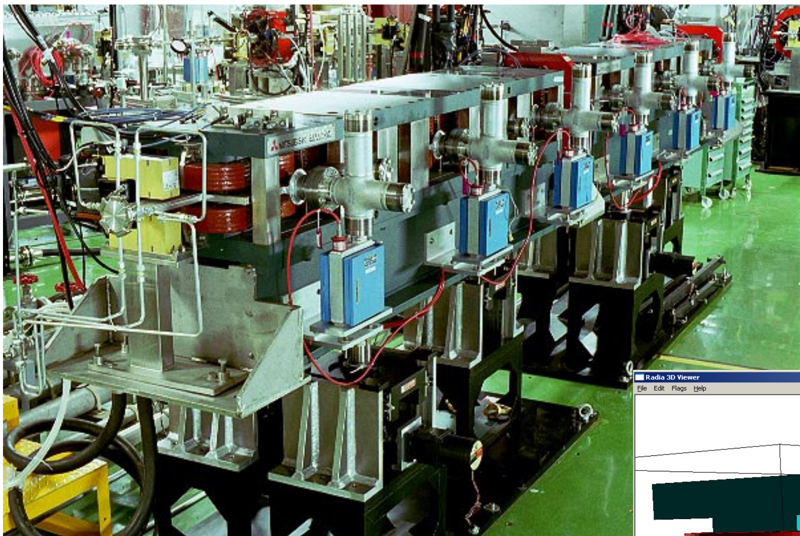
Usually, insertion devices are designed so that the integral of the field along the length of the device is zero: therefore, the overall geometry of the machine is not changed. However, since they produce radiation, they will contribute to the synchrotron radiation integrals, and hence affect the natural emittance of the lattice.

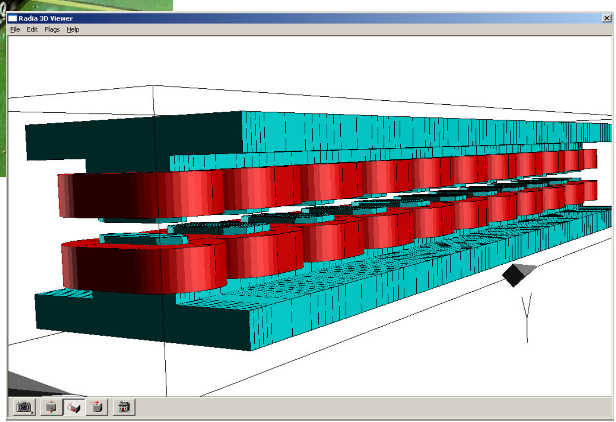
If a wiggler or undulator is inserted at a location with zero dispersion, then in the approximation that we neglect the dispersion generated by the device itself, there will be no contribution to I_5; however, there will be a non-zero contribution to I_2 (the energy loss of a particle).

Hence, since the natural emittance is given by the ratio I_5/I_2, wigglers and undulators can reduce the natural emittance of the beam. In effect, they enhance the radiation damping while making little contribution to the quantum excitation.

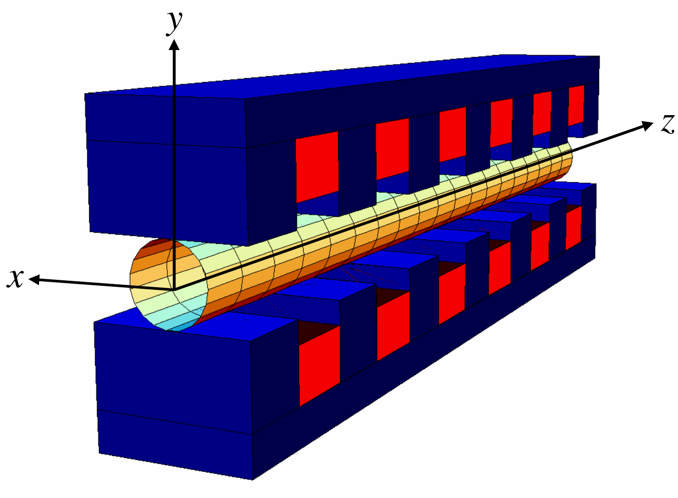
However, to obtain a reasonably accurate value for the natural emittance, we have to consider the dispersion generated by the insertion device itself.

An electromagnetic wiggler (from the KEK-ATF)





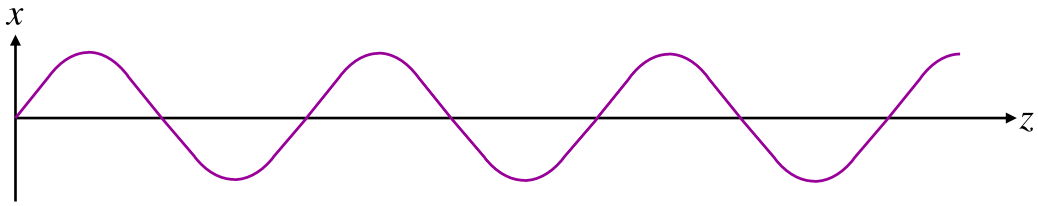
A simple model of a wiggler



$$B_y = B_w \sin(k_z z)$$

Peak field = B_w

Period = $\lambda_w = \dfrac{2\pi}{k_z}$



Wigglers increase the energy loss from synchrotron radiation

The total energy loss per turn is given in terms of the second synchrotron radiation integral:

$$U_0 = \frac{C_\gamma}{2\pi} E_0^4 I_2 \qquad\qquad I_2 = \oint \frac{1}{\rho^2}\, ds$$

The integral extends over the entire circumference of the ring. The contribution from the wigglers is:

$$I_{2w} = \int_0^{L_w} \frac{1}{\rho^2}\, ds = \frac{1}{(B\rho)^2} \int_0^{L_w} B^2\, ds \approx \frac{1}{(B\rho)^2} \frac{B_w^2 L_w}{2}$$

The approximation comes from the fact that we neglect end effects.

Note that I_{2w} depends only on the peak field and the total length of wiggler (and the beam energy), and is independent of the wiggler period.

Wiggler contribution to the natural emittance

The natural emittance depends on the second and fifth synchrotron radiation integrals:

$$\varepsilon_0 = C_q \gamma^2 \frac{I_5}{j_x I_2} \qquad C_q = 3.832 \times 10^{-13} \text{ m}$$

$$I_2 = \oint \frac{1}{\rho^2} ds \qquad I_5 = \oint \frac{\mathcal{H}_x}{|\rho|^3} ds \qquad \mathcal{H}_x = \gamma_x \eta_x^2 + 2\alpha_x \eta_x \eta_{px} + \beta_x \eta_{px}^2$$

The contribution of the wiggler to I_5 depends on the beta function in the wiggler. Let us assume that the beta function is constant (or changing slowly), so $\alpha_x \approx 0$.

Then, to calculate I_5, we just need to know the dispersion…

Dispersion generated in a wiggler

In a dipole of bending radius ρ and quadrupole gradient k_1, the dispersion obeys the equation:

$$\frac{d^2\eta_x}{ds^2} + K\eta_x = \frac{1}{\rho} \qquad K = \frac{1}{\rho^2} + k_1$$

Assuming that $k_1 = 0$ in the wiggler, we can write the equation for η_x as:

$$\frac{d^2\eta_x}{ds^2} + \frac{B_w^2}{(B\rho)^2} \eta_x \sin^2 k_w s = \frac{B_w}{B\rho} \sin k_w s$$

For $k_w \rho_w >> 1$, we can neglect the second term on the left, and we find:

$$\eta_x \approx -\frac{\sin k_w s}{\rho_w k_w^2} \qquad \eta_{px} \approx -\frac{\cos k_w s}{\rho_w k_w}$$

Wiggler contribution to the natural emittance

The wiggler contribution to I_5 can be written:

$$I_{5w} \approx \int_0^{L_w} \frac{\beta_x \eta_{px}^2}{|\rho|^3} ds \approx \frac{\langle \beta_x \rangle}{\rho_w^2 k_w^2} \int_0^{L_w} \frac{\cos^2 k_w s}{|\rho|^3} ds = \frac{\langle \beta_x \rangle}{\rho_w^5 k_w^2} \int_0^{L_w} \left| \sin^3 k_w s \right| \cos^2 k_w s \, ds$$

Using:

$$\left\langle \left| \sin^3 k_w s \right| \cos^2 k_w s \right\rangle = \frac{4}{15\pi}$$

we have:

$$I_{5w} \approx \frac{4}{15\pi} \frac{\langle \beta_x \rangle L_w}{\rho_w^5 k_w^2}$$

Natural emittance in a wiggler-dominated storage ring

Combining expressions for I_{2w} and I_{5w}, *then, in the case that the wiggler dominates the contributions to I_2 and I_5*, we can write for the natural emittance:

$$\varepsilon_0 \approx \frac{8}{15\pi} C_q \gamma^2 \frac{\langle \beta_x \rangle}{\rho_w^3 k_w^2}$$

Using short period wigglers, we can achieve small emittances, if the wiggler field is not too high, and the wigglers are placed at locations with small horizontal beta function.

Note that in the vast majority of electron storage rings, insertion devices only account for 10% - 20% of the synchrotron radiation energy loss: so the above formula cannot be used to calculate the emittance.

However, in the damping rings of a future linear collider, wigglers would account for around 90% of the synchrotron radiation energy loss. The natural emittance will be dominated by the wiggler parameters.

Using wigglers makes FODO lattices realistic alternatives to the (present baseline) TME lattice.

Wiggler contribution to the natural energy spread

The natural energy spread is given in terms of the second and third synchrotron radiation integrals:

$$\sigma_\delta^2 = C_q \gamma^2 \frac{I_3}{j_z I_2} \qquad I_3 = \oint \frac{1}{|\rho|^3} ds \qquad C_q = 3.832 \times 10^{-13} \text{ m}$$

Since I_3 does not depend on the dispersion, the wiggler potentially makes a significant contribution to the energy spread. Writing for the bending radius in the wiggler:

$$\frac{1}{\rho} = \frac{B}{B\rho} = \frac{B_w}{B\rho} \sin k_w s = \frac{1}{\rho_w} \sin k_w s$$

we find:

$$I_{3w} = \frac{1}{\rho_w^3} \int_0^{L_w} \left| \sin^3 k_w s \right| ds = \frac{4 L_w}{3\pi \rho_w^3}$$

If the wiggler dominates the synchrotron radiation energy loss, then the natural energy spread in the ring will be given by:

$$\sigma_\delta^2 \approx \frac{4}{3\pi} C_q \frac{\gamma^2}{\rho_w} = \frac{4}{3\pi} \frac{e}{mc} C_q \gamma B_w$$

Vertical emittance and coupling

So far, we have focused on the horizontal and longitudinal emittances.

If the vertical motion is not coupled to the horizontal (betatron coupling) or to the longitudinal (vertical dispersion), then the vertical emittance will be determined by the opening angle of the synchrotron radiation, ~ $1/\gamma^2$.

In a high energy storage ring (such as the ILC damping rings), the fundamental lower limit on the vertical emittance is much smaller than the vertical emittance achieved in operation.

In practice, the vertical emittance is determined by betatron coupling and vertical dispersion, which are generated by magnet alignment and tuning errors.

Vertical emittance and coupling

Vertical dispersion is generated by vertical steering errors.

Steering errors can come from vertical misalignments of the quadrupoles.

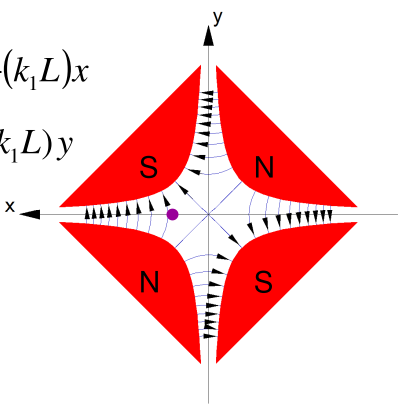
Vertical dispersion leads to the generation of vertical emittance in the same way that horizontal dispersion leads to the generation of horizontal emittance.

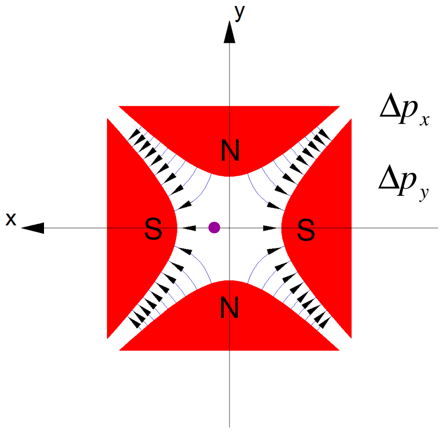
Betatron coupling is generated by quadrupole roll errors and vertical misalignments of sextupoles.

The consequence of betatron coupling is that the vertical motion depends on the horizontal, and vice-versa. Any effect that generates horizontal emittance will then also generate vertical emittance.

$$\Delta p_x = -(k_1 L)x$$

$$\Delta p_y = (k_1 L)y$$





$$\Delta p_x = (k_1 L)y$$

$$\Delta p_y = (k_1 L)x$$

Vertical emittance and coupling

Reducing the vertical emittance at the interaction point increases the luminosity. The goal in the damping rings is to achieve the lowest possible vertical emittance…

…although "dilution" of the vertical emittance in the downstream systems means that there is little benefit in trying to achieve the fundamental lower limit.

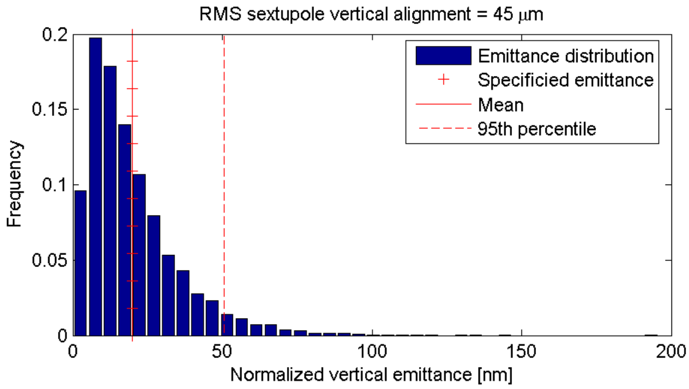
The present specification is for an extracted normalised vertical emittance of 20 nm, which corresponds to 2 pm geometric.

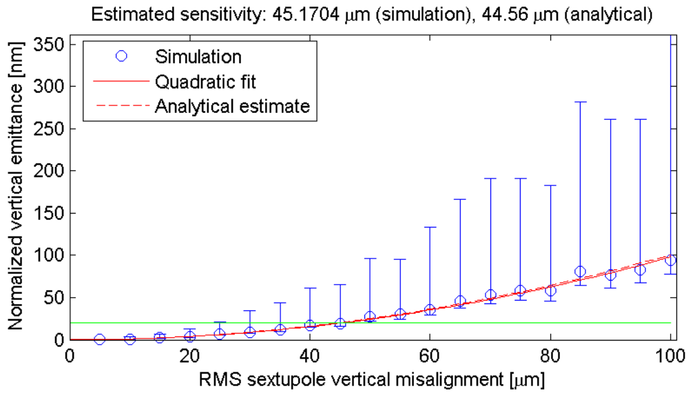
The lowest vertical emittance achieved in any storage ring to date is 4.5 pm, in the KEK ATF.

Tuning for ultra-low vertical emittance is a major issue for linear collider damping rings. Achieving and maintaining 2 pm vertical emittance will require:

- a lattice design with low sensitivity to magnet alignment errors.
- precise initial alignment of magnets (of order 100 µm for quadrupoles);
- effective techniques for precise characterisation and correction of errors generating vertical dispersion and betatron coupling;
- high-performance, high-stability instrumentation and diagnostics for measurement of beam position and beam size;
- highly stable magnet supports and instrumentation.

Example: sensitivity to sextupole alignment in ILC damping rings





Summary 1

The natural emittance in a storage ring is determined by the balance between the radiation damping (given by I_2) and the quantum excitation (given by I_5).

The quantum excitation depends on the lattice functions. Different "styles" of lattice can be used, depending on the emittance specification for the storage ring.

In general, for small bending angle θ the natural emittance can be written as:

$$\varepsilon_0 \approx F C_q \gamma^2 \theta^3$$

where θ is the bending angle of a single dipole, and the numerical factor F is determined by the lattice style:

Lattice style	F
90° FODO	$2\sqrt{2}$
180° FODO	1
Double-bend achromat (DBA) - see Appendix B	$1/4\sqrt{15}$
Theoretical minimum emittance (TME)	$1/12\sqrt{15}$

Summary 2

Wigglers can be used to:

- reduce the radiation damping times by increasing the energy loss per turn;
- reduce the natural emittance, if placed at locations with low beta function, very low dispersion, and if they have short period and not too high magnetic field.

The natural emittance in a wiggler-dominated storage ring is given by:

$$\varepsilon_0 \approx \frac{8}{15\pi} C_q \gamma^2 \frac{\langle \beta_x \rangle}{\rho_w^3 k_w^2}$$

Damping wigglers also contribute to the energy spread. The natural energy spread in a wiggler-dominated storage ring can be found from:

$$\sigma_\delta^2 \approx \frac{4}{3\pi} C_q \frac{\gamma^2}{\rho_w} = \frac{4}{3\pi} \frac{e}{mc} C_q \gamma B_w$$

Appendices

Appendix A: Quantum excitation of horizontal emittance

In deriving the equation of motion (Lecture 1, A4) for the action of a particle emitting synchrotron radiation, we made the classical approximation that in a time interval dt, the momentum of the radiation emitted dp goes to zero as dt goes to zero.

In reality, emission of radiation is quantized, so writing "$dp \to 0$" actually makes no sense.

Taking into account the quantization of radiation, the equation of motion for the action (Lecture 1, A1) should be written:

$$dJ_x = -w_1 \frac{dp}{P_0} + w_2 \left(\frac{dp}{P_0} \right)^2 \qquad \therefore \qquad \frac{dJ_x}{dt} = -w_1 \dot{N} \frac{\langle u \rangle}{P_0 c} + w_2 \dot{N} \frac{\langle u^2 \rangle}{P_0^2 c^2} \qquad \text{(A1)}$$

where $\dot{N}$ is the number of photons emitted per unit time.

The first term on the right hand side of (A1) just gives the same radiation damping as in the classical approximation. The second term on the right hand side of (A1) is an excitation term that we previously neglected…

Appendix A: Quantum excitation of horizontal emittance

Averaging around the circumference of the ring, the quantum excitation term can be written:

$$w_2 \dot{N} \frac{\langle u^2 \rangle}{P_0^2 c^2} \approx \frac{1}{C_0} \oint w_2 \dot{N} \frac{\langle u^2 \rangle}{P_0^2 c^2} \, ds$$

Using equation (Lecture 1, A3) for w_2, we find that (for $x << \eta_x$ and $p_x << \eta_{px}$) the excitation term can be written:

$$w_2 \dot{N} \frac{\langle u^2 \rangle}{P_0^2 c^2} \approx \frac{1}{2E_0^2 C_0} \oint \mathcal{H}_x \dot{N} \langle u^2 \rangle \, ds$$

where the "curly-H" function $\mathcal{H}_x$ is given by:

$$\mathcal{H}_x = \gamma_x \eta_x^2 + 2\alpha_x \eta_x \eta_{px} + \beta_x \eta_{px}^2$$

Appendix A: Quantum excitation of horizontal emittance

Including both (classical) damping and (quantum) excitation terms, and averaging over all particles in the bunch, we find that the horizontal emittance evolves as:

$$\frac{d\varepsilon_x}{dt} = -\frac{2}{\tau_x}\varepsilon_x + \frac{1}{2E_0^2 C_0}\oint \dot{N}\left\langle u^2 \right\rangle \mathcal{H}_x \, ds \qquad \text{(A2)}$$

We quote the result (from quantum radiation theory):

$$\dot{N}\left\langle u^2 \right\rangle = 2C_q \gamma^2 E_0 \frac{P_\gamma}{\rho} \qquad \text{(A3)}$$

where the "quantum constant" C_q is:

$$C_q = \frac{55}{32\sqrt{3}} \frac{\hbar}{mc} \approx 3.832\times 10^{-13} \text{ m}$$

Appendix A: Quantum excitation of horizontal emittance

Using equation (A3), and equation (Lecture 1, A5) for P_γ, and the results:

$$j_x \tau_x = 2\frac{E_0}{U_0} T_0 \qquad\qquad U_0 = \frac{C_\gamma}{2\pi} c E_0^4 I_2$$

we find that equation (A2) for the evolution of the emittance can be written:

$$\frac{d\varepsilon_x}{dt} = -\frac{2}{\tau_x}\varepsilon_x + \frac{2}{j_x \tau_x} C_q \gamma^2 \frac{I_5}{I_2}$$

where the fifth synchrotron radiation integral I_5 is given by:

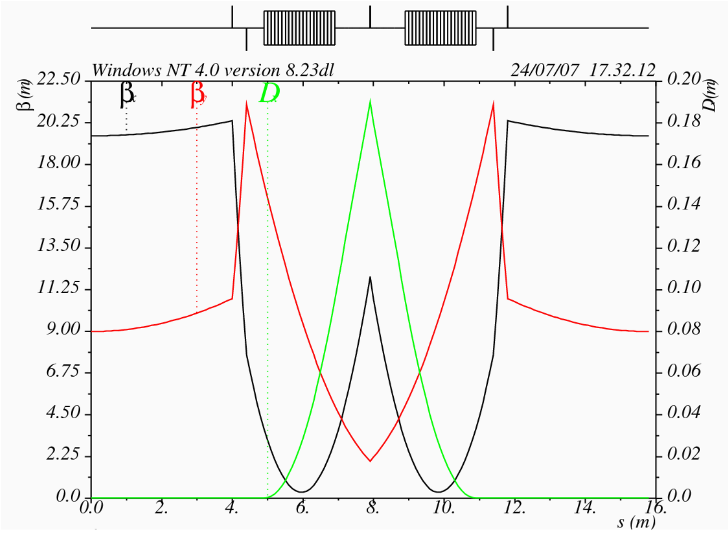
$$I_5 = \oint \frac{\mathcal{H}_x}{\left|\rho^3\right|} ds$$

Note that the excitation term is independent of the emittance: it does not simply modify the damping time, but leads to a non-zero equilibrium emittance.

Appendix B: Natural emittance in a DBA lattice

The double bend achromat (DBA) is a popular lattice style for third generation synchrotron light sources. DBA lattices produce lower natural emittance than FODO lattices using the same number of dipoles.

The "achromat" condition is that the dispersion vanishes at either end of a pair of dipoles (forming a unit cell in the lattice). Thus, there are zero dispersion straight sections which are good locations for insertion devices, i.e. undulators and wigglers.



Appendix B: Natural emittance in a DBA lattice

First of all, let us consider the constraints needed to achieve zero dispersion at either end of the cell.

Assuming that we start at one end of the cell with zero dispersion, then, by symmetry, the dispersion at the other end of the cell will also be zero if the central quadrupole simply reverses the gradient of the dispersion.

In the thin lens approximation, this condition can be written:

$$\begin{pmatrix} 1 & 0 \\ -1/f & 0 \end{pmatrix} \cdot \begin{pmatrix} \eta_x \\ \eta_{px} \end{pmatrix} = \begin{pmatrix} \eta_x \\ \eta_{px} - \dfrac{\eta_x}{f} \end{pmatrix} = \begin{pmatrix} \eta_x \\ -\eta_{px} \end{pmatrix}$$

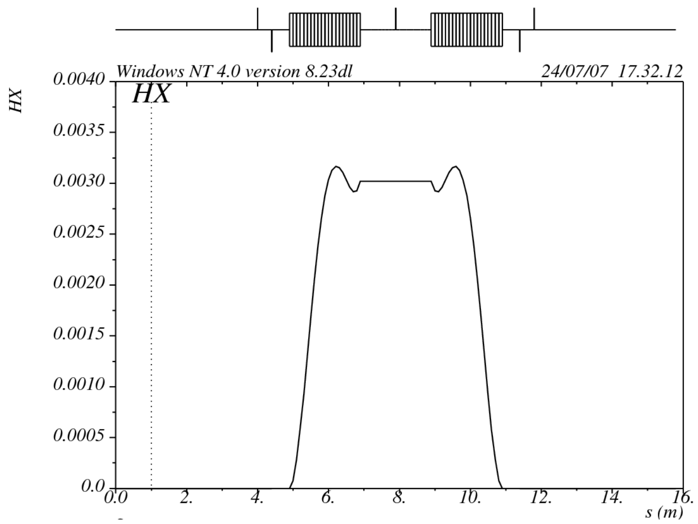
Hence, the central quadrupole must have focal length: $f = \dfrac{\eta_x}{2\eta_{px}}$

The actual value of the dispersion is determined by the dipole bending angle θ, the bending radius ρ, and the drift length L:

$$\eta_x = \rho(1 - \cos\theta) + L\sin\theta \qquad \eta_{px} = \sin\theta$$

Appendix B: Natural emittance in a DBA lattice

Is this type of lattice likely to have a lower natural emittance than a FODO lattice? We can get an idea by looking at the curly-H function.



Note that we use the same dipoles (bending radius and length) for our example in both cases (FODO and DBA). In the DBA lattice the curly-H function is reduced by a significant factor, compared to the FODO lattice.

Appendix B: Natural emittance in a DBA lattice

Let us calculate the minimum natural emittance of a DBA lattice, for given bending radius ρ and bending angle θ in the dipoles.

To do this, we need to calculate the minimum value of:

$$I_5 = \int \frac{\mathcal{H}_x}{\rho^3} ds$$

in one dipole, subject to the constraints:

$$\eta_0 = \eta_{p0} = 0$$

where η_0 and η_{p0} are the dispersion and the gradient of the dispersion at the entrance of a dipole.

We know how the dispersion and the Twiss parameters evolve through the dipole, so we can calculate I_5 for one dipole, for given initial values of the Twiss parameters α_0 and β_0.

Then, we simply have to minimise the value of I_5 with respect to α_0 and β_0.

Again, the algebra is rather formidable, and the full expression for I_5 is not especially enlightening…

Appendix B: Natural emittance in a DBA lattice

We find that, for given ρ and θ and with the constraints:

$$\eta_0 = \eta_{p0} = 0$$

the minimum value of I_5 is given by:

$$I_{5,\min} = \frac{1}{4\sqrt{15}} \frac{\theta^4}{\rho} + O(\theta^6)$$

which occurs for values of the Twiss parameters at the entrance to the dipole:

$$\beta_0 = \sqrt{\frac{12}{5}} L + O(\theta^3) \qquad \alpha_0 = \sqrt{15} + O(\theta^2)$$

where $L = \rho\theta$ is the length of a dipole.

Since:

$$I_2 = \int \frac{1}{\rho^2} ds = \frac{\theta}{\rho}$$

we can immediately write an expression for the minimum emittance in a DBA lattice…

Appendix B: Natural emittance in a DBA lattice

$$\varepsilon_{0,DBA,\min} = C_q \gamma^2 \frac{I_{5,\min}}{j_x I_2} \approx \frac{1}{4\sqrt{15}} C_q \gamma^2 \theta^3$$

The approximation is valid for small θ. Note that we have again assumed that, since there is no quadrupole component in the dipole, $j_x \approx 1$.

Compare the above expression with that for the minimum emittance in a FODO lattice:

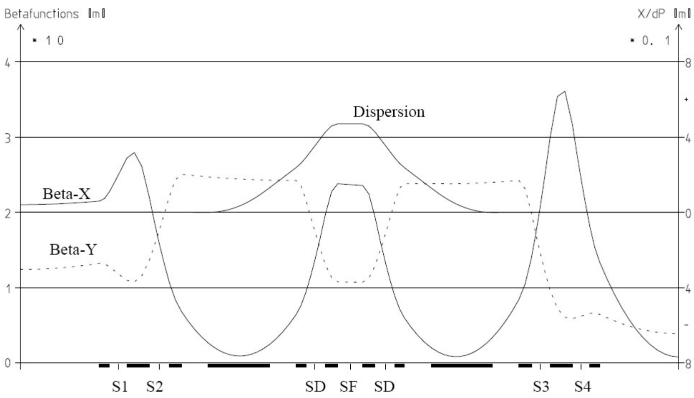
$$\varepsilon_{0,FODO,\min} \approx C_q \gamma^2 \theta^3$$

The minimum emittance in each case scales with the square of the beam energy, and with the cube of the bending angle of a dipole. However, the minimum emittance in a DBA lattice is smaller than that in a FODO lattice (for given energy and dipole bending angle) by a factor $4\sqrt{15} \approx 15.5$.

This is a significant improvement… but can we do even better?

Appendix B: DBA lattices in third generation synchrotron light sources

Lattices composed of DBA cells have been a popular choice for third generation synchrotron light sources.



Lattice functions in an early version of the ESRF lattice.

The DBA structure provides a lower natural emittance than a FODO lattice with the same number of dipoles

The long, dispersion-free straight sections provide ideal locations for insertion devices such as undulators and wigglers.

Appendix B: "Detuning" the DBA lattice

If an insertion device, such as an undulator or wiggler, is incorporated in a storage ring at a location with large dispersion, then the dipole fields in the device can make a significant contribution to the quantum excitation (I_5).

As a result, the insertion device can lead to an increase in the natural emittance of the storage ring.

By using a DBA lattice, we provide dispersion-free straights in which we can locate undulators and wigglers without blowing up the natural emittance.

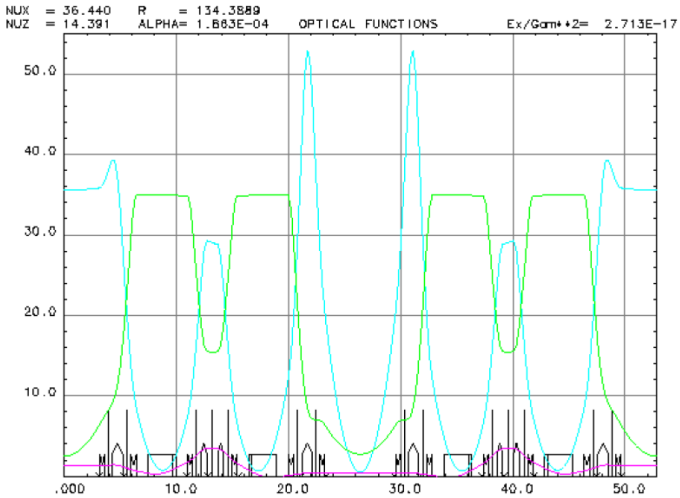
However, there is some tolerance. In many cases, it is possible to "detune" the lattice from the strict DBA conditions, thereby allowing some reduction in natural emittance at the cost of some dispersion in the straights.

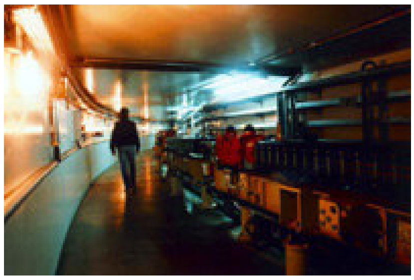
The insertion devices will then contribute to the quantum excitation; but depending on the lattice and the insertion devices, there may still be a net benefit in the reduction of the natural emittance compared to a lattice with zero dispersion in the straights.

Appendix B: “Detuning” the DBA lattice

Some light sources that were originally designed with zero-dispersion straights take advantage of tuning flexibility to operate routinely with dispersion in the straights, thus achieving lower natural emittance and providing better output for users.

For example, the ESRF…





Appendix C: Multi-bend achromats

In principle, it is possible to combine the DBA and TME lattices by having an arc cell consisting of more than two dipoles.

- The dipoles at either end of the cell have zero dispersion (and gradient of the dispersion) at their outside faces, thus satisfying the “achromat” condition.
- The lattice is tuned so that in the “central” dipoles, the Twiss parameters and dispersion satisfy the TME conditions.

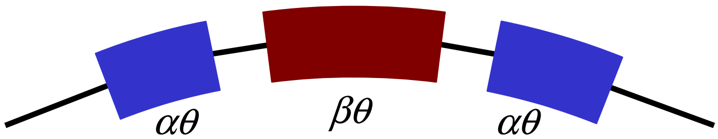
Since the lattice functions are different in the central dipoles compared to the end dipoles, we have additional degrees of freedom we can use to minimise the quantum excitation.

Therefore, it is possible to have cases where the end dipoles and central dipoles differ in:

- the bend angle (i.e. length of dipole), and/or
- the bend radius (i.e. strength of dipole).

Appendix C: Multi-bend achromats

For simplicity, let us consider the case where the dipoles all have the same bending radius (i.e. they all have the same field strength), but vary in length.



Assuming each arc cell has a fixed number, M, of dipoles, the bending angles must satisfy:

$$2\alpha + (M-2)\beta = M$$

Since the synchrotron radiation integrals are additive, for an M-bend achromat we can write:

$$I_{5,cell} \approx 2\frac{1}{4\sqrt{15}}\frac{(\alpha\theta)^4}{\rho} + (M-2)\frac{1}{12\sqrt{15}}\frac{(\beta\theta)^4}{\rho} = \frac{6\alpha^4 + (M-2)\beta^4}{12\sqrt{15}}\frac{\theta^4}{\rho}$$

$$I_{2,cell} = 2\frac{\alpha\theta}{\rho} + (M-2)\frac{\beta\theta}{\rho} = [2\alpha + (M-2)\beta]\frac{\theta}{\rho}$$

Appendix C: Multi-bend achromats

Hence, in an M-bend achromat,

$$\frac{I_{5,cell}}{I_{2,cell}} \approx \frac{1}{12\sqrt{15}}\left[\frac{6\alpha^4 + (M-2)\beta^4}{2\alpha + (M-2)\beta}\right]\theta^3$$

Minimising the ratio I_5/I_2 with respect to α gives:

$$\frac{\alpha}{\beta} = \frac{1}{\sqrt[3]{3}} \qquad \frac{6\alpha^4 + (M-2)\beta^4}{2\alpha + (M-2)\beta} \approx \frac{M+1}{M-1}$$

Hence, the natural emittance in an M-bend achromat is given by:

$$\varepsilon_0 \approx C_q\gamma^2 \frac{1}{12\sqrt{15}}\frac{M+1}{M-1}\theta^3 \qquad 2 < M < \infty$$

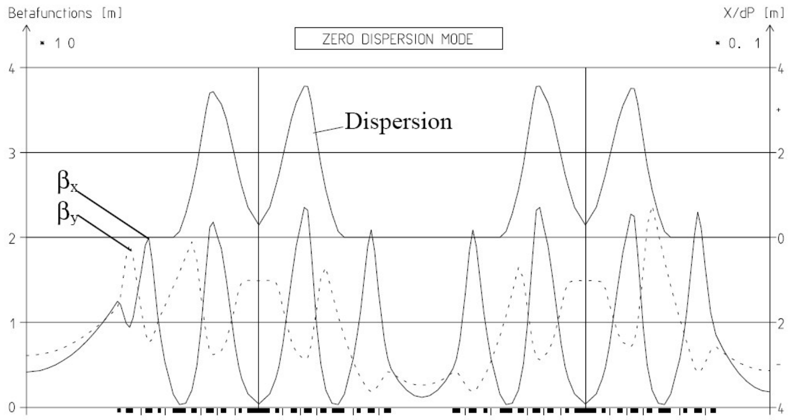
Note that θ is the *average* bending angle per dipole: the central bending magnets should be longer than the outer bending magnets by a factor $\sqrt[3]{3}$.

Of course, the emittance can always be reduced by "detuning" the achromat to allow dispersion in the straights…

Appendix C: Multi-bend achromats

The storage ring in the Swiss Light Source consists of 12 TBA (triple bend achromat) cells, has a circumference of 288 m, and beam energy 2.4 GeV.

In the "zero dispersion" mode, the natural emittance is 4.8 nm·rad.



Appendix C: Multi-bend achromats

Detuning the achromat to allow dispersion in the straights reduces the natural emittance in the SLS from 4.8 nm·rad to 3.9 nm·rad (a reduction of about 20% compared to the zero-dispersion case).