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Damping Rings

Lecture 1
Damping Ring Basics

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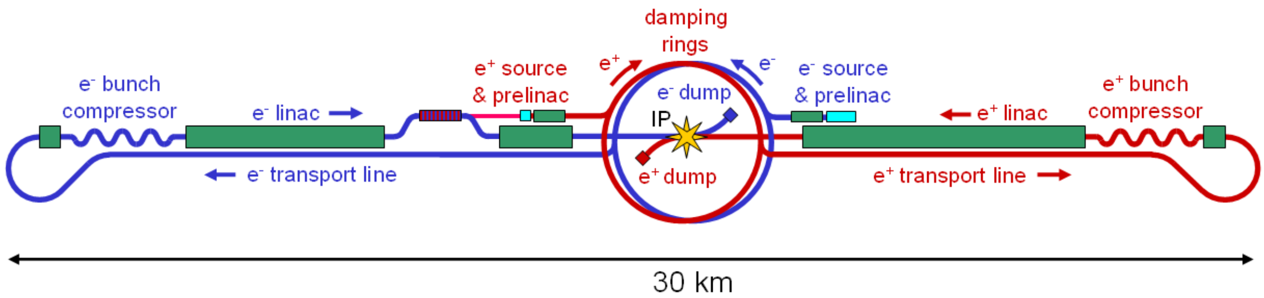
The objectives of this lecture are:

- to introduce the damping rings by explaining their role in a linear collider, and to describe the configuration of the ILC damping rings;
- to review the important basic concepts of beam dynamics in damping rings, including betatron and synchrotron motion, lattice functions, and coupling;
- to discuss radiation effects, and in particular, to derive an expression for the radiation damping times in a synchrotron.

Schematic of present ILC configuration

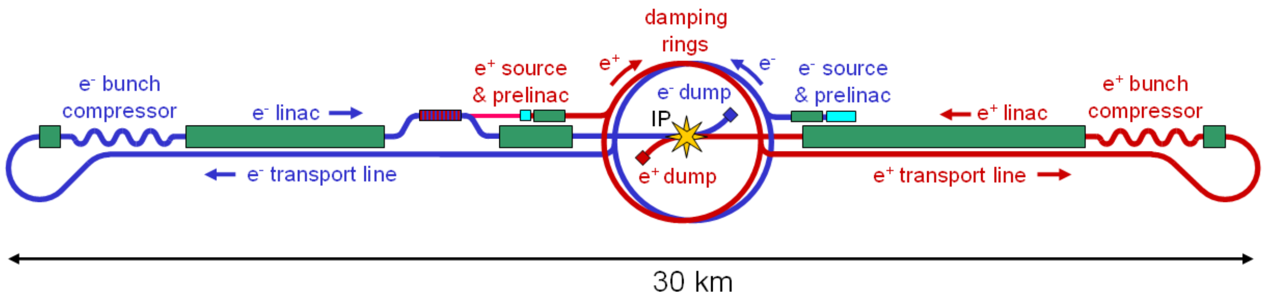


The RF power in the main linacs is pulsed at 5 Hz, with each pulse lasting for 1 ms.

During the machine pulse, a train of bunches is extracted from each damping ring, accelerated to 250 GeV by the main linacs, then collided at the interaction point.

Simultaneously, the sources produce new trains of bunches to fill the damping rings. The fresh bunches are stored in the damping rings during the 200 ms before the next machine pulse.

Schematic of present ILC configuration



The damping rings perform three essential functions:

- Reduce injected beam emittances by six orders of magnitude in the 200 ms interval between machine pulses.
- Remove jitter from the sources, providing a *highly stable beam* for tuning of downstream systems.
- Delay the beams from the source, allowing for feed-forwards.

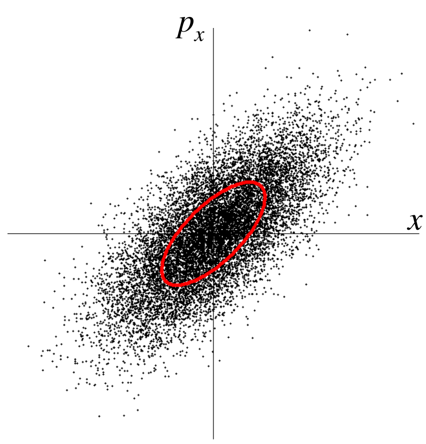
Emittance

The emittance of a bunch of particles is a measure of the volume of phase space occupied by the particles.

The horizontal rms emittance ε_x (in the absence of coupling) is defined by:

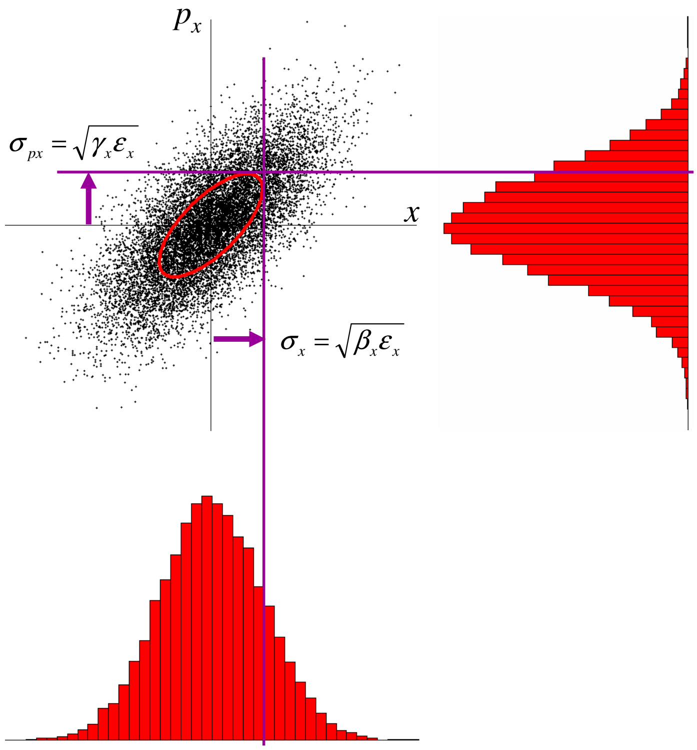
$$\varepsilon_x = \sqrt{\langle x^2 \rangle \langle p_x^2 \rangle - \langle x p_x \rangle^2}$$

where x is the horizontal coordinate of a particle, and p_x is the horizontal momentum of the particle normalised by the reference momentum $P_0 \approx E/c$.



Area of ellipse = $\pi\varepsilon_x$

Emittance



$$\varepsilon_x = \sqrt{\langle x^2 \rangle \langle p_x^2 \rangle - \langle x p_x \rangle^2}$$

Area of ellipse = $\pi \varepsilon_x$

$$\langle x^2 \rangle = \beta_x \varepsilon_x$$

$$\langle p_x^2 \rangle = \gamma_x \varepsilon_x$$

$$\langle x p_x \rangle = -\alpha_x \varepsilon_x$$

$$\beta_x \gamma_x - \alpha_x^2 = 1$$

Emittance and Liouville's Theorem

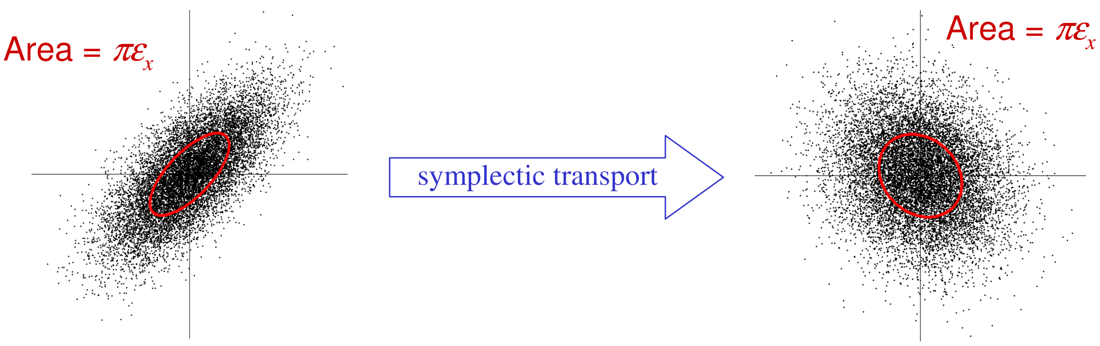
Liouville's theorem tells us that *under symplectic transport*, particle densities in phase space must be conserved.

The *symplectic condition* for Liouville's theorem to be satisfied is that the dynamics of individual particles must be governed by Hamilton's equations. This is the case for particles moving along an accelerator beam line if:

- we neglect dissipative effects like radiation;
- we keep a constant reference momentum, P_0.

Assuming that Liouville's theorem holds, the emittance of a bunch must be conserved as the bunch moves along an accelerator beam line.

The Twiss parameters β_x, α_x and γ_x change, but the emittance ε_x is constant.

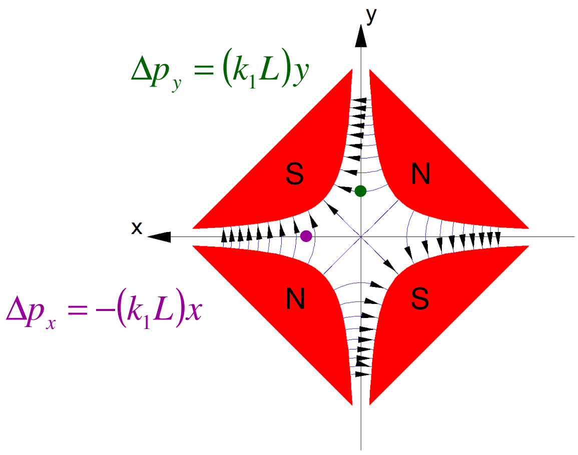


Variation of the Twiss parameters along a beam line

The transverse motion of a particle along a beam line is determined by the magnets along the beam line.

In particular, quadrupole magnets provide strong focusing, where the transverse deflection experienced by a particle passing through the magnet is proportional to the offset (coordinate) of the particle from the centre of the magnet.

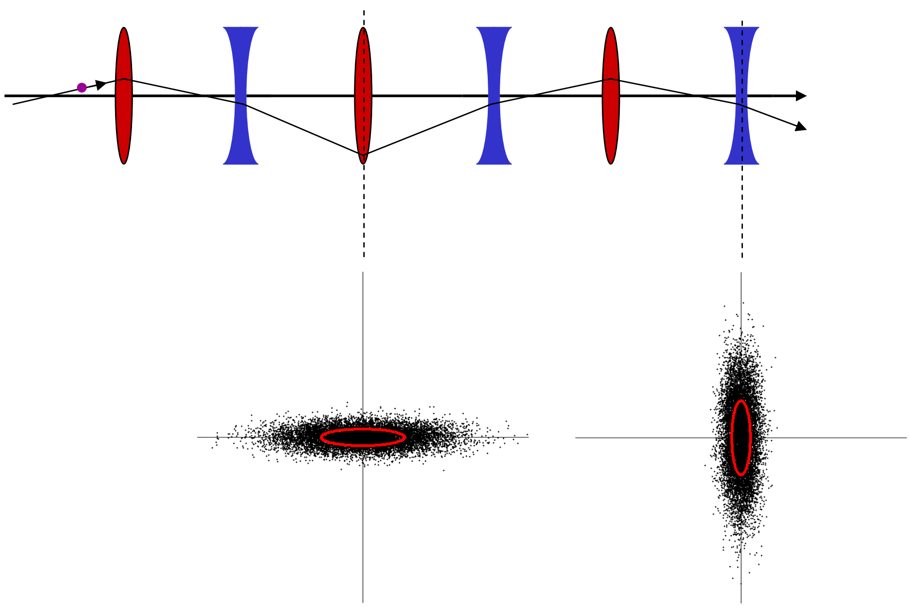
The transverse oscillations of a particle moving along an accelerator beam line are called *betatron oscillations*.



$$k_1 = \frac{e}{P_0} \frac{\partial B_y}{\partial x}$$

Variation of the Twiss parameters along a beam line

As particles move along a beam line, the shape of the emittance ellipse changes according to the strengths and positions of the magnets, though the area of the ellipse stays the same.



Normalised emittance

We stated in the previous slide that one of the conditions for conservation of emittance was that the reference momentum P_0 was constant.

It turns out that, when dealing with acceleration in linacs, we can generalise the concept of emittance to a quantity that is conserved under changes in the reference momentum:

$$\text{Normalised emittance} = \gamma\varepsilon_x$$

where γ is the relativistic factor, not to be confused with the Twiss parameter γ_x.

The emittance ε_x is sometimes referred to as the "geometric emittance" to emphasise it as a quantity distinct from the normalised emittance.

If we increase the beam energy (e.g. in a linac), usually we scale the reference momentum accordingly, in which case the geometric emittance goes down. This effect is referred to as *adiabatic damping*.

In linear colliders, we usually talk about the normalised emittance, since this is the conserved quantity in the linacs. But in the damping rings, we use both the normalised emittance and the geometric emittance.

The luminosity requirement drives the parameters

To generate high luminosity in a collider, we need:

- high bunch charges;
- high collision frequency;
- *small bunch sizes*.

$$\mathcal{L} = \frac{n_b N^2 f_{rep}}{4\pi\sigma_x\sigma_y} H_D$$

n_b is the number of bunches in a bunch train.

N is the number of particles in a single bunch.

f_{rep} is the machine pulse repetition rate.

σ_x and σ_y are the horizontal and vertical bunch sizes at the interaction point.

H_D is the "enhancement factor" (~ 1.5)

To achieve small transverse beam sizes, we need strong focusing (small beta functions), but we also need small emittances…

Some typical parameters at the interaction point

n_b	2×10¹⁰	limited by disruption effects at the IP.
N	3000	limited by current in linac and RF pulse length.
f_{rep}	5 Hz	limited by cryogenic cooling capacity.
σ_x	650 nm	optimised for high luminosity and low disruption.
σ_y	5 nm	as small as possible!

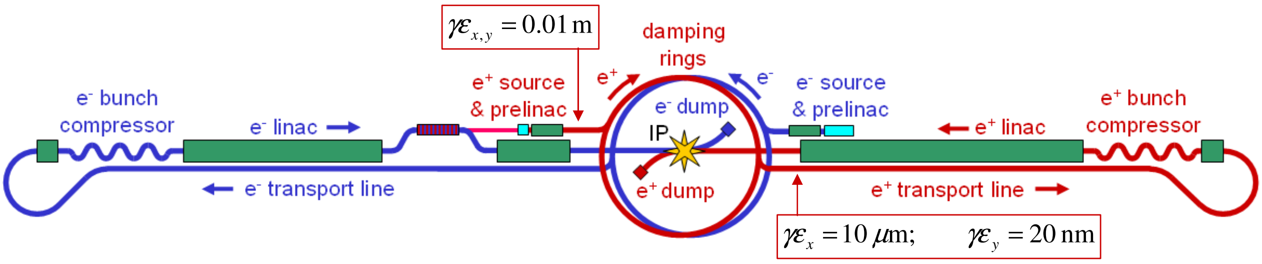
Assuming an enhancement factor $H_D \approx 1.5$, we find that with these parameters:

$$\mathcal{L} = \frac{n_b N^2 f_{rep}}{4\pi\sigma_x\sigma_y} H_D \approx 2\times10^{34}\ \mathrm{cm}^{-2}\mathrm{s}^{-1}$$

Beam parameters along the ILC

	e⁻ prelinac exit	e⁺ prelinac exit		Bunch compressor entrance	Bunch compressor exit	Main linac exit	IP
Beam energy	5 GeV	5 GeV		5 GeV	13.5 GeV	250 GeV	250 GeV
Bunch population (maximum)	2×10^{10}	2×10^{10}		2×10^{10}	2×10^{10}	2×10^{10}	2×10^{10}
Normalised horizontal emittance	45 µm	0.01 m		8 µm	8 µm	10 µm	10 µm
Horizontal beam size	~ 300 µm	~ 5 mm		~ 130 µm	~ 130 µm	~ 50 µm	650 nm
Normalised vertical emittance	45 µm	0.01 m		20 nm	20 nm	30 nm	40 nm
Vertical beam size	~ 300 µm	~ 5 mm		~ 5 µm	~ 5 µm	~ 2.5 µm	5 nm
RMS bunch length	~ 1 cm	~ 1cm		9 mm	200 µm	200 µm	200 µm

Damping rings in a linear collider



We need to reduce the emittances of each positron bunch from 0.01 m (normalised) at the exit of the positron source, to 8 um horizontally and 20 nm vertically (normalised) at the entrance to the main linac.

The reduction in emittance must be achieved in the 200 ms between machine pulses.

Synchrotrons naturally "damp" the emittance of a large injected beam, through synchrotron radiation processes. Radiation acts on betatron and synchrotron oscillations in a way analogous to friction acting on the oscillations of a pendulum.

Fortunately:

- the emittances that can be achieved in a (large) synchrotron are in the right regime for those required by a linear collider;
- radiation damping in a synchrotron happens on the right time scale (~ 10's of milliseconds) for a linear collider.

The linac and the damping rings

To achieve a luminosity of 2×10^{34} cm^{-2}s^{-1}, we need:

- a bunch population N of 2×10^{10} particles;
- a total of n_b = 3000 bunches per machine pulse.

The average current in the main linacs is limited to 9 mA.

This could be increased, but would add significant cost to the cryogenics.

The bunch separation in the main linacs must be:

$$T_{sep} = \frac{eN}{\langle I \rangle} \approx 350\,\text{ns}$$

A train of 3000 bunches with 350 ns bunch separation has a total duration of 1 ms, or, in other words, a total length of 300 km.

We must store this bunch train in a damping ring, in order to damp the emittances between machine pulses…

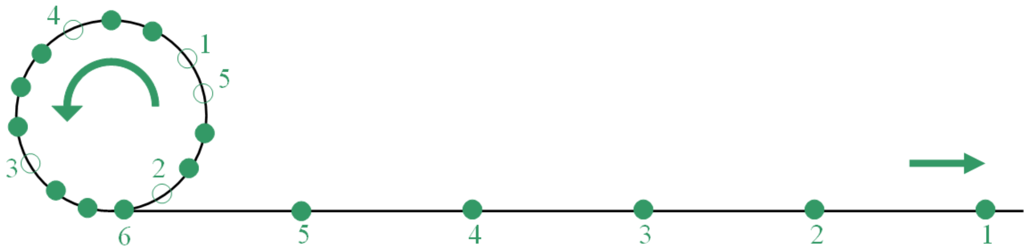
…but we are not likely to build a damping ring with circumference of 300 km!

Decompressing the bunch train from the damping rings to the main linac

To make the damping rings practicable, we must "compress" the bunch train.

To decompress the bunch train going into the main linac, we extract bunches one at a time from the damping rings, using a fast (~ 3 ns rise/fall time) kicker.

Consider a damping ring with h stored bunches, with bunch separation Δt. If we fire the extraction kicker to extract every n^{th} bunch, where n is *not* a factor of h, then we extract a continuous train of h bunches, with bunch spacing $n \times \Delta t$.



There are two complications:

We would like a continuous train of bunches in the linac, but the damping rings need to have regular gaps in the fill, for ion clearing.

The positrons are produced by the decompressed electron beam, so we have no control over the arrival of positron bunches to refill the damping ring. This places a constraint on beam line lengths in the ILC.

ILC damping rings design constraints and considerations

The upstream and downstream systems provide design constraints, in terms of the parameter specifications for the injected and extracted beams.

	Electron beam	Positron beam
Machine pulse repetition rate	5 Hz	
Number of bunches per pulse	3000	
Number of particles per bunch	2×10^{10}	
Injected normalised emittance	45 μm	0.01 m
Injected energy spread	0.1% rms	1% full width
Bunch spacing in main linac	330 ns	
Extracted normalised horizontal emittance	8 μm	
Extracted normalised vertical emittance	20 nm	
Extracted rms bunch length	6 mm	
Extracted rms energy spread	<0.15%	

We have to decide the damping rings circumference, beam energy, lattice style…

ILC damping rings design constraints and considerations: circumference

Optimising the damping ring design is complicated because all the choices are coupled. For example, in rings with large circumference, space-charge effects tend to be more severe, and one would therefore prefer a higher energy. But a higher energy increases the equilibrium emittances; and this affects the choice of lattice style…

For the circumference, cost is a major consideration. Smaller rings would be less expensive. But there are lower limits on the circumference from:

- Kicker performance. The smaller the ring, the smaller the separation between bunches, and the faster the kickers need to be to inject and extract individual bunches.
- Electron cloud and ion effects. With smaller bunch separations, the electron cloud and ion effects can be more severe.

Other considerations include operational reliability and tuning issues, which tend to favour smaller rings with fewer components.

The present damping ring designs have a circumference of 6.7 km, which pushes the limits on kicker performance and on electron cloud effects.

ILC damping rings design constraints and considerations: beam energy

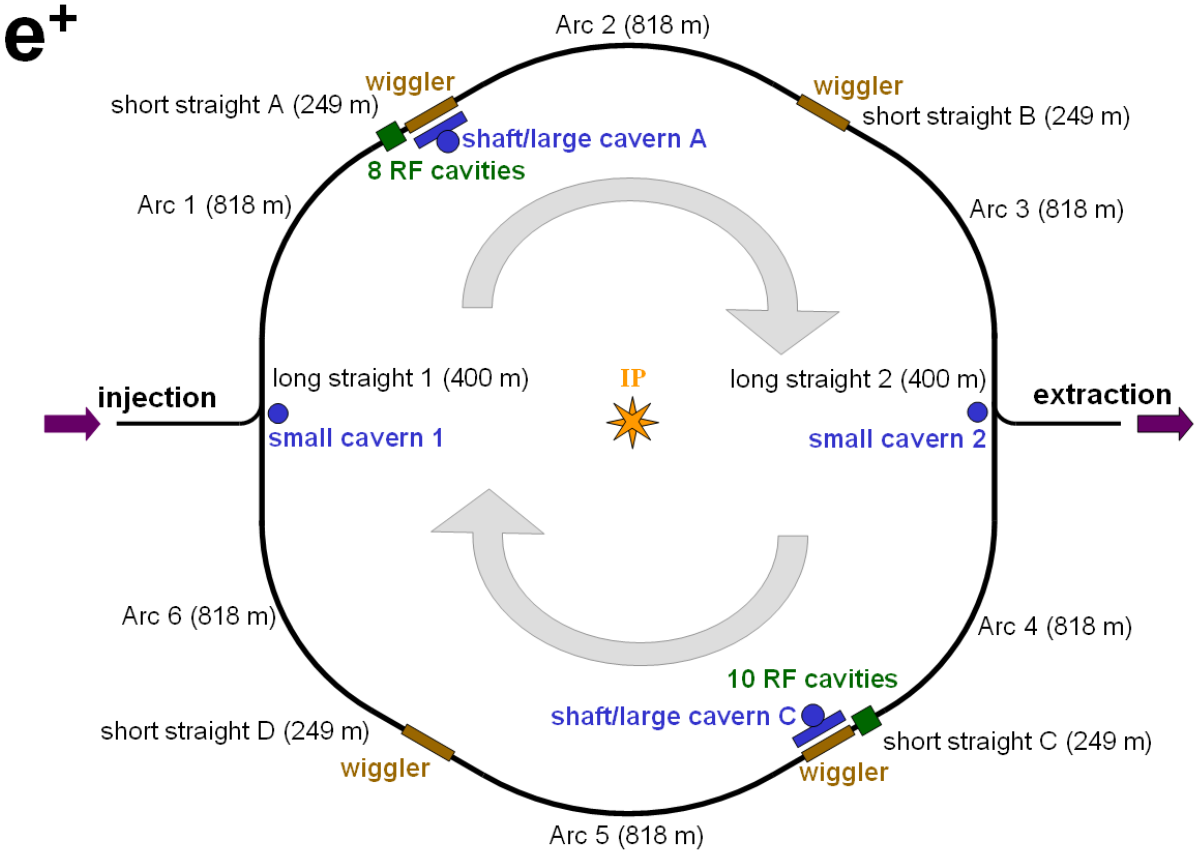
A higher beam energy is favoured by:

- Collective effects. Sensitivity to collective effects (instabilities, intrabeam scattering, space charge etc.) is reduced at higher beam energies.
- Acceptance. With a given normalised injected emittance, the beam size is smaller at higher energies (because of adiabatic damping as the beam is accelerated), and easier to capture in the ring.
- Damping rates. Higher beam energy results in faster damping from synchrotron radiation.

A lower beam energy is favoured by:

- Equilibrium emittances. Lower emittances (at least at low bunch charge) are easier to achieve in lower energy rings. The normalised natural emittance scales as the cube of the energy in a given lattice.
- Cost. Lower energy rings require less powerful magnets, and lower RF voltage.

The present damping rings design have a beam energy of 5 GeV, which appears to be a reasonable compromise between the competing effects.



ILC damping rings parameters compared with existing machines

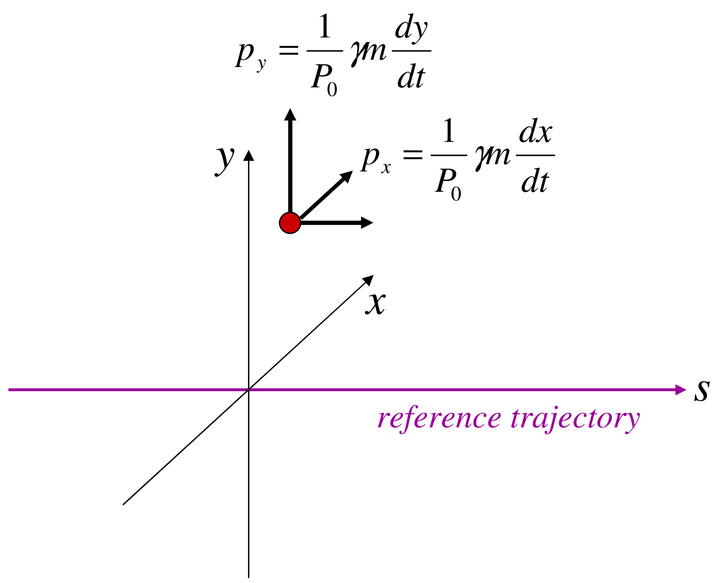
	KEK-ATF	LBNL-ALS	SLAC-PEP II LER	ILC DR	
Circumference	139 m	196 m	2200 m	6700 m	
Beam energy	1.28 GeV	1.9 GeV	3.1 GeV	5 GeV	
Average current	70 mA	400 mA	2450 mA	400 mA	
Bunch population	~ 10^{10}	0.6×10^{10}	7×10^{10}	1×10^{10}	2×10^{10}
Number of bunches	20	272	1588	5782	2767
Bunch spacing	2.8 ns	2 ns	4.2 ns	3.1 ns	6.2 ns
Transverse damping time	28 ms	9 ms	70 ms	25 ms	
Horizontal emittance, ε_x	1 nm	7 nm	30 nm	0.8 nm	
Vertical emittance, ε_y	4.5 pm	~ 10 pm	1400 pm	2 pm	
RMS bunch length	4 mm	7 mm	11 mm	9 mm	
RMS energy spread	0.055%	0.1%	0.07%	0.13%	
RF voltage	770 kV	1 MV	4 MV	24 MV	
RF frequency	714 MHz	500 MHz	476 MHz	650 MHz	

So far, we have seen that:

- the emittance is a measure of the phase space volume of a bunch of particles, that is conserved as a bunch travels along an accelerator beam line;
- to achieve a high luminosity, we need very small (~ nm) transverse beam sizes at the interaction point;
- to achieve nanometer beam sizes, we need bunches with much smaller transverse emittances than can be produced directly from the particle sources;
- we can use radiation damping in a synchrotron storage ring (a damping ring) to achieve the necessary emittance reduction on a time scale consistent with linear collider operation;
- the parameters for linac operation and luminosity production impose strong constraints on the damping rings configuration;
- parameter choices for the damping rings are often compromises between competing effects.

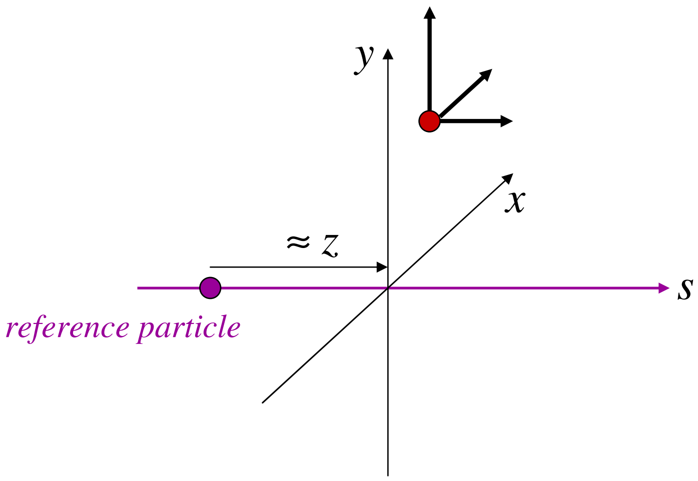
In the remainder of this first lecture, we shall study in some detail the process of radiation damping, and derive expressions for the radiation damping times in a synchrotron storage ring.

Coordinate system



P_0 = reference momentum

Longitudinal coordinate

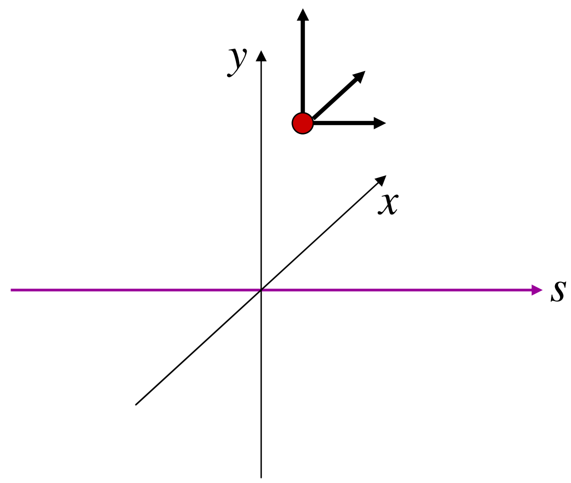


The reference particle is a particle travelling along the reference trajectory with momentum P_0 and velocity $\beta_0 c$.

If a particle is time τ *ahead of* the reference particle, then the longitudinal coordinate z is defined by:

$$z = c\tau$$

Energy deviation

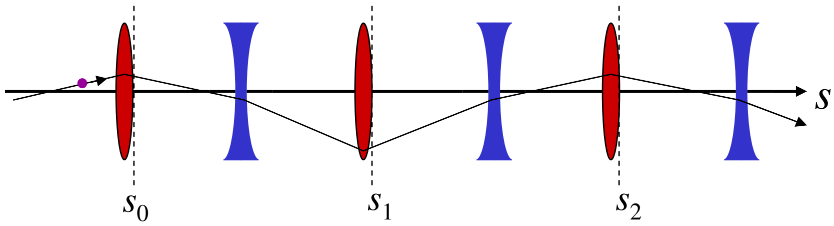


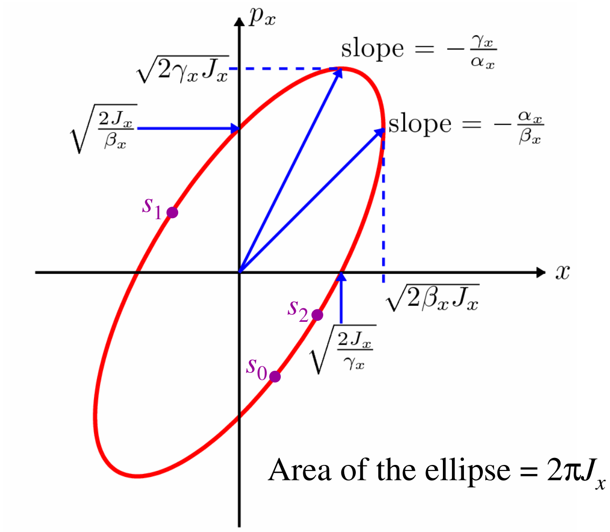
If the particle has total energy E, then the energy deviation δ is defined by:

$$\delta = \frac{E}{P_0 c} - \frac{1}{\beta_0}$$

For ultra-relativistic particles ($\beta \approx \beta_0 \approx 1$), we have: $\delta \approx \frac{\Delta E}{E_0}$

Twiss parameters and the particle *action*





In an *uncoupled* periodic beam line (i.e. in which motion in any plane is independent of motion in the other planes) particles trace out ellipses in phase space with each pass through the periodic cell. The shape of the ellipse defines the *Twiss parameters* at the observation point.

The area of the ellipse defines the *action* J_x of the particle.

The action is the amplitude of the motion of the particle as it moves along the beam line.

Cartesian variables and action-angle variables

Applying simple geometry to the phase space ellipse, we find that the action (for uncoupled motion) is related to the Cartesian variables for the particle by:

$$2J_x = \gamma_x x^2 + 2\alpha_x x p_x + \beta_x p_x^2$$

We also define the angle φ_x as follows:

$$\tan\varphi_x = -\beta_x \frac{p_x}{x} - \alpha_x$$

The action-angle variables provide an alternative to Cartesian variables for describing the dynamics of a particle moving along a beam line. The advantage of action-angle variables is that, under symplectic transport, the action of a particle is constant.

Note: if the beam line is coupled, then we need to make a coordinate transformation to the "normal mode" coordinates, in which the motion in one mode is independent of the motion in the other modes. Then we can apply the equations as above.

Action and emittance

The *action* J_x is a variable used to describe the amplitude of the motion of an individual particle. In terms of the action-angle variables, the Cartesian coordinate and momentum can be written:

$$x = \sqrt{2\beta_x J_x} \cos\varphi_x \qquad p_x = -\sqrt{\frac{2J_x}{\beta_x}}(\sin\varphi_x + \alpha_x \cos\varphi_x)$$

The *emittance* ε_x is the average amplitude of all particles in a bunch:

$$\varepsilon_x = \langle J_x \rangle$$

With this relationship between the emittance and the average action, we can obtain the following familiar relationships for the second-order moments of the bunch:

$$\langle x^2 \rangle = \beta_x \varepsilon_x \qquad \langle x p_x \rangle = -\alpha_x \varepsilon_x \qquad \langle p_x^2 \rangle = \gamma_x \varepsilon_x$$

Again, this is true for *uncoupled* motion.

Action and radiation

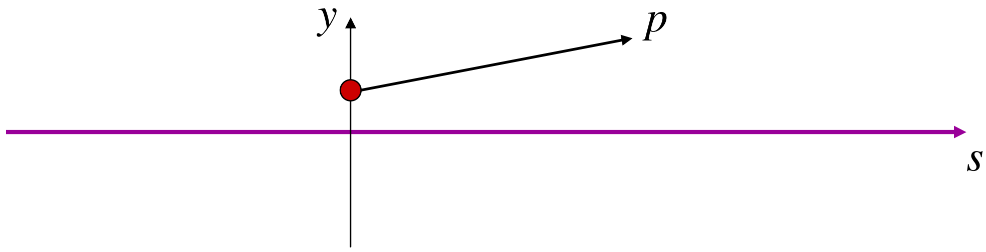
So far, we have considered only symplectic transport, i.e. motion of a particle in the electromagnetic fields of drifts, dipoles, quadrupoles etc. without any radiation.

However, we know that a charged particle moving through an electromagnetic field will (in general) undergo acceleration, and a charged particle undergoing acceleration will radiate electromagnetic waves.

What impact will the radiation have on the motion of the particle?

In answering this question, we will consider first the case of uncoupled vertical motion – for a particle in a storage ring, this turns out to be the simplest case.

Radiation damping of vertical emittance

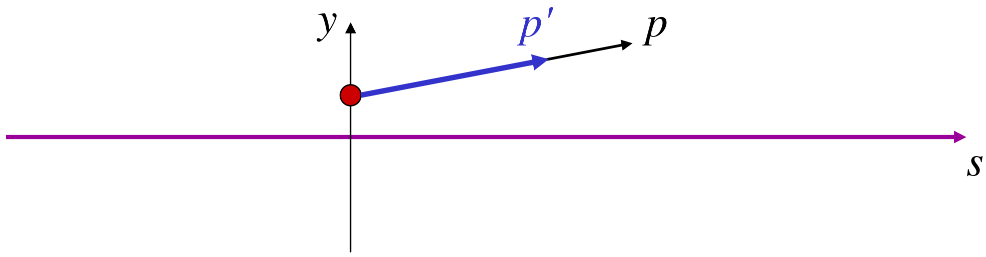


A relativistic particle will emit radiation with an opening angle of $1/\gamma$ with respect to its instantaneous direction of motion, where γ is the relativistic factor.

For an ultra-relativistic particle, $\gamma >> 1$, we can assume that the radiation is emitted directly along the instantaneous direction of motion of the particle.

Note: Generally, in storage rings we consider only radiation from the dipoles. Radiation is also emitted as particles move through quadrupoles, sextupoles etc., but the radiation power is much smaller, because the fields seen by the particles are much weaker.

Radiation damping of vertical emittance



The change in momentum of the particle is given by:

$$p' = p - dp \approx p\left(1 - \frac{dp}{P_0}\right)$$

where dp is the momentum carried by the radiation, and we assume that:

$$p \approx P_0$$

Since there is no change in direction of the particle, we must have:

$$p'_y \approx p_y\left(1 - \frac{dp}{P_0}\right)$$

Radiation damping of vertical emittance

After emission of radiation, the vertical momentum of the particle is:

$$p_y' = p_y \left(1 - \frac{dp}{P_0} \right)$$

Now we substitute this into the expression for the vertical betatron action (valid for *uncoupled* motion):

$$2J_y = \gamma_y y^2 + 2\alpha_y y p_y + \beta_y p_y^2$$

to find the change in the action resulting from the emission of radiation:

$$dJ_y = -\left(\alpha_y y p_y + \beta_y p_y^2 \right) \frac{dp}{P_0}$$

We average over all particles in the beam, to find:

$$\langle dJ_y \rangle = d\varepsilon_y = -\varepsilon_y \frac{dp}{P_0}$$

where we have used: $\langle y p_y \rangle = -\alpha_y \varepsilon_y$ $\quad \langle p_y^2 \rangle = \gamma_y \varepsilon_y$ $\quad$ and $\quad \beta_y \gamma_y - \alpha_y^2 = 1$

Radiation damping of vertical emittance

For a particle moving round a storage ring, we can integrate the loss in momentum around the ring. The emittance is conserved under symplectic transport; so if the non-symplectic (radiation) effects are slow, we can write:

$$d\varepsilon_y = -\varepsilon_y \frac{dp}{P_0} \qquad \therefore \qquad \frac{d\varepsilon_y}{dt} = -\frac{\varepsilon_y}{T_0} \oint \frac{dp}{P_0} \approx -\frac{U_0}{E_0 T_0} \varepsilon_y$$

where T_0 is the revolution period, and U_0 is the energy loss in one turn. The approximation is valid for an ultra-relativistic particle, which has $E \approx pc$.

We define the damping time τ_y:

$$\tau_y = 2 \frac{E_0}{U_0} T_0$$

so the evolution of the emittance is:

$$\varepsilon_y(t) = \varepsilon_y(0) \exp\left(-2 \frac{t}{\tau_y} \right)$$

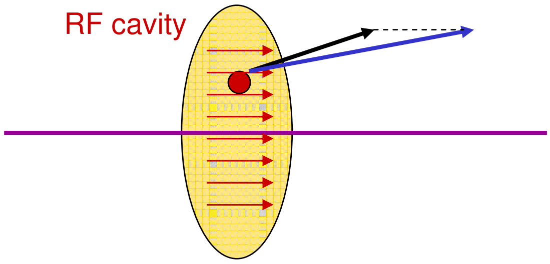
Typically, the damping time in a synchrotron storage ring is measured in tens of milliseconds, whereas the revolution period is measured in microseconds; so the radiation effects really are "slow".

Radiation damping of vertical emittance

Note that we made the assumption that the momentum of the particle was close to the reference momentum:

$$p \approx P_0$$

If the particle continues to radiate without any restoration of energy, we will reach a point where this assumption is no longer valid. However, electron storage rings contain RF cavities to restore the energy lost by synchrotron radiation. But then, we have to consider the change in momentum of a particle as it moves through an RF cavity.



Fortunately, RF cavities are usually designed with a longitudinal electric field, so that particles experience a change in longitudinal momentum as they pass through, without any change in transverse momentum.

Synchrotron radiation energy loss

To complete our calculation of the vertical damping time, we need to find the energy lost by a particle through synchrotron radiation on each turn through the storage ring. We quote the (classical) result that the power radiated by a particle of charge e and energy E in a magnetic field B is given by:

$$P_\gamma = \frac{C_\gamma}{2\pi} c^3 e^2 B^2 E^2$$

C_γ is a constant, given by:

$$C_\gamma = \frac{e^2}{3\varepsilon_0 \left(mc^2\right)^4} \approx 8.846\times10^{-5}\ \mathrm{m/GeV^3}$$

A charged particle with energy E in a magnetic field B follows a circular trajectory with radius ρ, given by:

$$B\rho = \frac{E}{ec}$$

Hence the synchrotron radiation power can be written:

$$P_\gamma = \frac{C_\gamma}{2\pi} c \frac{E^4}{\rho^2}$$

Synchrotron radiation energy loss

For a particle with the nominal energy, and traveling at (close to) the speed of light around the closed orbit, we can find the energy loss simply by integrating the radiation power around the ring:

$$U_0 = \oint P_\gamma \, dt = \oint P_\gamma \frac{ds}{c}$$

Using the previous expression for P_γ, we find:

$$U_0 = \frac{C_\gamma}{2\pi} E_0^4 \oint \frac{1}{\rho^2} ds$$

Conventionally, we define the *second synchrotron radiation integral*, I_2:

$$I_2 = \oint \frac{1}{\rho^2} ds$$

In terms of I_2, the energy loss per turn U_0 is written:

$$U_0 = \frac{C_\gamma}{2\pi} E_0^4 I_2$$

A short digression: the first synchrotron radiation integral

Note that I_2 is a property of the lattice (actually, of the reference trajectory), and does not depend on the properties of the beam.

Conventionally, there are five synchrotron radiation integrals defined, which are used to express in convenient form the dynamics of a beam emitting radiation.

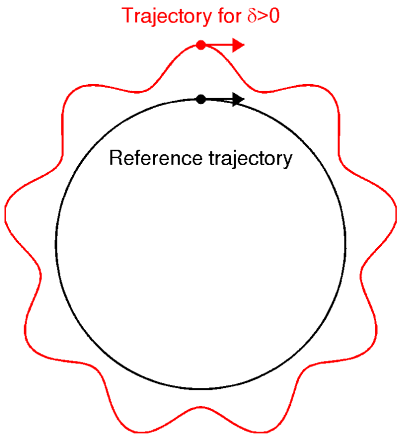
The first synchrotron radiation integral is not, however, directly related to the radiation effects. It is defined in terms of the horizontal dispersion η_x as:

$$I_1 = \oint \frac{\eta_x}{\rho} ds$$

The momentum compaction factor, α_p, gives the proportional change in circumference per change in momentum, and can be written:

$$\frac{\Delta C}{C_0} = \alpha_p \frac{\Delta P}{P_0} \qquad \alpha_p \equiv \frac{1}{C_0} \left.\frac{dC}{d\delta}\right|_{\delta=0} = \frac{1}{C_0} \oint \frac{1}{\rho} ds = \frac{1}{C_0} I_1$$

A short digression: the first synchrotron radiation integral



The dispersion η_x is defined as the change in the closed orbit with respect to a change in energy deviation:

$$\eta_x = \frac{dx_{co}}{d\delta}$$

Damping of horizontal emittance

Analysis of the radiation effects on the vertical emittance was relatively straightforward. When we consider the horizontal emittance, there are three complications that we need to address:

- The horizontal motion of a particle is often strongly coupled to the longitudinal motion (with the coupling described by the dispersion).
- Where the reference trajectory is curved (usually, in dipoles), the path length taken by a particle depends on the horizontal coordinate with respect to the reference trajectory.
- Dipole magnets are sometimes built with a gradient, so that the vertical field seen by a particle in a dipole depends on the horizontal coordinate of the particle.

Horizontal-longitudinal coupling

Coupling between transverse and longitudinal planes in a beam line is usually represented by the dispersion, η_x. So, in terms of the horizontal dispersion, the horizontal coordinate and momentum of a particle are given by:

$$x = \sqrt{2\beta_x J_x}\cos\varphi_x + \eta_x \delta$$

$$p_x = -\sqrt{\frac{2J_x}{\beta_x}}\left(\sin\varphi_x + \alpha_x \cos\varphi_x\right) + \eta_{px}\delta$$

When a particle emits radiation, we have to take into account:

- the change in momentum of the particle (because of the momentum carried by the radiation);
- the change in coordinate x and momentum p_x resulting from the change in energy deviation δ.

When we analysed the vertical motion, we ignored the second effect, because we assumed that the vertical dispersion was zero.

Damping of horizontal emittance

Taking all the above effects into account, we can proceed along the same lines as for the analysis of the vertical emittance. That is:

- Write down the changes in coordinate x and momentum p_x resulting from an emission of radiation with momentum dp (taking into account the additional effects of dispersion).
- Substitute expressions for the new coordinate and momentum into the expression for the horizontal betatron action, to find the change in action resulting from the radiation emission.
- Average over all particles in the beam, to find the change in the emittance resulting from radiation emission from each particle.
- Integrate around the ring (taking account of changes in path length and field strength with x in the bends) to find the change in emittance over one turn.

The algebra gets somewhat cumbersome, and is not especially enlightening: see Appendix A for more details. Here, we just quote the result…

Damping of horizontal emittance

The horizontal emittance decays exponentially:

$$\frac{d\varepsilon_x}{dt} = -\frac{2}{\tau_x}\varepsilon_x$$

where the horizontal damping time is given by:

$$\tau_x = \frac{2}{j_x}\frac{E_0}{U_0}T_0$$

The horizontal damping partition number j_x is given by:

$$j_x = 1 - \frac{I_4}{I_2}$$

where the fourth synchrotron radiation integral I_4 is given by:

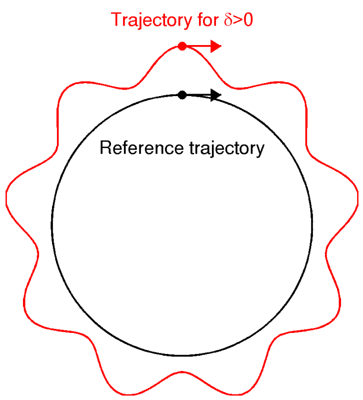
$$I_4 = \oint \frac{\eta_x}{\rho}\left(\frac{1}{\rho^2} + 2k_1\right)ds \qquad k_1 = \frac{e}{P_0}\frac{\partial B_y}{\partial x}$$

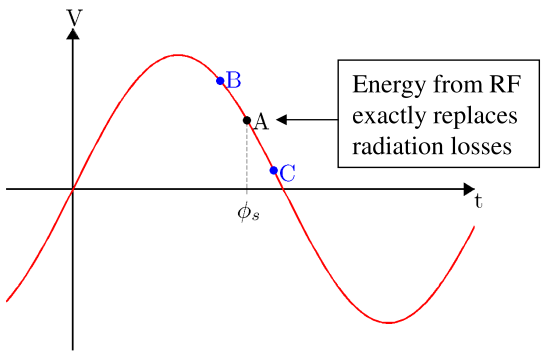
Longitudinal motion and synchrotron oscillations

As well as making transverse (betatron) oscillations as they travel round a synchrotron storage ring, particles also make longitudinal (synchrotron) oscillations.

Particles with momentum above the reference momentum take longer to go round the ring, because of the increased path length.

Particles taking longer to go round the ring slip back in RF phase, which leads to a reduction in their energy.





Damping of synchrotron oscillations

Generally, synchrotron oscillations are handled differently from betatron oscillations, because the synchrotron tune (the number of oscillations per revolution) in a storage ring is usually much less than 1, whereas the betatron tunes are much greater than 1.

To find the effects of radiation on synchrotron motion, we proceed as follows:

- We write down the equations of motion (for the variables z and δ) for a particle performing synchrotron motion, including the radiation energy loss.
- We express the energy loss per turn as a function of the energy deviation of the particle. This introduces a "damping term" into the equations of motion.
- Solving the equations of motion gives synchrotron oscillations (as expected) with amplitude that decays exponentially.

Damping of synchrotron oscillations

The change in energy deviation δ and longitudinal coordinate z for a particle in one turn around a storage ring are given by:

$$\Delta\delta = \frac{eV_{RF}}{E_0}\sin\left(\varphi_s - \frac{\omega_{RF} z}{c}\right) - \frac{U}{E_0}$$

$$\Delta z = -\alpha_p C_0 \delta$$

where V_{RF} is the RF voltage and ω_{RF} the RF frequency, E_0 is the reference energy of the beam, φ_s is the nominal RF phase, and U is the energy lost by the particle through synchrotron radiation.

If the revolution period is T_0, then we can write the longitudinal equations of motion for the particle:

$$\frac{d\delta}{dt} = \frac{eV_{RF}}{E_0 T_0}\sin\left(\varphi_s - \frac{\omega_{RF} z}{c}\right) - \frac{U}{E_0 T_0}$$

$$\frac{dz}{dt} = -\alpha_p c \delta$$

Damping of synchrotron oscillations

Let us assume that z is small compared to the RF wavelength, i.e. $\omega_{RF} z/c << 1$.

Also, the energy loss per turn is a function of the energy of the particle (particles with higher energy radiate higher synchrotron radiation power), so we can write (to first order in the energy deviation):

$$U = U_0 + \Delta E \left.\frac{dU}{dE}\right|_{E=E_0} = U_0 + E_0 \delta \left.\frac{dU}{dE}\right|_{E=E_0}$$

Further, we assume that the RF phase φ_s is set so that for $z = \delta = 0$, the RF cavity restores exactly the amount of energy lost by synchrotron radiation. The equations of motion then become:

$$\frac{d\delta}{dt} = -\frac{eV_{RF}}{E_0 T_0} \cos\varphi_s \frac{\omega_{RF}}{c} z - \frac{1}{T_0} \delta \left.\frac{dU}{dE}\right|_{E=E_0}$$

$$\frac{dz}{dt} = -\alpha_p c \delta$$

Damping of synchrotron oscillations

Combining these equations gives:

$$\frac{d^2\delta}{dt^2} + 2\alpha_E \frac{d\delta}{dt} + \omega_s^2 \delta = 0$$

This is the equation for a damped harmonic oscillator, with frequency ω_s and damping constant α_E given by:

$$\omega_s^2 = -\frac{eV_{RF}}{E_0} \cos\varphi_s \frac{\omega_{RF}}{T_0} \alpha_p$$

$$\alpha_E = \frac{1}{2T_0} \left.\frac{dU}{dE}\right|_{E=E_0}$$

Damping of synchrotron oscillations

If $\alpha_E << \omega_s$, the energy deviation and longitudinal coordinate damp as:

$$\delta(t) = \hat{\delta}\exp(-\alpha_E t)\sin(\omega_s t - \theta_0)$$

$$z(t) = \frac{\alpha_p c}{\omega_s}\hat{\delta}\exp(-\alpha_E t)\cos(\omega_s t - \theta_0)$$

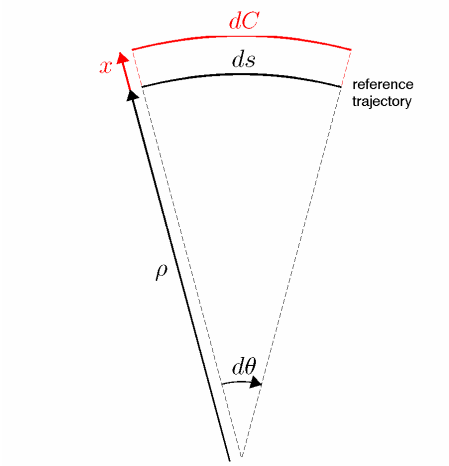
To find the damping constant α_E, we need to know how the energy loss per turn U depends on the energy deviation δ...

Damping of synchrotron oscillations

We can find the total energy lost by integrating over one revolution period:

$$U = \oint P_\gamma \, dt$$

To convert this to an integral over the circumference, we should recall that the path length depends on the energy deviation; so a particle with a higher energy takes longer to travel round the lattice.



$$dt = \frac{dC}{c}$$

$$dC = \left(1 + \frac{x}{\rho}\right)ds = \left(1 + \frac{\eta_x \delta}{\rho}\right)ds$$

$$U = \frac{1}{c}\oint P_\gamma \left(1 + \frac{\eta_x \delta}{\rho}\right)ds$$

Damping of synchrotron oscillations

With the energy loss per turn given by:

$$U = \frac{1}{c}\oint P_\gamma \left(1 + \frac{\eta_x}{\rho}\delta\right) ds$$

and the synchrotron radiation power given by:

$$P_\gamma = \frac{C_\gamma}{2\pi} c^3 e^2 B^2 E^2 = \frac{C_\gamma}{2\pi} c \frac{E^4}{\rho^2}$$

we find, after some algebra:

$$\left.\frac{dU}{dE}\right|_{E=E_0} = j_E \frac{U_0}{E_0}$$

where:

$$U_0 = \frac{C_\gamma}{2\pi} E_0^4 I_2 \qquad j_E = 2 + \frac{I_4}{I_2}$$

I_2 and I_4 are the same synchrotron radiation integrals that we saw before:

$$I_2 = \oint \frac{1}{\rho^2} ds \qquad I_4 = \oint \frac{\eta_x}{\rho}\left(\frac{1}{\rho^2} + 2k_1\right) ds \qquad k_1 = \frac{e}{P_0}\frac{\partial B_y}{\partial x}$$

Damping of synchrotron oscillations

Finally, we can write the longitudinal damping time:

$$\tau_z = \frac{1}{\alpha_E} = \frac{2}{j_z}\frac{E_0}{U_0} T_0$$

U_0 is the energy loss per turn for a particle with the reference energy E_0, following the reference trajectory. It is given by:

$$U_0 = \frac{C_\gamma}{2\pi} E_0^4 I_2$$

j_z is the longitudinal damping partition number, given by:

$$j_z = 2 + \frac{I_4}{I_2}$$

Damping of synchrotron oscillations

The longitudinal emittance is given by a similar expression to the horizontal and vertical emittances:

$$\varepsilon_z = \sqrt{\langle z^2\rangle\langle\delta^2\rangle - \langle z\delta\rangle^2}$$

In most storage rings, the correlation $\langle z\delta\rangle$ is negligible, so the emittance becomes:

$$\varepsilon_z \approx \sigma_z \sigma_\delta$$

Hence, the damping of the longitudinal emittance can be written:

$$\varepsilon_z(t) = \varepsilon_z(0)\exp\left(-2\frac{t}{\tau_z}\right)$$

Summary: synchrotron radiation damping

The energy loss per turn is given by:

$$U_0 = \frac{C_\gamma}{2\pi} E_0^4 I_2 \qquad C_\gamma = 8.846\times10^{-5}\ \text{m/GeV}^3$$

The emittances damp as:

$$\varepsilon(t) = \varepsilon(0)\exp\left(-\frac{2t}{\tau}\right)$$

The radiation damping times are given by:

$$\tau_x = \frac{2}{j_x}\frac{E_0}{U_0}T_0 \qquad \tau_y = \frac{2}{j_y}\frac{E_0}{U_0}T_0 \qquad \tau_z = \frac{2}{j_z}\frac{E_0}{U_0}T_0$$

The damping partition numbers are:

$$j_x = 1 - \frac{I_4}{I_2} \qquad j_y = 1 \qquad j_z = 2 + \frac{I_4}{I_2}$$

Summary: synchrotron radiation integrals

The first, second and fourth synchrotron radiation integrals are:

$$I_1 = \oint \frac{\eta_x}{\rho} ds$$

$$I_2 = \oint \frac{1}{\rho^2} ds$$

$$I_4 = \oint \frac{\eta_x}{\rho} \left(\frac{1}{\rho^2} + 2k_1 \right) ds \qquad k_1 = \frac{e}{P_0} \frac{\partial B_y}{\partial x}$$

Appendix

Appendix A: Damping of horizontal emittance

In this Appendix, we derive the expression for radiation damping of the horizontal emittance:

$$\frac{d\varepsilon_x}{dt} = -\frac{2}{\tau_x}\varepsilon_x$$

where:

$$\tau_x = \frac{2}{j_x}\frac{E_0}{U_0}T_0 \qquad j_x = 1 - \frac{I_4}{I_2}$$

To do this, we proceed as follows:

1. We find an expression for the change of horizontal action of a single particle when emitting radiation with momentum dp.
2. We integrate around the ring to find the change in action per revolution period.
3. We average the action over all particles in the bunch, to find the change in emittance per revolution period.

Appendix A: Damping of horizontal emittance

To begin, we note that, in the presence of dispersion, the action J_x is written:

$$2J_x = \gamma_x \tilde{x}^2 + 2\alpha_x \tilde{x}\tilde{p}_x + \beta_x \tilde{p}_x^2$$

where:

$$\tilde{x} = x - \eta_x \delta \qquad \tilde{p}_x = p_x - \eta_{px}\delta$$

After emission of radiation carrying momentum dp, the variables change by:

$$\delta \mapsto \delta - \frac{dp}{P_0} \qquad \tilde{x} \mapsto \tilde{x} + \eta_x \frac{dp}{P_0} \qquad \tilde{p}_x \mapsto \tilde{p}_x\left(1 - \frac{dp}{P_0}\right) + \eta_{px}(1-\delta)\frac{dp}{P_0}$$

The resulting change in the action is:

$$J_x \mapsto J_x + dJ_x$$

Appendix A: Damping of horizontal emittance

The change in the horizontal action is:

$$dJ_x = -w_1 \frac{dp}{P_0} + w_2 \left(\frac{dp}{P_0}\right)^2 \qquad \therefore \qquad \frac{dJ_x}{dt} = -w_1 \frac{1}{P_0}\frac{dp}{dt} + w_2 \frac{dp}{P_0^2}\frac{dp}{dt} \qquad \text{(A1)}$$

where, in the limit $\delta \to 0$:

$$w_1 = \alpha_x x p_x + \beta_x p_x^2 - \eta_x(\gamma_x x + \alpha_x p_x) - \eta_{px}(\alpha_x x + \beta_x p_x) \qquad \text{(A2)}$$

and:

$$w_2 = \frac{1}{2}(\gamma_x \eta_x^2 + 2\alpha_x \eta_x \eta_{px} + \beta_x \eta_{px}^2) - (\alpha_x \eta_x + \beta_x \eta_{px}) p_x + \frac{1}{2}\beta_x p_x^2 \qquad \text{(A3)}$$

Treating radiation as a classical phenomenon, we can take the limit $dp \to 0$ in the limit of small time interval, $dt \to 0$. In this approximation:

$$\frac{dJ_x}{dt} \approx -w_1 \frac{1}{P_0}\frac{dp}{dt} \approx -w_1 \frac{P_\gamma}{P_0 c}$$

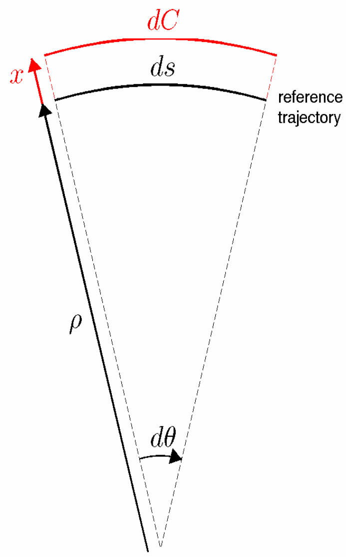
where P_γ is the *rate of energy loss* of the particle through radiation.

Appendix A: Damping of horizontal emittance

To find the *average* rate of change of horizontal action, we integrate over one revolution period:

$$\frac{dJ_x}{dt} = -\frac{1}{T_0}\oint w_1 \frac{P_\gamma}{P_0 c}\, dt$$

We have to be careful changing the variable of integration where the reference trajectory is curved:



$$dt = \frac{dC}{c} = \left(1 + \frac{x}{\rho}\right)\frac{ds}{c}$$

So:

$$\frac{dJ_x}{dt} = -\frac{1}{T_0 P_0 c^2}\oint w_1 P_\gamma \left(1 + \frac{x}{\rho}\right) ds \qquad \text{(A4)}$$

where the rate of energy loss is:

$$P_\gamma = \frac{C_\gamma}{2\pi} c^3 e^2 B^2 E^2 \qquad \text{(A5)}$$

Appendix A: Damping of horizontal emittance

We have to take into account the fact that the field strength in a dipole can vary with position. To first order in x we can write:

$$B = B_0 + x\frac{\partial B_y}{\partial x} \qquad \text{(A6)}$$

Substituting equation (A6) into (A5), and with the use of (A2), we find (after some algebra!) that, averaging over all particles in the beam:

$$\oint \left\langle w_1 P_\gamma \left(1+\frac{x}{\rho}\right)\right\rangle ds = cU_0\left(1-\frac{I_4}{I_2}\right)\varepsilon_x \qquad \text{(A7)}$$

where:

$$U_0 = \frac{C_\gamma}{2\pi}cE_0^4 I_2 \qquad I_2 = \oint \frac{1}{\rho^2}\,ds \qquad I_4 = \oint \frac{\eta_x}{\rho}\left(\frac{1}{\rho^2}+2k_1\right)ds$$

and k_1 is the quadrupole gradient in the dipole field:

$$k_1 = \frac{e}{P_0}\frac{\partial B_y}{\partial x}$$

Appendix A: Damping of horizontal emittance

Combining equations (A4) and (A7) we have:

$$\frac{d\varepsilon_x}{dt} = -\frac{1}{T_0}\frac{U_0}{E_0}\left(1-\frac{I_4}{I_2}\right)\varepsilon_x$$

Defining the horizontal damping time, τ_x:

$$\tau_x = \frac{2}{j_x}\frac{E_0}{U_0}T_0 \qquad j_x = 1-\frac{I_4}{I_2}$$

the evolution of the horizontal emittance can be written:

$$\frac{d\varepsilon_x}{dt} = -\frac{2}{\tau_x}\varepsilon_x$$

The quantity j_x is called the horizontal damping partition number. For most lattices, if there is no gradient in the dipoles, then j_x is very close to 1.