Fourth International Accelerator School for Linear Colliders

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Course A4: Damping Ring Design and Physics Issues

Lecture 5

Vertical Emittance

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Ultra-low vertical emittance is critical for generating luminosity

In this lecture, we shall discuss:

- The fundamental lower limit on the vertical emittance in a storage ring.
- Generation of orbit distortion, coupling and vertical dispersion from magnet alignment errors.
- Generation of vertical emittance from coupling and vertical dispersion.

Our main goal will be to understand the alignment tolerances for the quadrupoles and sextupoles, needed to achieve the very demanding vertical emittance specifications for a damping ring.

We shall briefly mention, without going into detail, tuning and correction methods for achieving very low vertical emittance.

Natural emittance in a storage ring

Recall that the natural (horizontal) emittance in a storage ring is given by:

$$\varepsilon_0 = C_q \gamma^2 \frac{I_5}{I_2 - I_4}$$

where:

- C_q is a physical constant: $C_q = \frac{55}{32\sqrt{3}} \frac{\hbar}{mc} \approx 3.832 \times 10^{-13}$ m
- γ is the relativistic factor;
- I_5 is the fifth synchrotron radiation integral, characterising the quantum excitation of the emittance;
- I_2 is the second synchrotron radiation integral, characterising the energy loss per turn of particles in the ring;
- I_4 is the fourth synchrotron radiation integral, characterising a shift in the balance of damping from the horizontal to the longitudinal plane.

$$I_2 = \oint \frac{1}{\rho^2} ds; \qquad I_4 = \oint \frac{\eta_x}{\rho} \left(\frac{1}{\rho^2} + 2k_1 \right) ds \qquad k_1 = \frac{e}{P_0} \frac{\partial B_y}{\partial x};$$

$$I_5 = \oint \frac{\mathcal{H}_x}{|\rho|^3} ds \qquad \mathcal{H}_x = \gamma_x \eta_x^2 + 2\alpha_x \eta_x \eta_{px} + \beta_x \eta_{px}^2$$

Vertical emittance in a storage ring

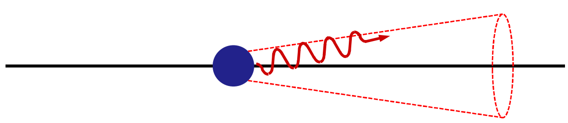
If we assume that the horizontal and vertical motion are independent of each other, then, in principle, we can apply the same analysis to the vertical motion as to the horizontal motion.

This implies that, if we build a ring that is completely "flat" so that:

$$\eta_y = \eta_{p_y} = 0 \qquad \therefore \qquad \mathcal{H}_y = 0$$

everywhere, then $I_{5y}=0$, and the vertical emittance will damp to zero.

However, this is not quite true, since in deriving our expression for the natural emittance, we assumed that all photons were emitted directly along the instantaneous direction of motion of the electron.



In fact, photons are emitted with a distribution with an angular width $1/\gamma$ about the direction of motion of the electron. This leads to some vertical "recoil" that excites vertical betatron motion, resulting in a non-zero vertical emittance.

Fundamental lower limit on the vertical emittance

A detailed analysis* leads to the formula for the fundamental lower limit on the vertical emittance in a storage ring:

$$\varepsilon_{y,\min} = \frac{13}{55}\frac{C_q}{j_y I_2}\int \frac{\beta_y}{|\rho|^3}ds$$

To make a rough estimate of the value of this quantity in the ILC damping rings, let us write:

$$\varepsilon_{y,\min} \approx \frac{1}{4}\frac{\langle\beta_y\rangle C_q}{j_y I_2}\int \frac{1}{|\rho|^3}ds = \frac{\langle\beta_y\rangle}{4}\frac{j_z}{j_y}\frac{\sigma_\delta^2}{\gamma^2}$$

Using some typical values ($\beta_y = 20$ m, $j_z = 2$, $j_y = 1$, $\sigma_\delta = 10^{-3}$, $\gamma = 10^4$) we find that:

$$\varepsilon_{y,\min} \approx 0.1\,\mathrm{pm}$$

This is an order of magnitude smaller than the specified value of 2 pm.
In practice, the vertical emittance is dominated by other effects…

* T. Raubenheimer, SLAC Report 387 (1992).

Vertical emittance in practice

In practice, vertical emittance in a (nominally flat) storage ring is dominated by two effects:

- residual vertical dispersion (couples longitudinal and vertical motion);
- betatron coupling (coupling between horizontal and vertical motion).

The dominant causes of residual vertical dispersion and betatron coupling are magnet alignment errors, in particular:

- tilts of the dipoles around the beam axis;
- vertical alignment errors of the quadrupoles;
- tilts of the quadrupoles around the beam axis;
- vertical alignment errors of the sextupoles.

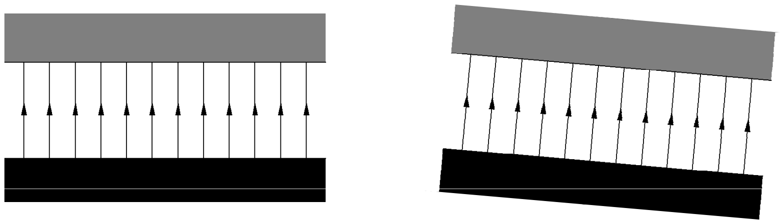
In the rest of this lecture, we shall discuss the impact of these errors on the vertical emittance, and consider ways to minimize the vertical emittance by correcting for these errors.

Steering errors

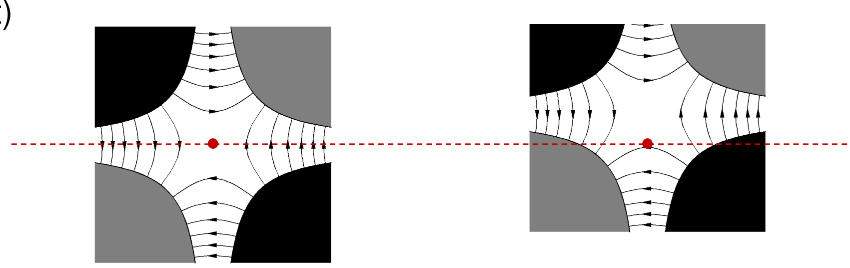
Steering errors lead to distortion of the closed orbit, which generates vertical dispersion, and (through beam offset in the sextupoles) betatron coupling.

A vertical steering error may be generated by:

- rotation of a dipole, so the field is not exactly vertical



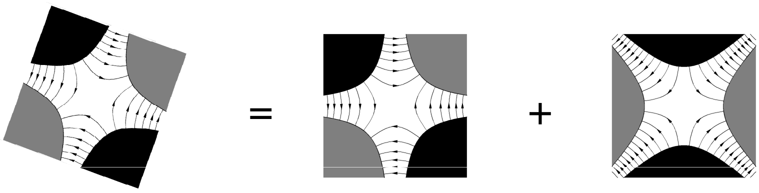
- vertical misalignment of a quadrupole, so that there is a horizontal magnetic field at the location of the reference trajectory ("ideal" closed orbit)



Coupling errors: quadrupole rotations

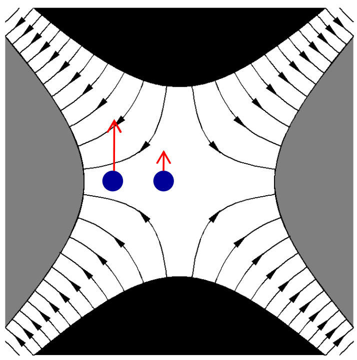
Coupling errors lead to transfer of horizontal betatron motion and dispersion into the vertical plane: in both cases, the result is an increase in vertical emittance.

Coupling may result from rotation of a quadrupole, so that the field contains a skew component.

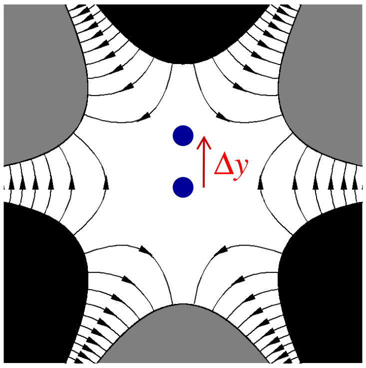


The result is that when particles pass through the magnet, they experience a vertical "kick" that depends on their horizontal offset.

This means that quantum excitation of the horizontal emittance feeds into the vertical plane, blowing up the vertical emittance.



Coupling errors: sextupole misalignments



A vertical beam offset in a sextupole has the same effect as a skew quadrupole.

To understand this, recall that a sextupole field is given by:

$$B_x = k_2 xy \qquad B_y = \tfrac{1}{2} k_2 \left(x^2 - y^2\right)$$

A vertical offset can be represented by $y \to y + \Delta y$:

$$B_x = \boxed{k_2 \Delta y\, x} + k_2 xy \qquad B_y = \boxed{-k_2 \Delta y\, y} + \tfrac{1}{2} k_2 \left(x^2 - y^2\right) - \tfrac{1}{2} k_2 \Delta y^2$$

The first terms in each expression represent a skew quadrupole of strength:

$$k_s = k_2 \Delta y$$

Vertical emittance and magnet alignment

When designing and building a storage ring, we need to know how accurately the magnets must be aligned, to keep the vertical emittance below some specified limit.

To estimate the alignment tolerances requires some understanding of the errors leading to closed orbit distortion, vertical dispersion and coupling.

We shall tackle each of these effects in this lecture, beginning with closed orbit distortion.

Our goal will be to relate the dynamical quantities:

- closed orbit distortion,
- vertical dispersion,
- betatron coupling,

and (ultimately):

- vertical emittance,

to the alignment errors on the magnets.

Closed orbit distortion from steering errors

It is helpful to work with action-angle variables, J_y and ϕ_y, in terms of which we can write the coordinate and momentum of a particle at any point:

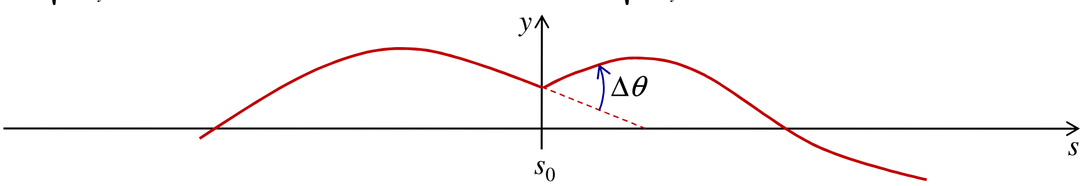
$$y = \sqrt{2\beta_y J_y}\cos\phi_y \qquad\qquad p_y = -\sqrt{\frac{2J_y}{\beta_y}}\left(\sin\phi_y + \alpha_y\cos\phi_y\right)$$

Suppose there is a steering error at some location $s = s_0$, which gives a "kick" $\Delta\theta$ to the vertical momentum. The trajectory of a particle will close on itself (i.e. will be the closed orbit) if:

$$\sqrt{2\beta_{y0}J_{y0}}\cos\left(\phi_{y0} + \mu_y\right) = \sqrt{2\beta_{y0}J_{y0}}\cos\phi_{y0}$$

and:

$$-\sqrt{\frac{2J_{y0}}{\beta_{y0}}}\left(\sin\left(\phi_{y0}+\mu_y\right) + \alpha_{y0}\cos\left(\phi_{y0}+\mu_y\right)\right) = -\sqrt{\frac{2J_{y0}}{\beta_{y0}}}\left(\sin\phi_{y0} + \alpha_{y0}\cos\phi_{y0}\right) - \Delta\theta$$



Closed orbit distortion from steering errors

Solving the equations on the previous slide for the action and angle at s_0:

$$J_{y0} = \frac{\beta_{y0}\Delta\theta^2}{8\sin^2\pi\nu_y} \qquad\qquad \phi_{y0} = \pi\nu_y$$

where $\nu_y = \mu_y/2\pi$ is the vertical tune. Note that if the tune is an integer, there is no solution for the closed orbit: even the smallest steering error will kick the beam out of the ring.

Note that with the above expression for the action and the angle at $s = s_0$, we can write the coordinate of the closed orbit at any point in the ring:

$$y_{co}(s) = \frac{\sqrt{\beta_y(s_0)\beta_y(s)}}{2\sin\pi\nu_y}\Delta\theta\cos\left(\pi\nu_y + \mu_y(s;s_0)\right)$$

Where $\mu_y(s;s_0)$ is the phase advance from s_0 to s.

Finally, we need to add up all the steering errors around the ring. This leads to the final expression for the closed orbit:

$$y_{co}(s) = \frac{\sqrt{\beta_y(s)}}{2\sin\pi\nu_y}\int_0^C \sqrt{\beta_y(s')}\frac{d\theta}{ds'}\cos\left(\pi\nu_y + \mu_y(s;s')\right)ds'$$

Closed orbit distortion from quadrupole alignment errors

It is often helpful to understand the size of the closed orbit error that may be expected from random quadrupole misalignments of a given magnitude. We can derive an expression for this from the closed orbit equation.

For a quadrupole of integrated focusing strength k_1L, vertically misaligned from the reference trajectory by ΔY, the steering is:

$$\frac{d\theta}{ds} = (k_1L)\Delta Y$$

The square of the closed orbit distortion can then be written:

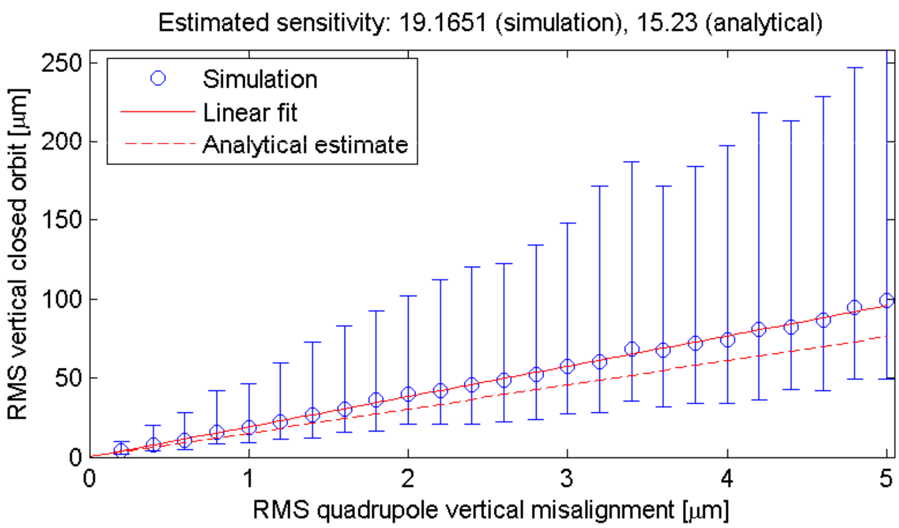
$$\frac{y_{co}{}^2(s)}{\beta_y(s)} = \frac{1}{4\sin^2 \pi\nu_y} \int_0^C \int_0^C \sqrt{\beta_y(s')\beta_y(s'')}(k_1L)_{s'}(k_1L)_{s''}\Delta Y_{s'}\Delta Y_{s''} \cos(\pi\nu_y + \mu_y(s;s'))\cos(\pi\nu_y + \mu_y(s;s''))\, ds' ds''$$

If we average over many seeds of random alignment errors, and assume that the quadrupole alignment errors are uncorrelated, this becomes:

$$\left\langle \frac{y_{co}^2(s)}{\beta_y(s)} \right\rangle = \frac{\langle \Delta Y^2 \rangle}{8\sin^2 \pi\nu_y} \sum_{\text{quads}} \beta_y (k_1L)^2$$

Closed orbit distortion from quadrupole alignment errors

"Orbit amplification factors" are commonly in the range 10 - 100. Of course, the amplification factor is a statistical quantity, over many different sets of misalignments. In any particular case, the orbit distortion may be much larger or smaller than expected from the rms quadrupole alignment error.



Closed orbit errors and vertical emittance

In the context of low-emittance tuning, vertical closed orbit errors are of concern for two reasons:

- Vertical steering generates vertical dispersion, which is a source of vertical emittance.
- Vertical orbit errors contribute to vertical beam offset in the sextupoles, which effectively generates skew quadrupole fields: skew quadrupole fields lead to betatron coupling.

We have seen that we can analyse the beam dynamics to understand the closed orbit distortion that arises from quadrupole alignment errors of a given magnitude.

Recall that our goal is to relate quantities such as orbit distortion, vertical dispersion, coupling, and vertical emittance to the alignment errors on the magnets.

We continue with betatron coupling…

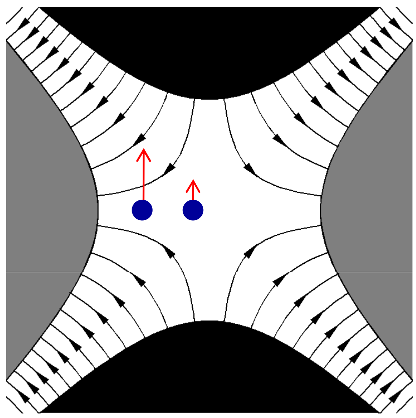
Betatron coupling

Betatron coupling describes the effects that can arise when the vertical motion depends on the horizontal motion, and vice-versa.

For example, a skew quadrupole will give rise to betatron coupling.

A particle passing through a skew quadrupole will receive a vertical "kick", the size of which depends on the horizontal offset of the particle with respect to the centre of the magnet.

Skew quadrupole fields in storage rings often arise from quadrupole tilts, and sextupole vertical alignment errors.



A full treatment of betatron coupling can get quite complex. Also, there are many different formalisms that can be used.

Here, we shall follow a relatively straightforward analysis, to derive an expression for the equilibrium emittances in the presence of coupling.

Hamilton's equations

Our treatment of betatron coupling will be based on Hamiltonian mechanics. In this framework, the equations of motion for a single particle are derived from the Hamiltonian, which is a function of the particle's coordinates and momenta (and, in general, time or distance along the reference trajectory).

$$H = H(\phi_x, J_x, \phi_y, J_y; s)$$

$$\frac{dJ_x}{ds} = -\frac{\partial H}{\partial \phi_x} \qquad \frac{dJ_y}{ds} = -\frac{\partial H}{\partial \phi_y}$$

$$\frac{d\phi_x}{ds} = \frac{\partial H}{\partial J_x} \qquad \frac{d\phi_y}{ds} = \frac{\partial H}{\partial J_y}$$

For example, the Hamiltonian for a particle moving along a linear, uncoupled beam line can be written:

$$H = \frac{J_x}{\beta_x} + \frac{J_y}{\beta_y}$$

Note that the angle is the "coordinate" and the action is the "momentum".

Hamilton's equations in a skew quadrupole

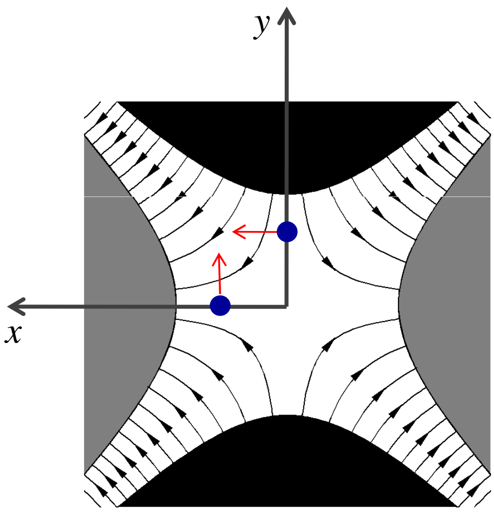
Although it is often helpful to use action-angle variables in Hamiltonian mechanics, it is sometimes easier to start with cartesian variables.

For example, the equations of motion in a skew quadrupole can be written:

$$\frac{dp_x}{ds} = k_s y \qquad \frac{dp_y}{ds} = k_s x$$

$$\frac{dx}{ds} = p_x \qquad \frac{dy}{ds} = p_y$$

$$k_s = \frac{1}{B\rho}\frac{\partial B_x}{\partial x}$$



These equations can be derived from the Hamiltonian:

$$H = \frac{1}{2}p_x^2 + \frac{1}{2}p_y^2 - k_s xy$$

Equations of motion in a coupled beam line

We are interested in the case where there are skew quadrupoles distributed around a storage ring.

The "focusing" effect of a skew quadrupole is represented by a term in the Hamiltonian:

$$k_s xy = 2k_s\sqrt{\beta_x\beta_y}\sqrt{J_xJ_y}\cos\phi_x\cos\phi_y$$

This implies that the Hamiltonian for a beam line with distributed skew quadrupoles can be written:

$$H = \frac{J_x}{\beta_x} + \frac{J_y}{\beta_y} - 2k_s(s)\sqrt{\beta_x\beta_y}\sqrt{J_xJ_y}\cos\phi_x\cos\phi_y$$

The beta functions and the skew quadrupole strength are functions of position, s. This makes it difficult to solve the equations of motion. Therefore, we simplify the problem by "averaging" the Hamiltonian:

$$H = \omega_x J_x + \omega_y J_y - 2\kappa\sqrt{J_xJ_y}\cos\phi_x\cos\phi_y$$

Here, ω_x, ω_y and κ are constants…

Equations of motion in a coupled beam line

$$H = \omega_x J_x + \omega_y J_y - 2\kappa\sqrt{J_xJ_y}\cos\phi_x\cos\phi_y$$

ω_x, ω_y are the betatron frequencies, given by:

$$\omega_{x,y} = \frac{1}{C}\int_0^C \frac{ds}{\beta_{x,y}}$$

For reasons that will become clear shortly, we re-write the coupling term:

$$H = \omega_x J_x + \omega_y J_y - \kappa_-\sqrt{J_xJ_y}\cos(\phi_x - \phi_y) - \kappa_+\sqrt{J_xJ_y}\cos(\phi_x + \phi_y)$$

The constants $\kappa_\pm$ represent the skew quadrupole strength averaged around the ring. However, we need to take into account that the kick from a skew quadrupole depends on the betatron phase. Thus, we write:

$$\kappa_\pm e^{i\chi} = \frac{1}{C}\int_0^C e^{i(\mu_x \pm \mu_y)} k_s\sqrt{\beta_x\beta_y}\, ds$$

Equations of motion in a coupled beam line

Now suppose that $\kappa_- >> \kappa_+$. In that case, we can simplify things further by dropping one term from the Hamiltonian:

$$H = \omega_x J_x + \omega_y J_y - \kappa_- \sqrt{J_x J_y} \cos(\phi_x - \phi_y)$$

We can now write down the equations of motion:

$$\frac{dJ_x}{ds} = -\frac{\partial H}{\partial \phi_x} = \kappa_- \sqrt{J_x J_y} \sin(\phi_x - \phi_y)$$

$$\frac{dJ_y}{ds} = -\frac{\partial H}{\partial \phi_y} = -\kappa_- \sqrt{J_x J_y} \sin(\phi_x - \phi_y)$$

$$\frac{d\phi_x}{ds} = \frac{\partial H}{\partial J_x} = \omega_x + \frac{\kappa_-}{2}\sqrt{\frac{J_y}{J_x}} \cos(\phi_x - \phi_y)$$

$$\frac{d\phi_y}{ds} = \frac{\partial H}{\partial J_y} = \omega_y + \frac{\kappa_-}{2}\sqrt{\frac{J_x}{J_y}} \cos(\phi_x - \phi_y)$$

Equations of motion in a coupled beam line

The equations of motion are rather difficult to solve. Fortunately, however, we do not require the general solution. In fact, we are only interested in the properties of some special cases.

First of all, we note that the sum of the actions is constant:

$$\frac{dJ_x}{ds} + \frac{dJ_y}{ds} = 0 \qquad \therefore \qquad J_x + J_y = \text{constant}$$

This is true in all cases.

Going further, we notice that if $\phi_x = \phi_y$, then the rate of change of each action falls to zero, i.e.:

$$\text{if} \qquad \phi_x = \phi_y \qquad \text{then} \qquad \frac{dJ_x}{ds} = \frac{dJ_y}{ds} = 0$$

This implies that if we can find a solution to the equations of motion with $\phi_x = \phi_y$ for all s, then the actions will remain constant.

Fixed point solutions to the equations of motion in a coupled beam line

From the equations of motion, we find that if:

$$\phi_x = \phi_y \qquad \text{and} \qquad \frac{d\phi_x}{ds} = \frac{d\phi_y}{ds}$$

then:

$$\frac{J_y}{J_x} = \frac{\sqrt{1+\kappa_-^2/\Delta\omega^2} - 1}{\sqrt{1+\kappa_-^2/\Delta\omega^2} + 1}$$

where $\Delta\omega = \omega_x - \omega_y$.

If we further use $J_x + J_y = J_0$, where J_0 is a constant, then we have the *fixed point solution*:

$$J_x = \frac{1}{2}\left(1 + \frac{1}{\sqrt{1+\kappa_-^2/\Delta\omega^2}}\right) J_0$$

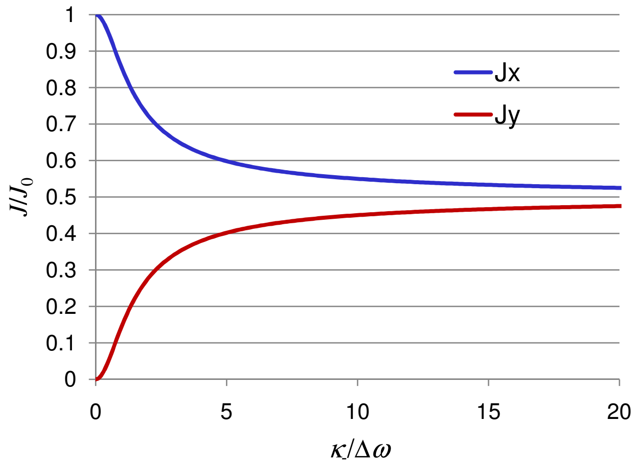
$$J_y = \frac{1}{2}\left(1 - \frac{1}{\sqrt{1+\kappa_-^2/\Delta\omega^2}}\right) J_0$$

Fixed point solutions to the equations of motion in a coupled beam line

Note the behaviour of the fixed-point actions as we vary the "coupling strength" κ_-, and the betatron tunes (betatron frequencies).

The fixed-point actions are well separated for $\kappa_- \ll \Delta\omega$, but approach each other for $\kappa_- \gg \Delta\omega$.

The condition at which the tunes are equal (or differ by an exact integer) is known as the *difference coupling resonance.*



Equilibrium emittances

Recall that the emittance may be defined as the betatron action averaged over all particles in the beam:

$$\varepsilon_x = \langle J_x \rangle \qquad \text{and} \qquad \varepsilon_y = \langle J_y \rangle$$

Now, synchrotron radiation will damp the beam towards an equilibrium distribution. In this equilibrium, we expect the betatron actions of the particles to change only slowly, i.e. on the timescale of the radiation damping, which is much longer than the timescale of the betatron motion.

In that case, the actions of most particles must be in the correct ratio for a fixed-point solution to the equations of motion. Then, if we assume that $\varepsilon_x + \varepsilon_y = \varepsilon_0$, where ε_0 is the natural emittance of the storage ring, we must have for the equilibrium emittances:

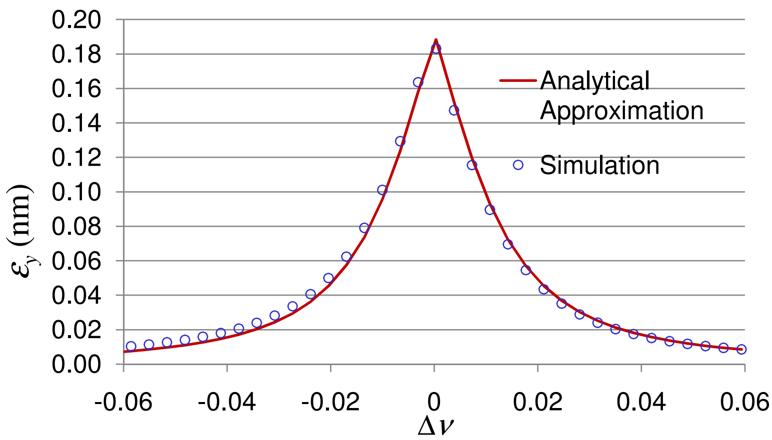
$$\varepsilon_x = \frac{1}{2}\left(1 + \frac{1}{\sqrt{1+\kappa_-^2/\Delta\omega^2}}\right)\varepsilon_0$$

$$\varepsilon_y = \frac{1}{2}\left(1 - \frac{1}{\sqrt{1+\kappa_-^2/\Delta\omega^2}}\right)\varepsilon_0$$

Equilibrium emittances

It may appear that our model is rather simplistic: in fact, we did gloss over many subtle issues. However, we can test our final result by performing a "tune scan" in a real lattice with a known skew quadrupole strength. This is most easily done in simulation…

For the results shown below, we use a single skew quadrupole at a zero-dispersion location (why?) in a DBA lattice. The vertical emittance is computed using Chao's method in MAD8: this takes into account many details that we overlooked, and provides a more reliable answer. Nevertheless, it appears that our formula works reasonably well.



Measuring the coupling strength

There turns out to be an elegant technique for measuring the coupling strength κ_-. If we take a particle close to the fixed-point solution, so that we can assume $\phi_x = \phi_y$, then the Hamiltonian becomes:

$$H = \omega_x J_x + \omega_y J_y - \kappa_- \sqrt{J_x J_y} = \left(\sqrt{J_x} \quad \sqrt{J_y}\right) \cdot \begin{pmatrix} \omega_x & -\frac{1}{2}\kappa_- \\ -\frac{1}{2}\kappa_- & \omega_y \end{pmatrix} \cdot \begin{pmatrix} \sqrt{J_x} \\ \sqrt{J_y} \end{pmatrix}$$

The normal modes can be constructed using the eigenvectors of the matrix in the final expression. The frequencies of these normal modes are the eigenvalues of this matrix. These are the frequencies at which a particle (or a beam of particles) will resonate, if driven by an external oscillator.

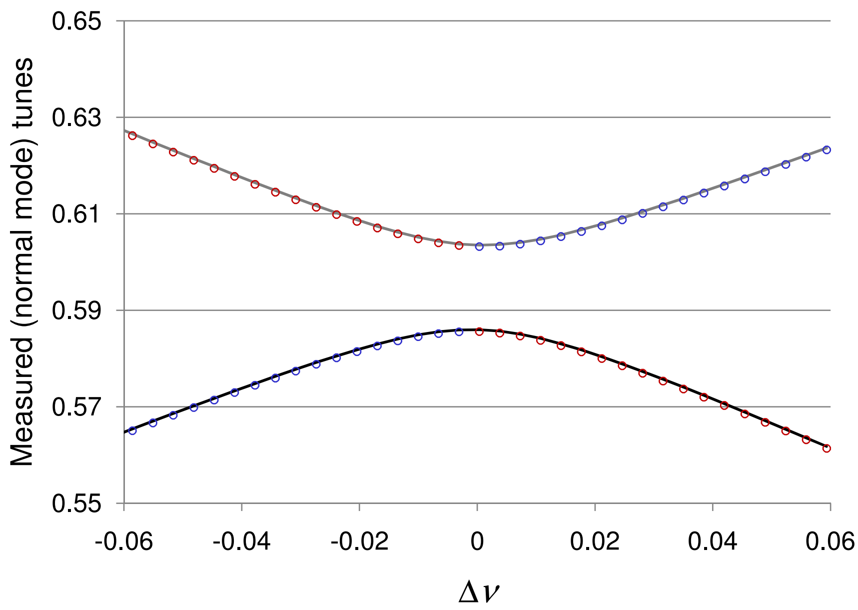
The normal mode frequencies are:

$$\omega_\pm = \frac{1}{2}\left(\omega_x + \omega_y \pm \sqrt{\kappa_-^2 + \Delta\omega^2}\right)$$

Notice that as $\Delta\omega \to 0$, the *measured* tunes are separated by κ_-.

Measuring the coupling strength

Again, we can compare the results of a simulation (circles, in the plot below) with the analytical formula (lines). Again, it seems that the simple model works reasonably well.



Difference and sum coupling resonances

Recall that skew quadrupoles introduced two terms into the Hamiltonian:

$$H = \omega_x J_x + \omega_y J_y - \kappa_- \sqrt{J_x J_y} \cos(\phi_x - \phi_y) - \kappa_+ \sqrt{J_x J_y} \cos(\phi_x + \phi_y)$$

We have assumed that the third term dominates over the fourth. In this case, we have characterised the behaviour as a *difference resonance*:

- The sum of the actions is constant.
- At a fixed-point solution, the angle variables remain equal, and the actions are in a fixed ratio determined by $\kappa_-/\Delta\omega$.

What happens if the fourth term dominates over the third? In this case, the behaviour is completely different, as can be seen by writing down the equations of motion. In particular, there are no fixed point solutions; and the actions can grow indefinitely.

The fourth term can have a strong effect if the *sum* of the tunes is close to an integer: sum resonances are to be avoided.

Coupling, vertical emittance and magnet alignment

Major sources of coupling in storage rings are quadrupole tilts and sextupole alignment. Using the theory just developed, we can make estimates of the alignment tolerances on these magnets, for given optics and specified vertical emittance.

It is sufficient to find an expression for $\kappa/\Delta\omega$ in terms of the optical functions, magnet parameters, and rms alignment.

We start with:

$$\frac{\kappa e^{i\chi}}{\Delta\omega} = \frac{1}{2\pi \Delta\nu} \int_0^C e^{i(\mu_x - \mu_y)} k_s \sqrt{\beta_x \beta_y}\, ds$$

Taking the modulus squared, and using (for sextupoles) $k_s = k_2 \Delta y$:

$$\left(\frac{\kappa}{\Delta\omega}\right)^2 \approx \frac{\langle \Delta Y_S^2 \rangle}{4\pi^2 \Delta\nu^2} \sum_{\text{sextupoles}} \beta_x \beta_y (k_2 l)^2$$

Coupling, vertical emittance and magnet alignment

For example, let us take the case of the ILC damping rings. We require:

$$\frac{\varepsilon_y}{\varepsilon_0} = \frac{1}{2}\left(1 - \frac{1}{\sqrt{1+\kappa^2/\Delta\omega^2}}\right) < 5\times10^{-3}$$

This gives: $$\frac{\kappa}{\Delta\omega} < 0.1$$

For a recent design of the lattice (DCO3, August 2009), we have:

$$\sum_{\text{sextupoles}} \beta_x\beta_y(k_2 l)^2 \approx 3.6\times10^4\,\text{m}^{-2} \qquad \text{and} \qquad \Delta\nu \approx 0.2$$

Thus, from:

$$\left(\frac{\kappa}{\Delta\omega}\right)^2 \approx \frac{\langle \Delta Y_S^2\rangle}{4\pi^2\Delta\nu^2}\sum_{\text{sextupoles}} \beta_x\beta_y(k_2 l)^2$$

we find:

$$\sqrt{\langle \Delta Y_S^2\rangle} < 660\,\mu\text{m}$$

Vertical dispersion

An alignment tolerance of 660 μm on the sextupoles seems reasonably relaxed… however:

- The closed orbit distortion will make a significant contribution to the trajectory offsets in the sextupoles. Assuming an orbit amplification factor of 20, this implies an alignment tolerance on the quadrupoles of order 20 μm.
- We have not made any allowance for vertical dispersion, which will also make a significant contribution to the vertical emittance.

Let us now address the second of these points: the vertical dispersion.

Our goal will be to understand the vertical dispersion that is generated by magnet alignment errors; and to estimate the contribution that this makes to the vertical emittance.

Vertical dispersion

We begin by writing the equation of motion for the trajectory of a particle with momentum P:

$$\frac{d^2 y}{ds^2} = \frac{e}{P} B_x$$

For small energy deviation δ, P is related to the reference momentum P_0 by:

$$P \approx (1+\delta) P_0$$

We can write for the horizontal field (to first order in the derivatives):

$$B_x \approx B_{0x} + y\frac{\partial B_x}{\partial y} + x\frac{\partial B_x}{\partial x}$$

If we consider a particle following an off-momentum closed orbit, so that:

$$y = \eta_y \delta \qquad \text{and} \qquad x = \eta_x \delta$$

then we find, combining the above equations, to first order in δ:

$$\frac{d^2\eta_y}{ds^2} - k_1\eta_y \approx -k_{0s} + k_{1s}\eta_x$$

Vertical dispersion

The "equation of motion" for the dispersion is:

$$\frac{d^2\eta_y}{ds^2} - k_1\eta_y \approx -k_{0s} + k_{1s}\eta_x$$

This is similar to the "equation of motion" for the closed orbit:

$$\frac{d^2 y_{co}}{ds^2} - k_1 y_{co} \approx -k_{0s} + k_{1s} x_{co}$$

We can therefore immediately generalise the relationship between the closed orbit and the quadrupole misalignments, to apply to the dispersion:

$$\left\langle \frac{\eta_y^2(s)}{\beta_y(s)} \right\rangle = \frac{\left\langle \Delta Y_Q^2 \right\rangle}{8\sin^2 \pi\nu_y} \sum_{\text{quads}} \beta_y (k_1 L)^2 + \frac{\left\langle \Delta Y_S^2 \right\rangle}{8\sin^2 \pi\nu_y} \sum_{\text{sexts}} \eta_x^2 \beta_y (k_2 L)^2$$

Here, we assume that the skew dipoles k0s come from vertical alignment errors on the quadrupoles with mean square $\langle \Delta Y_Q^2 \rangle$, and the skew quadrupoles k1s come from vertical alignment errors on the sextupoles, with mean square $\langle \Delta Y_S^2 \rangle$, and that all alignment errors are uncorrelated.

Vertical dispersion and vertical emittance

The final step is to relate the vertical dispersion to the vertical emittance. This is not too difficult. First, the same formula that we obtained (in an earlier lecture) for the horizontal emittance applies:

$$\varepsilon_y = C_q \gamma^2 \frac{I_{5y}}{j_y I_2}$$

where j_y is the vertical damping partition number (=1, usually), and the synchrotron radiation integrals are given by:

$$I_{5y} = \oint \frac{\mathcal{H}_y}{|\rho|^3} ds \qquad \text{where} \qquad \mathcal{H}_y = \gamma_y \eta_y^2 + 2\alpha_y \eta_y \eta_{py} + \beta_y \eta_{py}^2$$

$$I_2 = \oint \frac{1}{\rho^2} ds$$

If the vertical dispersion is generated randomly, then it will, in general, not be correlated with the curvature $1/\rho$ of the reference trajectory. This means (in contrast to the horizontal dispersion) that we can write:

$$I_{5y} \approx \langle \mathcal{H}_y \rangle \oint \frac{1}{|\rho|^3} ds = \langle \mathcal{H}_y \rangle I_3$$

Vertical dispersion and vertical emittance

Hence, we can write for the vertical emittance:

$$\varepsilon_y \approx C_q \gamma^2 \langle \mathcal{H}_y \rangle \frac{I_3}{j_y I_2}$$

It is convenient to express the right hand side in terms of the natural rms energy spread:

$$\sigma_\delta^2 = C_q \gamma^2 \frac{I_3}{j_z I_2}$$

which gives:

$$\varepsilon_y \approx \frac{j_z}{j_y} \langle \mathcal{H}_y \rangle \sigma_\delta^2$$

Now, we note the similarity between the action:

$$2J_y = \gamma_y y^2 + 2\alpha_y y p_y + \beta_y p_y^2$$

and the $\mathcal{H}$-function:

$$\mathcal{H}_y = \gamma_y \eta_y^2 + 2\alpha_y \eta_y \eta_{py} + \beta_y \eta_{py}^2$$

This implies that we can write:

$$\eta_y = \sqrt{\beta_y \mathcal{H}_y} \cos\phi_{\eta y} \qquad \therefore \qquad \left\langle \frac{\eta_y^2}{\beta_y} \right\rangle = \frac{1}{2} \langle \mathcal{H}_y \rangle$$

Dispersion and coupling contributions to the vertical emittance

Combining the equations on the previous slide gives our result:

$$\varepsilon_y \approx 2\frac{j_z}{j_y}\left\langle\frac{\eta_y^2}{\beta_y}\right\rangle\sigma_\delta^2$$

Combining this with our previous result (extended to include quadrupole tilts):

$$\left\langle\frac{\eta_y^2}{\beta_y}\right\rangle \approx \frac{\langle\Delta Y_Q^2\rangle}{8\sin^2\pi\nu_y}\sum_{\text{quads}}\beta_y(k_1L)^2 + \frac{\langle\Delta\Theta_Q^2\rangle}{8\sin^2\pi\nu_y}\sum_{\text{quads}}\eta_x^2\beta_y(k_1L)^2 + \frac{\langle\Delta Y_S^2\rangle}{8\sin^2\pi\nu_y}\sum_{\text{sexts}}\eta_x^2\beta_y(k_2L)^2$$

to relate the *expected* vertical emittance *generated by dispersion* with the rms quadrupole and sextupole alignment errors.

To this, we need to add the vertical emittance generated by coupling:

$$\varepsilon_y \approx \frac{1}{2}\left(1-\frac{1}{\sqrt{1+\kappa^2/\Delta\omega^2}}\right)\varepsilon_0$$

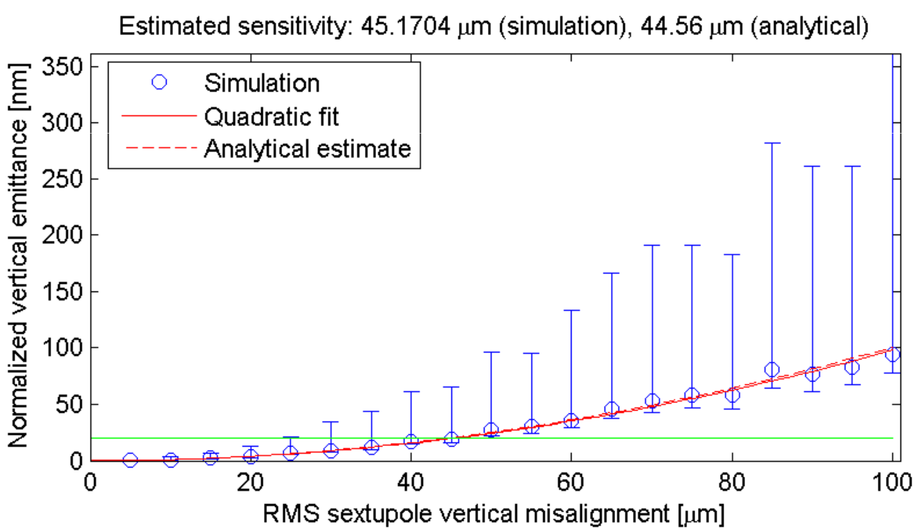
where (including the quadrupole tilts with mean square $\langle\Delta\Theta_Q^2\rangle$):

$$\left(\frac{\kappa}{\Delta\omega}\right)^2 \approx \frac{\langle\Delta\Theta_Q^2\rangle}{4\pi^2\Delta\nu^2}\sum_{\text{quads}}\beta_x\beta_y(k_1l)^2 + \frac{\langle\Delta Y_S^2\rangle}{4\pi^2\Delta\nu^2}\sum_{\text{sexts}}\beta_x\beta_y(k_2l)^2$$

Vertical emittance

We should emphasise that the vertical emittance calculated from the expressions on the previous slide is an "expectation value": it gives the value expected for the statistical mean, taken over many seeds of errors.

For any given seed the vertical emittance can differ significantly from the mean value; nevertheless, the analytical expressions can predict quite well the variation in the mean as a function of rms alignment error. For example, for an old (2005) lattice...



Alignment tolerances in the ILC damping ring lattices

As an example, let us compare the emittance generated by various alignment errors in two different lattice designs for the ILC damping rings.

We shall assume the following arbitrary (and probably very optimistic) alignment errors:

rms quadrupole vertical alignment error	20 μm
rms quadrupole rotation error	150 μrad
rms sextupole vertical alignment error	80 μm

With these errors, we can calculate the various contributions to the vertical emittance (specified limit, 2 pm):

	OCS8 (DBA)	DCO3 (FODO)
Dispersion, quadrupole vertical alignment	0.1 pm	0.07 pm
Dispersion, quadrupole rotation	1.3 pm	0.29 pm
Dispersion, sextupole vertical alignment	2.3 pm	1.8 pm
Coupling, quadrupole rotation	0.03 pm	0.003 pm
Coupling, sextupole vertical alignment	0.03 pm	0.01 pm

Alignment tolerances in the ILC damping ring lattices: Comments

- It appears that the vertical emittance in the DCO3 (FODO) lattice is somewhat less sensitive to alignment errors than that in the OCS8 (DBA) lattice.
  - This is a consequence of the stronger quadrupole and sextupole magnet strengths in the DCO3 lattice.
- The dispersion appears to make a larger contribution to the vertical emittance than the coupling.
  - This is a little surprising, and should not be regarded as a general feature of storage rings. Nevertheless, it does emphasise the importance of minimising the vertical dispersion for achieving a small vertical emittance.
- The assumed alignment errors seem to put us in the right regime for achieving a vertical emittance of 2 pm. But...
  - We have not included some errors (e.g. dipole rolls) that may be significant; and have not considered combined effects (e.g. additional contribution to beam offset in the sextupoles from orbit distortion).
- ...it is too optimistic to assume that the specified vertical emittance of 2 pm could be achieved by survey alignment alone.
  - A systematic and rigorous approach to orbit, dispersion and coupling correction will certainly be necessary.

Correction techniques

Correction of alignment and optics errors is a complex topic: there are many different techniques in use, and new ones are still being developed.

However, without going into details, we can outline a typical procedure:

1. Beam-based alignment.
   - The goal is to determine the offsets of the BPMs with respect to the quadrupoles. This can be achieved by looking for an orbit where changes in quadrupole strength have no effect on the orbit.
2. Optics correction.
   - The goal is to minimise the beta-beat. This may be achieved using a technique such as orbit response matrix (ORM) analysis.
3. Orbit and dispersion correction.
   - The goal is to minimise the closed orbit distortion and the vertical dispersion. This is typically achieved using the response matrix between the steering magnets and the orbit and dispersion.
4. Coupling correction.
   - Typically, one uses skew quadrupoles to minimise the orbit response in one plane to changes in steering in the other plane.

Summary

In this lecture we have seen that:

- There is a fundamental lower limit on the vertical emittance in a storage ring, arising from the non-zero opening angle of the synchrotron radiation. However...
- ...in practice, the vertical emittance in a storage ring is dominated by effects arising from magnet alignment errors.
- Magnet alignment errors lead to orbit distortion, vertical dispersion, and betatron coupling.
- By analysing the dynamics, we can estimate (for a given lattice design) the closed orbit distortion, vertical dispersion and betatron coupling arising from various types of magnet alignment errors.
- Given the vertical dispersion and betatron coupling in a given lattice design, we can estimate the vertical emittance.
- The alignment tolerances for the magnets in a linear collider damping ring will be very tight. A rigorous and systematic correction procedure will be needed to achieve the specified vertical emittance.