Nonlinear Single-Particle Dynamics in High Energy Accelerators

Part 7: Normal Form Analysis

This course consists of eight lectures:

1. Introduction – some examples of nonlinear dynamics
2. Basic mathematical tools and concepts
3. Representations of dynamical maps
4. Integrators I
5. Integrators II
6. Canonical Perturbation Theory
7. Normal form analysis
8. A case study

We have seen how nonlinear dynamics can play an important role in some diverse and common accelerator systems. Nonlinear effects have to be taken into account when designing such systems.

A number of powerful tools for analysis of nonlinear systems can be developed from Hamiltonian mechanics. Using these tools, the solutions to the equations of motion for a particle moving through a component in an accelerator beamline may be represented in various ways.

In the case that the Hamiltonian can be written as a sum of integrable terms, the algebra associated with Lie transforms allows construction of an explicit symplectic integrator that is accurate to some specified order.

In an accelerator, the "global" dynamics that arise from a sequence of components in a beam line are of interest. However, the s-dependent Hamiltonian that describes such a beam line explicitly is usually too complicated to solve explicitly.

However, in a periodic beam line, we can use perturbation theory to construct a generating function that transforms the Hamiltonian into a simpler form. The generating function removes (to some order) terms that drive resonances (as long as the tune is not too close to those resonances).

We can solve the equations of motion for the new Hamiltonian, then relate the old variables to the new variables using the generating function.

Instead of describing the dynamics in a beam line using an s-dependent Hamiltonian, we can construct a map, for example, in the form of a Lie transformation. Such a map may be constructed by concatenating the maps for individual elements.

It may be difficult to understand, simply by inspecting the map, interesting features of the dynamics represented by the map. However, we can carry out a procedure similar to perturbation theory to construct a transformation that puts the map into a simpler form. Particular aspects of the dynamics (for example, the strengths of different resonances) may then be extracted from the transformation.

In the context of maps, such a procedure is called *normal form analysis*. We shall first give an example in the context of linear dynamics; and will then see how it applies for nonlinear maps.

Normal form analysis of linear dynamics

In general, a map (in one degree of freedom) for one period of a periodic linear system may be expressed as a matrix:

$$M = \begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{pmatrix}. \tag{1}$$

If the map is symplectic, then $\beta\gamma - \alpha^2 = 1$.

Normal form analysis of a linear system involves finding a transformation to variables in which the map appears as a pure rotation.

Consider the matrix:

$$N = \begin{pmatrix} \frac{1}{\sqrt{\beta}} & 0 \\ \frac{\alpha}{\sqrt{\beta}} & \sqrt{\beta} \end{pmatrix}. \tag{2}$$

If M is symplectic, then we find that:

$$N \cdot M \cdot N^{-1} = R, \tag{3}$$

where R is a pure rotation matrix:

$$R = \begin{pmatrix} \cos\mu & \sin\mu \\ -\sin\mu & \cos\mu \end{pmatrix}. \tag{4}$$

Note that N is not unique in transforming M into a pure rotation. Any matrix $R' \cdot N$ will have the same effect, where R' is a rotation. However, (2) is the conventional choice.

Using N, we can define "normalised variables" $\vec{x}_N$:

$$\vec{x}_N = N \cdot \vec{x} = N \cdot \begin{pmatrix} x \\ p \end{pmatrix}. \tag{5}$$

In passing through one periodic section of the beam line, the normalised variables transform as:

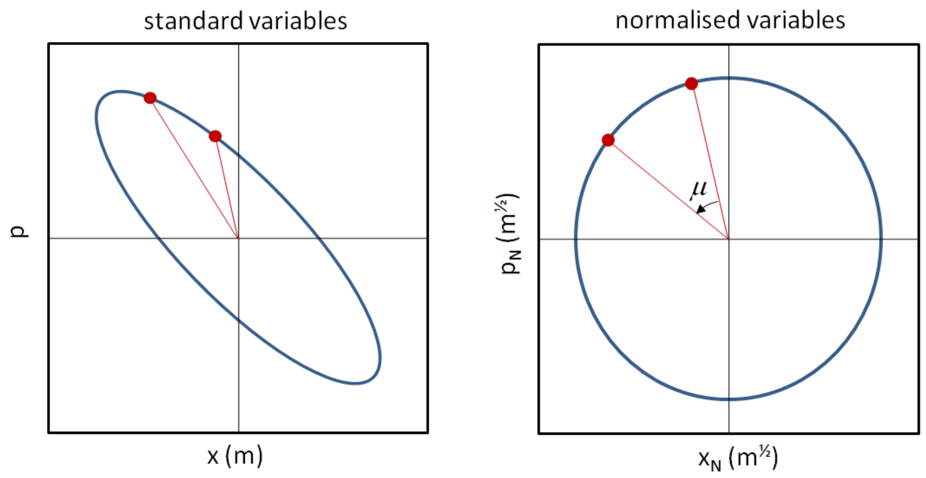
$$\vec{x}_N \mapsto N \cdot M \cdot \vec{x} = N \cdot M \cdot N^{-1} \cdot N \cdot \vec{x}. \tag{6}$$

Then, using (3) and (5), we find:

$$\vec{x}_N \mapsto R \cdot \vec{x}_N. \tag{7}$$

Note that N is symplectic: therefore, it represents a canonical transformation, and the normalised variables $\vec{x}_N$ are canonical variables.

In the normalised variables, the linear map is very simple: particles map out circles (rather than, in general, ellipses) in phase space, with a phase advance of μ each period.



The Twiss parameters are contained in the normalising transformation N; the phase advance is contained in the normalised map R.

For linear dynamics in more than one degree of freedom, normal form analysis leads to a "natural" generalisation of the Twiss parameters.

The normalising transformation can be constructed from the eigenvectors of the one-period map: this applies in any number of degrees of freedom. The generalised Twiss parameters are then identified with the eigenvectors, and the phase advances with the eigenvalues.

For example, a matched distribution (that is invariant under the one-period map) is constructed in one degree of freedom by:

$$\Sigma = \begin{pmatrix} \langle x^2 \rangle & \langle xp \rangle \\ \langle xp \rangle & \langle p^2 \rangle \end{pmatrix} = \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix} \varepsilon, \tag{8}$$

where ε is the emittance (an invariant under transport along the beam line).

In three degrees of freedom, we can generalise (8) to:

$$\Sigma_{ij} = \langle x_i x_j \rangle = \sum_{k=\mathrm{I,II,III}} \beta_{ij}^k \varepsilon_k. \tag{9}$$

where Σ_{ij} is the (i, j) component of the symmetric beam distribution matrix, there are three invariant emittances ε_k, and the phase space vector is:

$$\vec{x} = \begin{pmatrix} x \\ p_x \\ y \\ p_y \\ z \\ \delta \end{pmatrix}. \tag{10}$$

The three 6×6 matrices β^k contain the generalised Twiss parameters, obtained from the eigenvectors of the one-period map. For more details, see the previous lectures on linear dynamics.

Now we turn to normal form analysis of nonlinear dynamics. Our goal is to find a transformation that puts the map into as simple a form as possible.

We shall assume that the map is represented as a Lie transformation, in a Dragt-Finn factorisation:

$$\mathcal{M} = \mathcal{R}\, e^{:f_3:}\, e^{:f_4:} \cdots \tag{11}$$

f_3 is a homogeneous polynomial of order 3 in the phase space variables $\vec{x}$, f_4 a homogeneous polynomial of order 4, and so on.

For simplicity, let us assume that M is of order 4 in the generator (or that the map may be truncated to this order, without losing important features of the dynamics). Then:

$$\mathcal{M} = \mathcal{R}\, e^{:f_3:}\, e^{:f_4:}. \tag{12}$$

In action-angle variables, the linear part of the map may be written as:

$$\mathcal{R} = e^{:-\mu J:}. \tag{13}$$

In general, $\mathcal{R}$ is not a "pure" rotation: the linear phase space will be an ellipse rather than a circle. However, we already have a transformation (2) that will transform the linear phase space into a circle, so we can assume that the linear normalisation can be carried out without difficulty.

Now, let us construct the map M_3 given by:

$$\mathcal{M}_3 = e^{:F_3:}\,\mathcal{M}\,e^{:-F_3:}. \tag{14}$$

Our goal is to find the generator F_3 that removes resonance driving terms from $e^{:f_3:}$. This is equivalent to the first-order perturbation theory that we studied in the previous lecture.

$\mathcal{M}_3$ will then be a "simpler" map than $\mathcal{M}$. Assuming we can continue the procedure, to remove resonance driving terms to successively higher orders, we will obtain (we hope) a map whose dynamics can be solved very easily.

We can then understand the dynamics of the original map in terms of the dynamics of the normalised map, and the normalising transformations.

Let us write (14) in full as:

$$\mathcal{M}_3 = e^{:F_3:}\,\mathcal{R}\,e^{:f_3:}\,e^{:f_4:}\,e^{:-F_3:}. \tag{15}$$

Inserting identity transformations $\mathcal{R}\,\mathcal{R}^{-1}$ and $e^{:-F_3:}\,e^{:F_3:}$, we obtain:

$$\mathcal{M}_3 = \mathcal{R}\,\mathcal{R}^{-1}\,e^{:F_3:}\,\mathcal{R}\,e^{:f_3:}\,e^{:-F_3:}\,e^{:F_3:}\,e^{:f_4:}\,e^{:-F_3:}. \tag{16}$$

Now, we use the result:

$$e^{:h:}\,e^{:g:}\,e^{:-h:} = e^{:e^{:h:}g:}, \tag{17}$$

to write:

$$\mathcal{M}_3 = \mathcal{R}\,e^{:\mathcal{R}^{-1}F_3:}\,e^{:f_3:}\,e^{:-F_3:}\,e^{:e^{:F_3:}f_4:}. \tag{18}$$

Now we use the Baker-Campbell-Hausdorff formula:

$$e^{:A:}\,e^{:B:} = e^{:C:}, \tag{19}$$

where:

$$C = A + B + \frac{1}{2}[A,B] + \cdots \tag{20}$$

$[\circ,\circ]$ is the Poisson bracket:

$$[A,B] = \frac{\partial A}{\partial \vec{q}}\frac{\partial B}{\partial \vec{p}} - \frac{\partial A}{\partial \vec{p}}\frac{\partial B}{\partial \vec{q}}. \tag{21}$$

This allows us to combine some of the factors in (18):

$$\mathcal{M}_3 = \mathcal{R}\,e^{:\mathcal{R}^{-1}F_3 + f_3 - F_3 + O(4):}\,e^{:e^{:F_3:}f_4:}. \tag{22}$$

If F_3 is a polynomial of order 3 in the dynamical variables, then $O(4)$ represents terms of order 4 and higher in the dynamical variables.

With some further manipulation (involving the Zassenhaus formula, and the BCH formula once again) to combine the terms $O(4)$ into the final factor in the map, we obtain:

$$\mathcal{M}_3 = \mathcal{R}\, e^{:f_3^{(1)}:}\, e^{:f_4^{(1)}:}, \tag{23}$$

where:

$$f_3^{(1)} = \mathcal{R}^{-1} F_3 + f_3 - F_3, \tag{24}$$

and $f_4^{(1)}$ is a polynomial containing terms of fourth order (and higher) in the dynamical variables. The form of $f_4^{(1)}$ depends on f_3, f_4 and F_3.

We shall not pursue the normalisation to higher order, so we do not concern ourselves further with $f_4^{(1)}$, other than to note that, with the BCH formula and the Zassenhaus formula, we have the appropriate tools to construct $f_4^{(1)}$ in any given case.

The solution to (24) is:

$$F_3 = \left(\mathcal{I} - \mathcal{R}^{-1}\right)^{-1} \left(f_3 - f_3^{(1)}\right), \tag{25}$$

where $\mathcal{I}$ is the identity transformation.

Since f_3 is periodic in the angle variable ϕ, we can write:

$$f_3 = \sum_m \tilde{f}_{3,m}(J) e^{im\phi}. \tag{26}$$

Then, if:

$$f_3^{(1)} = \tilde{f}_{3,0}(J) \tag{27}$$

F_3 is given by:

$$F_3 = \sum_{m \neq 0} \frac{\tilde{f}_{3,m}(J) e^{im\phi}}{1 - e^{-im\mu}}. \tag{28}$$

That (28) is indeed the solution for F_3 can easily be checked, using:

$$\mathcal{R}^{-1}h(\phi, J) = h(\phi - \mu, J), \tag{29}$$

for any function h.

Note that we cannot construct a transformation that will remove terms in f_3 (26) that are independent of the angle variable ϕ. But these terms simply lead to a tune shift with amplitude, and do not limit our ability to solve the dynamics.

The normalised map is therefore:

$$\mathcal{M}_3 = e^{:F_3:}\,\mathcal{M}\,e^{:-F_3:} = \mathcal{R}\,e^{:\tilde{f}_{3,0}:}\,e^{:f_4^{(1)}:}. \tag{30}$$

To lowest order in the nonlinear perturbation, the normalised map contains only a tune shift with amplitude: and resonance driving terms have been pushed to higher order.

The normalising transformation is, from (28):

$$F_3 = \sum_{m \neq 0} \frac{\tilde{f}_{3,m}(J)e^{im\phi}}{1 - e^{-im\mu}}. \tag{31}$$

We can interpret the quantities $\tilde{f}_{3,m}(J)$ as "resonance strengths".

If there is a Fourier mode m in f_3 such that $m\mu = 2\pi \times$integer, then the generator of the transformation diverges, and the perturbation has a large effect.

As an example, let us consider the case of an octupole perturbation in an otherwise linear, periodic beam line (we've looked at a sextupole often enough!)

The map for one period is:

$$\mathcal{M} = \mathcal{R}\, e^{:f_4:}, \tag{32}$$

where:

$$f_4 = -\frac{1}{24} k_3 \ell\, x^4. \tag{33}$$

In action-angle variables, the generator of the perturbation can be written:

$$f_4 = -\frac{1}{6} k_3 \ell\, \beta^2 J^2 \cos^4 \phi \tag{34}$$

$$= -\frac{1}{48} k_3 \ell\, \beta^2 J^2 \left(3 + 4\cos 2\phi + \cos 4\phi\right). \tag{35}$$

From (35), we see that the generator for the nonlinear part of the normalised map is:

$$\tilde{f}_{4,0} = -\frac{1}{16} k_3 \ell\, \beta^2 J^2, \tag{36}$$

so the normalised map itself is:

$$\mathcal{M}_4 = \mathcal{R}\, e^{:\tilde{f}_{4,0}:} = e^{:-\mu J - \frac{1}{16} k_3 \ell\, \beta^2 J^2:} \approx e^{:F_4:} \mathcal{M} e^{:-F_4:}. \tag{37}$$

Note that the normalised map just gives a rotation in phase space, but that the rotation angle depends on the amplitude of the particle.

The generator for the normalising transformation is:

$$F_4 = -\frac{1}{96} k_3 \ell\, \beta^2 J^2 \left\{ \frac{4\left[\cos 2\phi - \cos 2(\phi + \mu)\right]}{1 - \cos 2\mu} + \frac{\left[\cos 4\phi - \cos 4(\phi + \mu)\right]}{1 - \cos 4\mu} \right\}. \tag{38}$$

We can define "normalised variables" (ϕ_N, J_N) as:

$$\begin{pmatrix} \phi_N \\ J_N \end{pmatrix} = e^{:F_4:} \begin{pmatrix} \phi \\ J \end{pmatrix}. \tag{39}$$

The normalised variables simply evolve according to the normalised map:

$$\begin{pmatrix} \phi_N \\ J_N \end{pmatrix} \mapsto \mathcal{M}_4 \begin{pmatrix} \phi_N \\ J_N \end{pmatrix} \tag{40}$$

(where we need to replace J in the generator for $\mathcal{M}_4$ by J_N).

To track particles through the beam line we can either apply the full nonlinear map (32), or:

1. transform to normalised variables using (39);

2. track through as many periods as required, using the normalised map (37);

3. transform back to the original variables using the inverse of the transformation in equation (39).

For the final step, note that the inverse of the transformation $e^{:F_4:}$ is simply $e^{:-F_4:}$.

The normalising transformation acting on the action-angle variables can be represented as:

$$e^{:F_4:}\phi = \phi + \frac{\partial F_4}{\partial J} + \cdots \tag{41}$$

$$e^{:F_4:}J = J - \frac{\partial F_4}{\partial \phi} + \cdots \tag{42}$$

If:

$$\frac{\partial F_4}{\partial J} \ll 1, \quad \text{and} \quad \frac{\partial F_4}{\partial \phi} \ll J, \tag{43}$$

then the normalising transformation will be close to the identity, and the dynamics are essentially those of the normalised map.

Whether the conditions (43) are satisfied depends on: the amplitude J, the strength of the octupole $k_3\ell$ and the beta function β at the octupole, and the proximity to a resonance that is driven by the octupole.

The resonances driven by the octupole can be seen from (35). The driving terms are $\sim \cos 2\phi$ and $\sim \cos 4\phi$: these terms drive the half-integer and quarter-integer resonances, $2\nu =$ integer and $4\nu =$ integer, where the linear part of the map gives a phase advance $\mu = 2\pi\nu$.

Let us first consider the case where we are far from resonance; for example $\mu \approx 2\pi/3$. This is a third-integer resonance, but is not driven by the octupole. We will take $k_3\ell = 4800\,\text{m}^{-3}$, and $\beta = 1\,\text{m}$.

Assuming that the normalising transformation is close to the identity, let us work out the effect of the normalised map on the action-angle variables.

Retaining only lowest order nonlinear terms, the normalised map is:

$$\mathcal{M}_4 = e^{:-\mu J:}\, e^{:-\frac{1}{16}k_3\ell\,\beta^2 J^2:}. \tag{44}$$

Since the generators of each factor in $\mathcal{M}_4$ include only the action variable J (and are independent of the angle variable ϕ), we may combine them using the BCH formula to give:

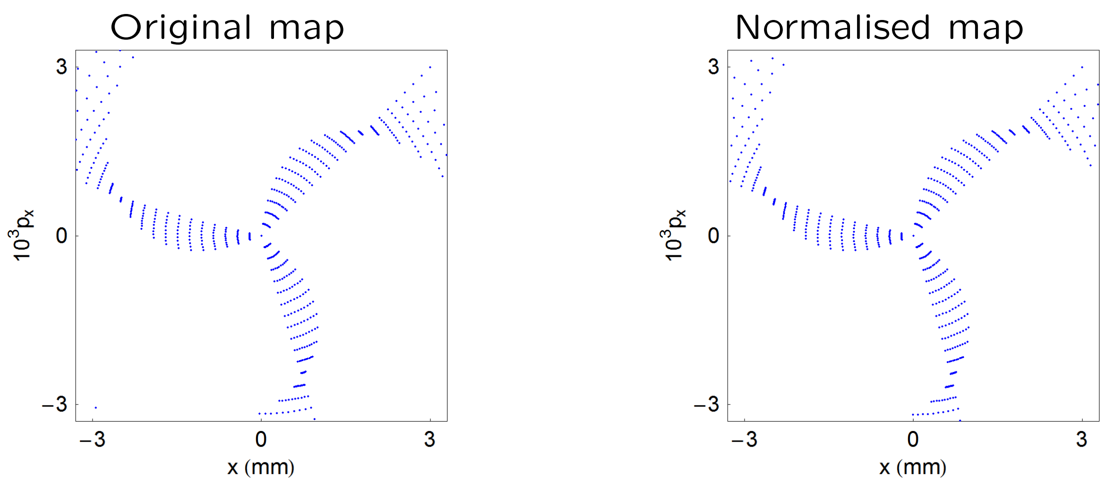
$$\mathcal{M}_4 = e^{:-\mu J - \frac{1}{16}k_3\ell\,\beta^2 J^2:}. \tag{45}$$

The (normalised) map for one periodic section of the beam line is given by:

$$J \mapsto J \tag{46}$$

$$\phi \mapsto \phi + \mu + \frac{1}{8}k_3\ell\,\beta^2 J. \tag{47}$$

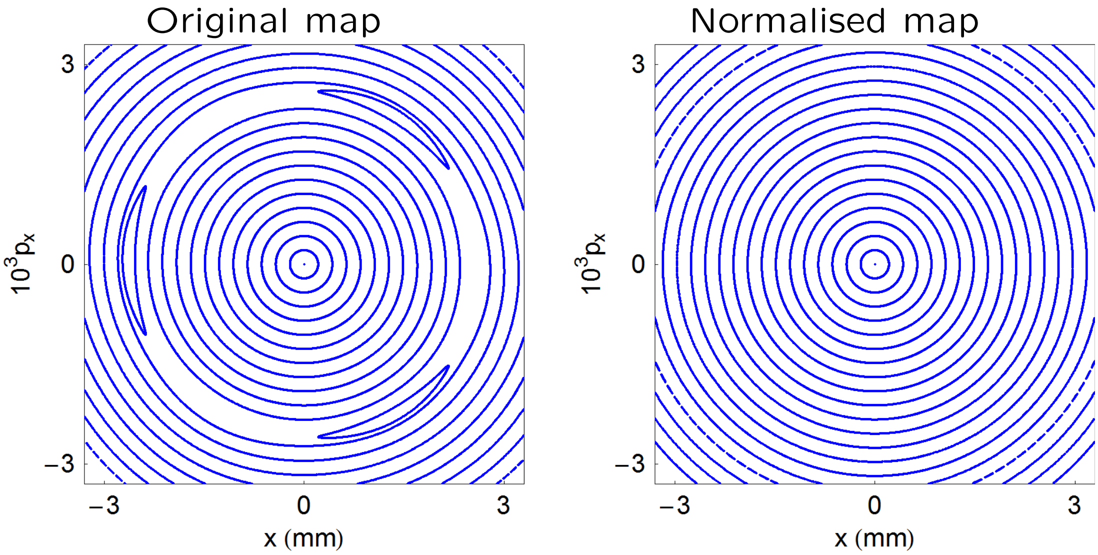
If $k_3\ell$ is positive, the tune increases with increasing amplitude J. We can illustrate the effect by setting the (linear) tune $\mu = 0.330 \times 2\pi$ and tracking a set of particles with different actions through 30 periods:



In this case, the normalising transformation is indeed close to the identity, at least up to the amplitudes shown, and we see clearly the effect of the tune shift with amplitude.

Nonlinear normal form example: octupole perturbation

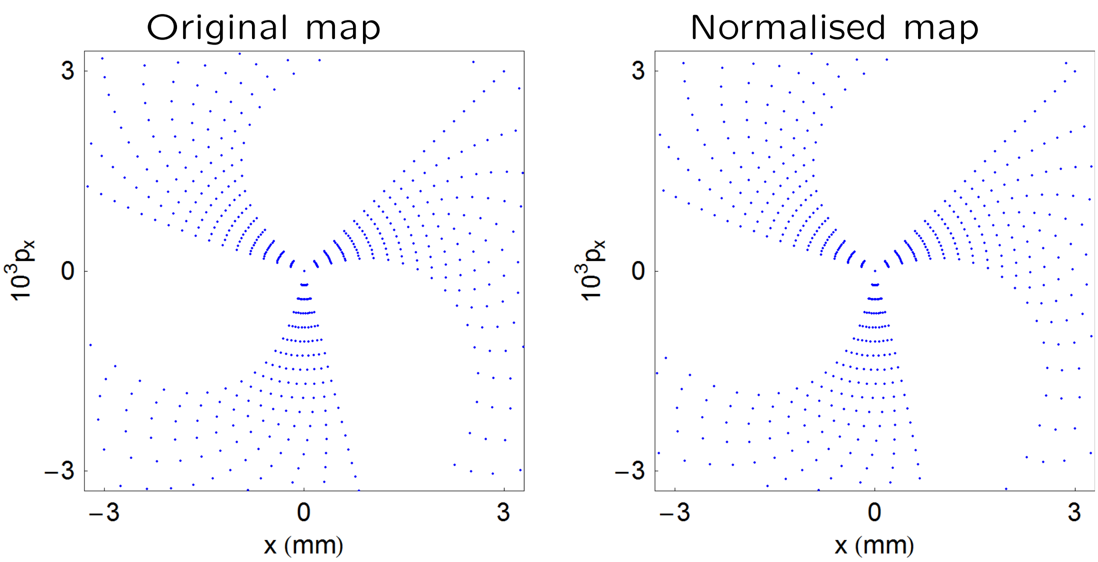
Tracking through 2500 turns, again with tune 0.330:



It appears that the third-integer resonance is weakly driven in the original map; of course, this behaviour cannot be reproduced in the normalised map.

Nonlinear normal form example: octupole perturbation

If we track 30 turns, but this time with tune just above the third integer (0.336), we again see clearly the effect of the tune shift with amplitude:



If we adjust the tune so that it is close to a fourth integer, then, since this resonance is driven by the octupole, we expect to see some significant nonlinear distortion of phase space. In this case, we can attempt to reconstruct the phase space from the normalising transformation as follows.

Since the original map is related to the normalised map by:

$$\mathcal{M}_4 \approx e^{:F_4:}\mathcal{M}e^{:-F_4:}, \tag{48}$$

we must have:

$$\mathcal{M} \approx e^{:-F_4:}\mathcal{M}_4 e^{:F_4:}. \tag{49}$$

If we write:

$$\mathcal{M}_4 = e^{:m_4:}, \tag{50}$$

then, by using the rules for a similarity transformation:

$$\mathcal{M} \approx \exp : e^{:-F_4:} m_4 :. \tag{51}$$

But for any map $\mathcal{G}$ with generator g:

$$\mathcal{G} = e^{:g:}, \tag{52}$$

the function g must be an invariant, i.e.:

$$\mathcal{G}\, g = e^{:g:}\, g = g. \tag{53}$$

This is because the Poisson bracket of any function with itself vanishes:

$$:g:g = [g, g] = 0. \tag{54}$$

Therefore, since:

$$\mathcal{M}_4 = e^{:-\mu J - \mu_2 J^2:}, \tag{55}$$

(where μ_2 represents the tune shift with amplitude), we must have that:

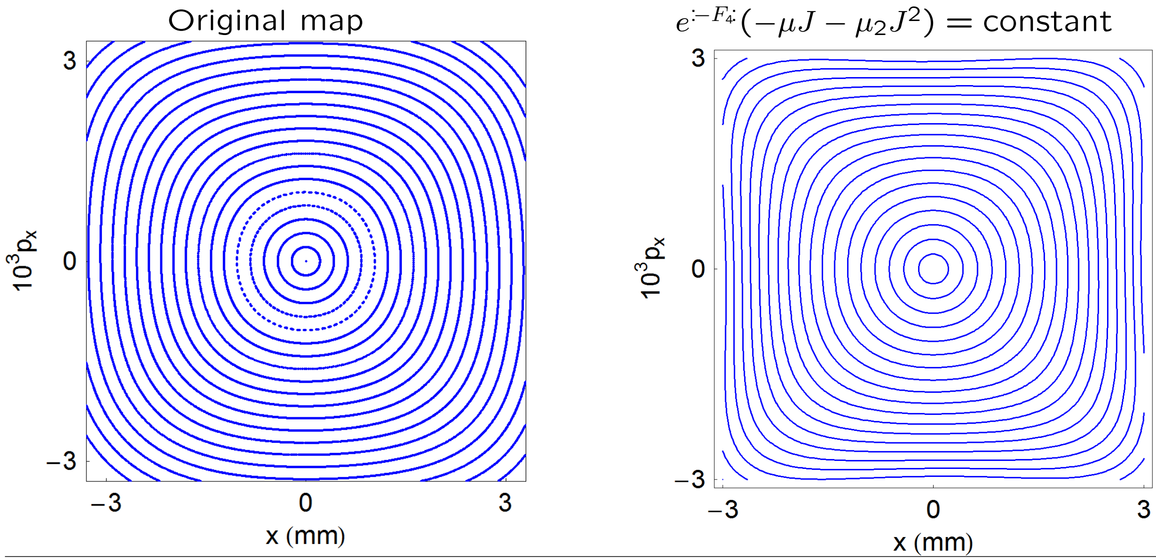
$$e^{:-F_4:}(-\mu J - \mu_2 J^2) \tag{56}$$

is an (approximate) invariant of the original map $\mathcal{M}$.

We can then plot a phase space portrait without any tracking at all. We just need to draw contours of:

$$e^{:-F_4:}(-\mu J - \mu_2 J^2) = \text{constant}. \tag{57}$$

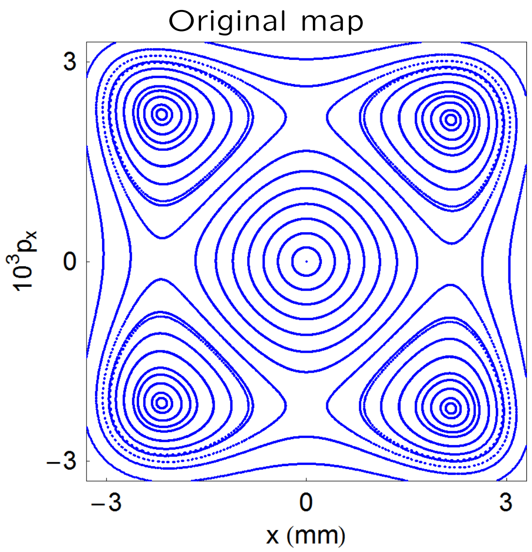
Let us compare such a contour plot with the results of tracking with the original map, for a tune = 0.252:

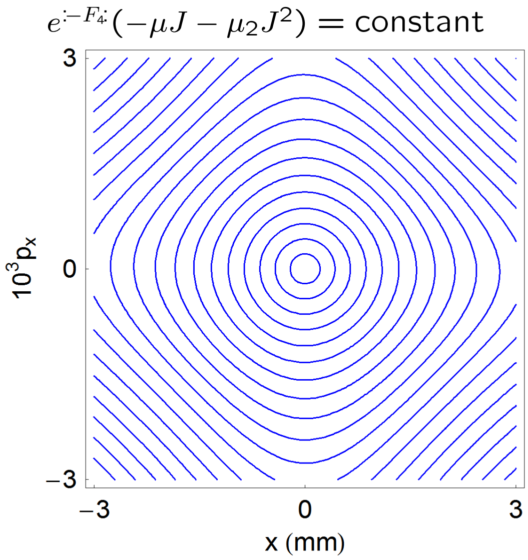


Above the quarter-integer resonance, the tune shift with amplitude takes us away from resonance, and we get reasonable (not perfect) agreement between the tracking plot and the contour plot.

Just below the quarter-integer resonance, however, the story is rather different...

The contour plot is unable to reproduce the resonant "islands" that we find from the original map. The reason for the difference between the plots, is that we are too close to the resonance for the generating transformation to describe the dynamics accurately.





Summary

Normal form analysis is the analogue for (discrete) maps of perturbation theory for (continuous) Hamiltonian systems. The goal is to find a transformation that puts the map into as simple a form as possible.

We have described a procedure for constructing a canonical transformation to normalise the lowest-order nonlinear part of the map.

We find a similar situation to perturbation theory: we cannot remove pure tune shifts with amplitude, and the normalisation fails if the tune is on a resonance driven by the nonlinear terms.

Applying normal form analysis allows us to: (i) identify tune-shifts with amplitude (from the terms remaining in the normalised map); and (ii) characterise the nonlinear distortion of phase space (from the normalising transformation).

For a far more complete and rigorous treatment of normal form analysis, including extension of the technique to normalise terms beyond leading order in the nonlinear part of the map, see:

E. Forest, "Beam dynamics: a new attitude and framework," Harwood Academic Publishers (1998).

In the final lecture, we shall look at a case study of nonlinear dynamics.

We shall see how many of the techniques described in previous lectures may be applied in a "real" situation: analysis of the impact of a long wiggler on the dynamics in the ILC damping rings.

As well as reviewing some of the techniques we have already seen, we shall also discuss frequency map analysis, and its application to storage ring beam dynamics.

Apply normal form analysis to a sextupole perturbation.