Nonlinear Single-Particle Dynamics in High Energy Accelerators

Part 6: Canonical Perturbation Theory

This course consists of eight lectures:

1. Introduction – some examples of nonlinear dynamics
2. Basic mathematical tools and concepts
3. Representations of dynamical maps
4. Integrators I
5. Integrators II
6. Canonical Perturbation Theory
7. Normal form analysis
8. Some numerical techniques

In the previous lectures...

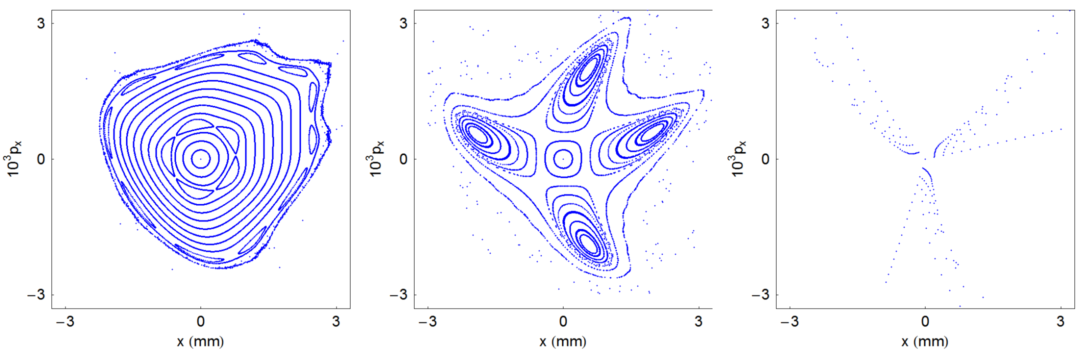
We have seen how nonlinear dynamics can play an important role in some diverse and common accelerator systems. Nonlinear effects have to be taken into account when designing such systems.

A number of powerful tools for analysis of nonlinear systems can be developed from Hamiltonian mechanics. Using these tools, the solutions to the equations of motion for a particle moving through a component in an accelerator beamline may be represented in various ways, including: (truncated) power series; Lie transform; (implicit) generating function.

In the case that the Hamiltonian can be written as a sum of integrable terms, the algebra associated with Lie transforms allows construction of an explicit symplectic integrator that is accurate to some specified order.

In this lecture...

We now start to consider what happens when nonlinear elements are combined in an accelerator beam line. In particular, we shall aim to understand features of phase space portraits that can be constructed by tracking multiple particles multiple times through a periodic beam line (e.g. a storage ring).



There are two common approaches to analysis of nonlinear periodic systems:

1. Canonical perturbation theory, based on canonical transformation of an s-dependent Hamiltonian.

2. Normal form analysis, based on Lie transformation of a single-turn map.

In fact, both these approaches are really attempting to do the same thing: the goal is to find a transformation that puts the Hamiltonian, or the map, into the simplest possible form.

In this lecture, we shall develop the first approach, canonical perturbation theory.

For clarity, we shall work in one degree of freedom. The extension to multiple degrees of freedom lengthens the algebra, but does not add any significant new features.

In action-angle variables, the Hamiltonian for a periodic beam line can be written as:

$$H = \frac{J}{\beta} + \epsilon V(\phi, J, s), \tag{1}$$

where the perturbation term V contains the nonlinear features of the dynamics, and ϵ is a small parameter. Note that V depends on the independent variable, s (the distance along the reference trajectory).

In general, the equations of motion following from (1) are difficult to solve.

Our goal is to find canonical transformations that remove the nonlinear term from the Hamiltonian, to progressively higher order in ϵ.

The equations of motion in the new variables will be easy to solve, at least in the approximation that terms of higher order in ϵ may be neglected. In fact, since we explicitly remove any angle-dependence from the Hamiltonian, the solutions to the equations of motion will satisfy:

$$J_1 = \text{constant} \tag{2}$$

where J_1 is the action variable in the transformed coordinates. Expressing the original coordinates J and ϕ in terms of J_1 allows us easily to construct a phase space portrait.

For reasons that will become clearer shortly, let us generalise the Hamiltonian (1), by replacing the linear term by a general function of the action variable:

$$H = H_0(J) + \epsilon V(\phi, J, s). \tag{3}$$

Then, we define ω_β as:

$$\omega_\beta = \frac{dH_0}{dJ}. \tag{4}$$

For purely linear motion, $H_0 = J/\beta$, so $\omega_\beta = 1/\beta$ which is the betatron frequency (the betatron phase advance per metre of beam line). But in general, ω_β is a function of the betatron amplitude (i.e. a function of the betatron action, J).

Let us consider a generating function of the second kind:

$$F_2(\phi, J_1, s) = \phi J_1 + \epsilon G(\phi, J_1, s). \tag{5}$$

The new and old dynamical variables and Hamiltonians are related by:

$$\begin{aligned} J &= \frac{\partial F_2}{\partial \phi} = J_1 + \epsilon \frac{\partial G}{\partial \phi}, && (6) \\ \phi_1 &= \frac{\partial F_2}{\partial J_1} = \phi + \epsilon \frac{\partial G}{\partial J_1}, && (7) \\ H_1 &= H + \frac{\partial F_2}{\partial s} = H + \epsilon \frac{\partial G}{\partial s}. && (8) \end{aligned}$$

We see that if ϵ is small, then the transformation is close to the identity.

The new Hamiltonian can be written as:

$$H_1 = H_0\left(J_1 + \epsilon \frac{\partial G}{\partial \phi}\right) + \epsilon V\left(\phi, J_1 + \epsilon \frac{\partial G}{\partial \phi}, s\right) + \epsilon \frac{\partial G}{\partial s}. \tag{9}$$

Note that in this form, the Hamiltonian is expressed in terms of the "mixed" variables ϕ and J_1. Eventually, to solve the equations of motion, we will need to substitute for ϕ, so that the Hamiltonian is expressed purely in terms of the new variables (ϕ_1, J_1). For now, however, it is convenient to leave the Hamiltonian in the mixed form.

By adding and subtracting appropriate terms, we can rewrite the Hamiltonian as:

$$\begin{aligned} H_1 &= H_0(J_1) + H_0\left(J_1 + \epsilon\frac{\partial G}{\partial \phi}\right) - H_0(J_1) \\ &\quad + \epsilon V\left(\phi, J_1 + \epsilon\frac{\partial G}{\partial \phi}, s\right) - \epsilon V(\phi, J_1, s) + \epsilon V(\phi, J_1, s) \\ &\quad + \epsilon\frac{\partial G}{\partial s}, \qquad (10) \\ &\approx H_0(J_1) + \epsilon\left[\omega_\beta(J_1)\frac{\partial G}{\partial \phi} + \frac{\partial G}{\partial s} + V(\phi, J_1, s)\right] \\ &\quad + \epsilon^2\frac{\partial}{\partial J_1}V(\phi, J_1, s). \qquad (11) \end{aligned}$$

Note that in the last step, we have used the definition $\omega_\beta(J_1) = H_0'(J_1)$.

We see that if we can find a generating function $F_2 = \phi J_1 + \epsilon G(\phi, J_1, s)$ where G satisfies:

$$\omega_\beta(J_1)\frac{\partial G}{\partial \phi} + \frac{\partial G}{\partial s} + V(\phi, J_1, s) = 0, \qquad (12)$$

then the terms in ϵ in the Hamiltonian H_1 are second-order and higher:

$$H_1 \approx H_0(J_1) + \epsilon^2\frac{\partial}{\partial J_1}V(\phi, J_1, s). \qquad (13)$$

In an accelerator beam line, the perturbation V can be written in terms of the coordinate x. Then, since x is a periodic function of ϕ, it must be the case that V is also a periodic function of ϕ. Therefore, we can write V as a sum over modes:

$$V(\phi, J_1, s) = \sum_m \tilde{V}_m(J_1, s)\, e^{im\phi}. \tag{14}$$

We assume that the function G appearing as a term in the generating function F_2 is also periodic in ϕ:

$$G(\phi, J_1, s) = \sum_m \tilde{G}_m(J_1, s)\, e^{im\phi}. \tag{15}$$

Then, equation (12) becomes:

$$\left(im\omega_\beta(J_1) + \frac{\partial}{\partial s} \right) \tilde{G}_m = -\tilde{V}_m. \tag{16}$$

By substitution into equation (16), we see that the solution for $\tilde{G}_m(J_1, s)$ is:

$$\tilde{G}_m(J_1, s) = \frac{i}{2\sin\left(\frac{1}{2}m\omega_\beta L\right)} \int_s^{s+L} e^{im\omega_\beta\left(s' - s - \frac{L}{2}\right)} \tilde{V}_m(J_1, s')\, ds', \tag{17}$$

where L is the length of one periodic section of the beam line.

Strictly speaking, this solution assumes that the betatron frequency ω_β is constant along the beam line (i.e. is independent of s). However, we can generalise the result. If we define the phase $\psi(s)$ and the "tune" ν:

$$\psi(s) = \int_0^s \omega_\beta\, ds, \qquad \nu = \frac{1}{2\pi} \int_s^{s+L} \omega_\beta\, ds, \tag{18}$$

then we can write the expression for $\tilde{G}_m(J_1, s)$:

$$\tilde{G}_m(J_1, s) = \frac{i}{2\sin(\pi m\nu)} \int_s^{s+L} e^{im[\psi(s') - \psi(s) - \pi\nu]} \tilde{V}_m(J_1, s')\, ds'. \tag{19}$$

Finally, the function $G_m(\phi, J_1, s)$ can be written:

$$G(\phi, J_1, s) = \sum_m \frac{i}{2\sin(\pi m \nu)} \int_s^{s+L} e^{im[\phi + \psi(s') - \psi(s) - \pi\nu]} \tilde{V}_m(J_1, s')\, ds'. \tag{20}$$

Equation (20) is an important result: it tells us how to construct a canonical transformation that removes (to first order) an s-dependent perturbation from the Hamiltonian.

We can already see some interesting properties. For example, the expression for G gets large when $m\nu$ is close to an integer. Even a small perturbation can have a large effect when the lattice is tuned to a resonance: but the impact also depends on the resonance *strength*, i.e. the integral in (20). If the resonance strength is zero, then we can sit right on a resonance without adverse effect.

Note that if $V(\phi, J_1, s)$ has a non-zero average over ϕ, then $\tilde{V}_0(J_1, s)$ is non-zero. This implies there is a resonance strength for $m = 0$, given by:

$$\int_s^{s+L} \tilde{V}_0(s')\, ds'. \tag{21}$$

But for $m = 0$, $m\nu$ is an integer (zero) for all ν.

Therefore, no matter what the tune of the lattice, we cannot construct a generating function to remove any term in the perturbation that is independent of ϕ.

However, such terms may be absorbed into $H_0(J)$: this is the reason why we introduced $H_0(J)$ (as a generalisation of J/β) in the first place.

The canonical transformation (5):

$$F_2(\phi, J_1, s) = \phi J_1 + \epsilon G(\phi, J_1, s)$$

removes the perturbation term in the Hamiltonian, so that:

$$H_1 = H_0(J_1) + O(\epsilon^2). \tag{22}$$

Since H_1 is independent of ϕ_1 (to first order in ϵ), we can write:

$$J_1 = \text{constant} + O(\epsilon^2). \tag{23}$$

The original action variable is then given by (6):

$$J = J_1 + \epsilon \frac{\partial G}{\partial \phi} = J_0 + \epsilon \frac{\partial}{\partial \phi} G(\phi, J_0, s) + O(\epsilon^2), \tag{24}$$

where J_0 is a constant. At a given location in the beam line (i.e. for a given s), we can construct a phase space portrait by plotting J as a function of ϕ (between 0 and 2π) for different values of J_0.

Equipped with equation (20), we are now ready to look at some examples. Before proceeding to a nonlinear case (sextupole perturbation), we will look at a linear perturbation, namely, quadrupole focusing. This will allow us to compare quantities such as the tune shift and phase space (i.e. beta function) distortion that we obtain from perturbation theory, with the expressions that we may obtain from a purely linear theory.

In cartesian variables, the Hamiltonian is:

$$H = \frac{p^2}{2} + K(s)\frac{x^2}{2} + \epsilon k(s)\frac{x^2}{2}. \tag{25}$$

The first two terms can be written in action-angle variables as $J/\beta(s)$. The perturbation term (in ϵ) can be transformed into action-angle variables using:

$$x = \sqrt{2\beta(s)J}\cos\phi. \tag{26}$$

Thus, we have:

$$H = \frac{J}{\beta(s)} + \frac{1}{2}\epsilon k(s)\beta(s)J(1 + \cos 2\phi). \tag{27}$$

We see immediately that the perturbation introduces a term independent of ϕ. We therefore define:

$$H_0(J) = \frac{J}{\beta(s)} + \frac{1}{2}\epsilon k(s)\beta(s)J, \tag{28}$$

so that the Hamiltonian is written:

$$H = H_0(J) + \frac{1}{2}\epsilon k(s)\beta(s)J\cos 2\phi. \tag{29}$$

The tune of the lattice (i.e. the phase advance across one periodic section of length L) is given by:

$$\nu = \frac{1}{2\pi}\int_s^{s+L} \omega_\beta \, ds = \frac{1}{2\pi}\int_s^{s+L} \frac{dH_0}{dJ}\, ds. \tag{30}$$

From equation (28), we see that the tune with the perturbation is:

$$\nu = \frac{1}{2\pi}\int_s^{s+L} \frac{ds}{\beta(s)} + \frac{\epsilon}{4\pi}\int_s^{s+L} k(s)\beta(s)\, ds. \tag{31}$$

The change in tune resulting from the perturbation is:

$$\Delta\nu = \frac{1}{2\pi}\int_s^{s+L} \frac{ds}{\beta(s)} + \frac{\epsilon}{4\pi}\int_s^{s+L} k(s)\beta(s)\, ds. \tag{32}$$

This is the same expression we would have obtained using standard linear (matrix) theory.

To find the change in beta function, we need to use equation (20). This will give us a (canonical) transformation to new variables J_1, ϕ_1; to first order in the perturbation (ϵ), J_1 is constant.

First, we need the Fourier transform of the (ϕ dependent part of the) perturbation. From equation (14)

$$\tilde{V}_m(J_1, s) = \frac{1}{2\pi} \int_0^{2\pi} e^{-im\phi} V(\phi, J_1, s)\, d\phi, \tag{33}$$

where:

$$V(\phi, J_1, s) = \frac{1}{2} k(s)\beta(s) J_1 \cos 2\phi. \tag{34}$$

Thus:

$$\tilde{V}_{\pm 2}(J_1, s) = \frac{1}{4} k(s)\beta(s) J_1, \tag{35}$$

with all other $\tilde{V}_m$ equal to zero.

Substituting for $\tilde{V}_{\pm 2}(J_1, s)$ into equation (20) gives:

$$G(\phi, J_1, s) = -\frac{J_1}{4 \sin(2\pi\nu)} \int_s^{s+L} \sin 2 \left(\phi + \psi(s') - \psi(s) - \pi\nu \right) k(s')\beta(s')\, ds'. \tag{36}$$

Note that the phase advance ψ and the tune ν include the phase shift resulting from the perturbation, i.e.

$$\psi(s) = \int_0^s \frac{ds}{\beta(s)} + \frac{\epsilon}{2} \int_0^s k(s)\beta(s)\, ds, \tag{37}$$

and (32):

$$\nu = \frac{1}{2\pi} \int_s^{s+L} \frac{ds}{\beta(s)} + \frac{\epsilon}{4\pi} \int_s^{s+L} k(s)\beta(s)\, ds. \tag{38}$$

At a given location in the beam line, the invariant curves in phase space (the "contour lines" in a phase space portrait) are given by (24):

$$J = J_0 + \epsilon \frac{\partial}{\partial \phi} G(\phi, J_0, s) + O(\epsilon^2), \tag{39}$$

where J_0 is a constant, and ϕ varies from 0 to 2π.

Using (36), we find:

$$J = J_0 - \frac{\epsilon J_0}{2\sin(2\pi\nu)} \int_s^{s+L} \cos 2\left(\phi + \psi(s') - \psi(s) - \pi\nu\right) k(s')\beta(s')\, ds' + O(\epsilon^2). \tag{40}$$

The effect of a quadrupole perturbation may be represented as a distortion of the beta functions. This can be calculated by considering a particle with a given x coordinate, and with $\phi = 0$. In terms of the unperturbed action J and beta function β, we would write:

$$x = \sqrt{2\beta J}. \tag{41}$$

In terms of the perturbed action J_0 and beta function β_0 we have:

$$x = \sqrt{2\beta_0 J_0}. \tag{42}$$

Therefore, since $\beta J = \beta_0 J_0$, we have:

$$\frac{\Delta\beta}{\beta} = -\frac{\Delta J}{J} = -\frac{J_0 - J}{J}. \tag{43}$$

So, from (40), the change in the beta function resulting from a *small* quadrupole perturbation is:

$$\frac{\Delta\beta}{\beta} = -\frac{\epsilon}{2\sin(2\pi\nu)} \int_s^{s+L} \cos 2\left(\phi + \psi(s') - \psi(s) - \pi\nu\right) k(s')\beta(s')\, ds'. \tag{44}$$

As a second example, let us consider the perturbation introduced by sextupole components in the lattice. Again, we follow closely the analysis of Ruth. And, to keep things simple, we again consider only one degree of freedom.

The Hamiltonian is given by:

$$H = \frac{p^2}{2} + K(s)\frac{x^2}{2} + \epsilon k_2(s)\frac{x^3}{6}. \tag{45}$$

In action-angle variables, the Hamiltonian becomes:

$$\begin{aligned} H &= \frac{J}{\beta(s)} + \epsilon k_2(s)\frac{(2\beta(s)J)^{\frac{3}{2}}}{6}\cos^3\phi, & (46) \\ &= H_0(J,s) + \epsilon V(\phi, J, s), & (47) \end{aligned}$$

where:

$$H_0(J,s) = \frac{J}{\beta(s)}, \quad \text{and} \quad V(\phi,J,s) = k_2(s)\frac{(2\beta(s)J)^{\frac{3}{2}}}{6}\cos^3\phi.$$

We can write the perturbation term as:

$$V(\phi,J,s) = \frac{1}{24}k_2(s)\left(2\beta(s)J\right)^{\frac{3}{2}}\left(\cos 3\phi + 3\cos\phi\right). \tag{48}$$

Using equation (20), the generating function that will remove the perturbation to first order in ϵ is $F = \phi J_1 + \epsilon G$, where:

$$\begin{aligned} G = -\frac{(2J_1)^{\frac{3}{2}}}{16}\Bigg\{ &\frac{1}{\sin\pi\nu}\int_s^{s+L} k_2(s')\beta(s')^{\frac{3}{2}}\sin\left[\phi + \psi(s') - \psi(s) - \pi\nu\right]\,ds' \\ &+ \frac{1}{3\sin 3\pi\nu}\int_s^{s+L} k_2(s')\beta(s')^{\frac{3}{2}}\sin 3\left[\phi + \psi(s') - \psi(s) - \pi\nu\right]\,ds'\Bigg\} \end{aligned} \tag{49}$$

where:

$$\psi(s) = \int_s^{s+L}\frac{ds'}{\beta(s')}. \tag{50}$$

The first-order invariant is given by equation (24):

$$J = J_0 + \epsilon \frac{\partial}{\partial \phi} G(\phi, J_0, s) + O(\epsilon^2).$$

If we know the sextupole distribution along the beamline, we can evaluate G from (49). Equation (24) then gives us the invariant, which allows us to plot the phase space.

As a specific example, consider a single thin sextupole located at $s = 0$. The sextupole strength $k_2(s)$ can be represented by a Dirac delta function:

$$k_2(s) = k_2 \ell \, \delta(0). \tag{51}$$

We shall plot the phase space at $s = 0$. From equation (49), we find that:

$$G = -\frac{(2J_1)^{\frac{3}{2}}}{16} k_2 \ell \, \beta(0)^{\frac{3}{2}} \left[\frac{\sin(\phi - \pi\nu)}{\sin \pi\nu} + \frac{\sin 3(\phi - \pi\nu)}{3 \sin 3\pi\nu} \right]. \tag{52}$$

From equation (24) the action variable J at $s = 0$ is related to the (first order) invariant J_1 by:

$$\begin{aligned} J &= J_1 + \epsilon \frac{\partial G}{\partial \phi} \\ &= J_1 - \epsilon \frac{(2J_1)^{\frac{3}{2}}}{16} k_2 \ell \, \beta(0)^{\frac{3}{2}} \left[\frac{\cos(\phi - \pi\nu)}{\sin \pi\nu} + \frac{\cos 3(\phi - \pi\nu)}{\sin 3\pi\nu} \right]. \end{aligned} \tag{53}$$

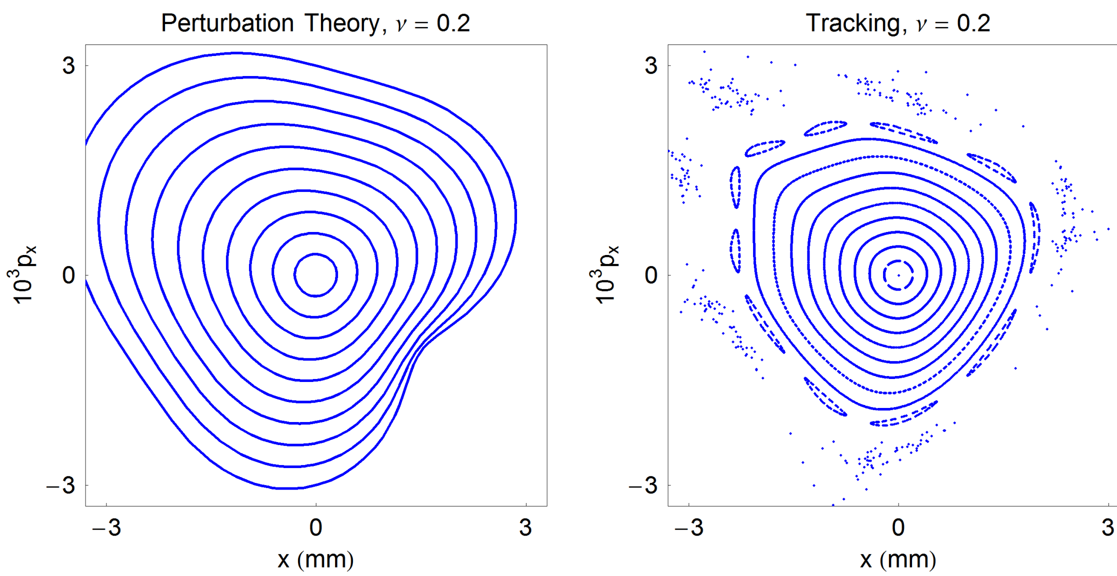
We shall choose values $k_2\ell = 600\,\mathrm{m}^{-}2$, and $\beta(0) = 1\,\mathrm{m}$, and plot the "contours" in phase space obtained from:

$$x = \sqrt{2\beta J}\cos\phi, \qquad p = -\sqrt{\frac{2J}{\beta}}\sin\phi, \tag{54}$$

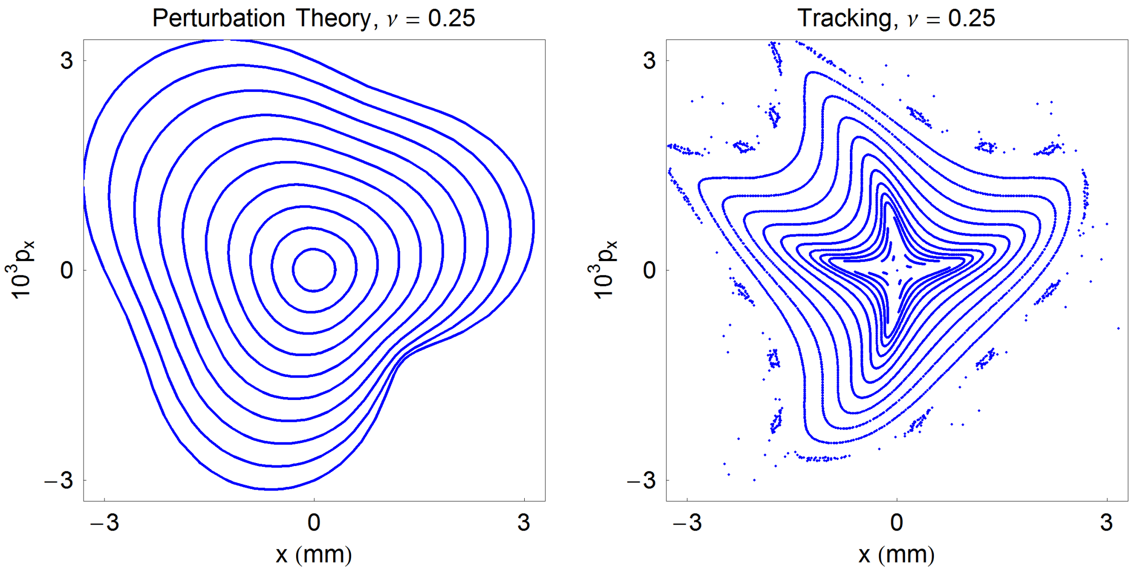
for $0 < \phi < 2\pi$, and for a set of values of J_1.

The phase space depends on the tune, ν. For a range of values of the tune, we shall compare the phase space plot obtained from perturbation theory, with the plots obtained by "tracking" particles, applying sextupole "kicks" with a phase advance between one kick and the next.

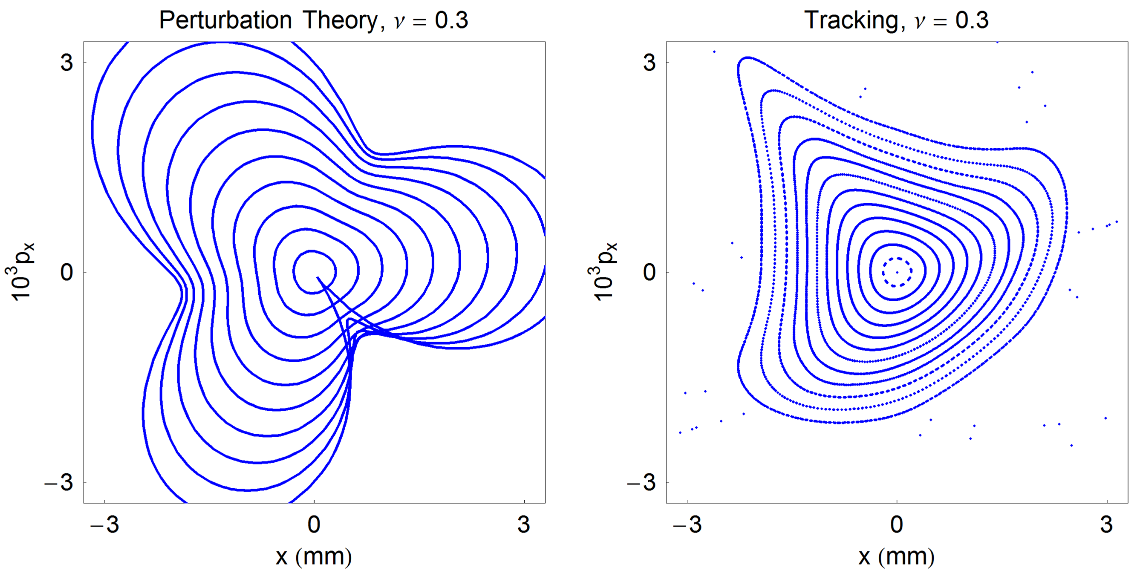
First-order perturbation theory: sextupole example



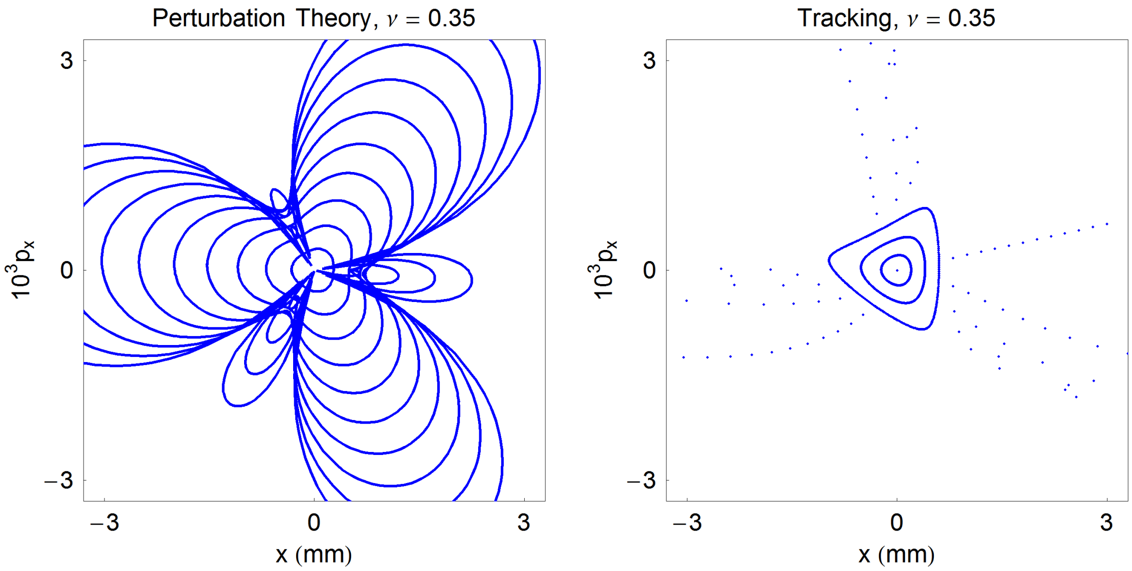
First-order perturbation theory: sextupole example



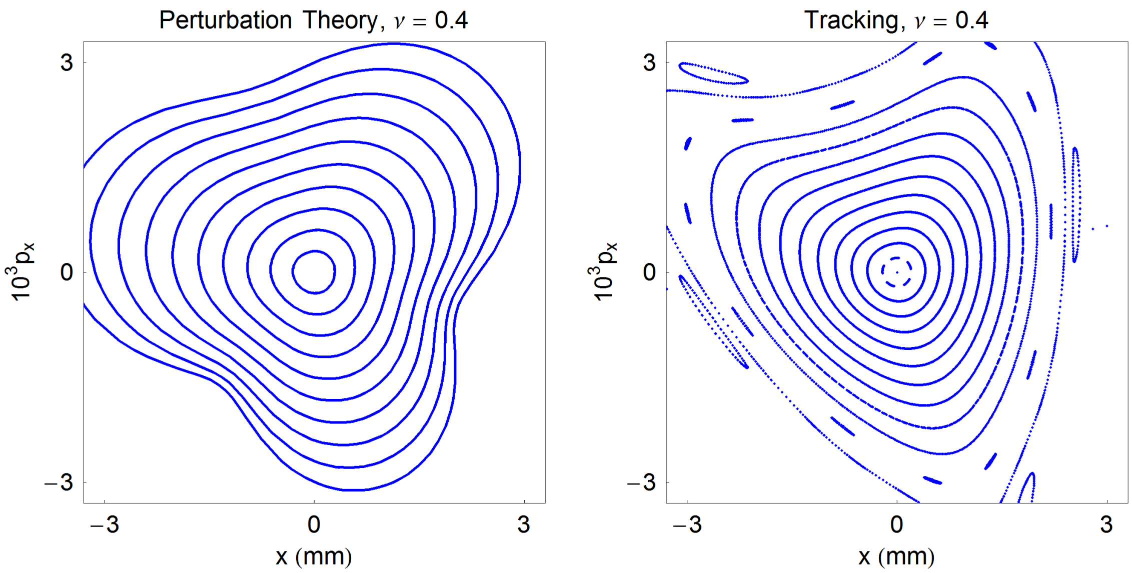
First-order perturbation theory: sextupole example



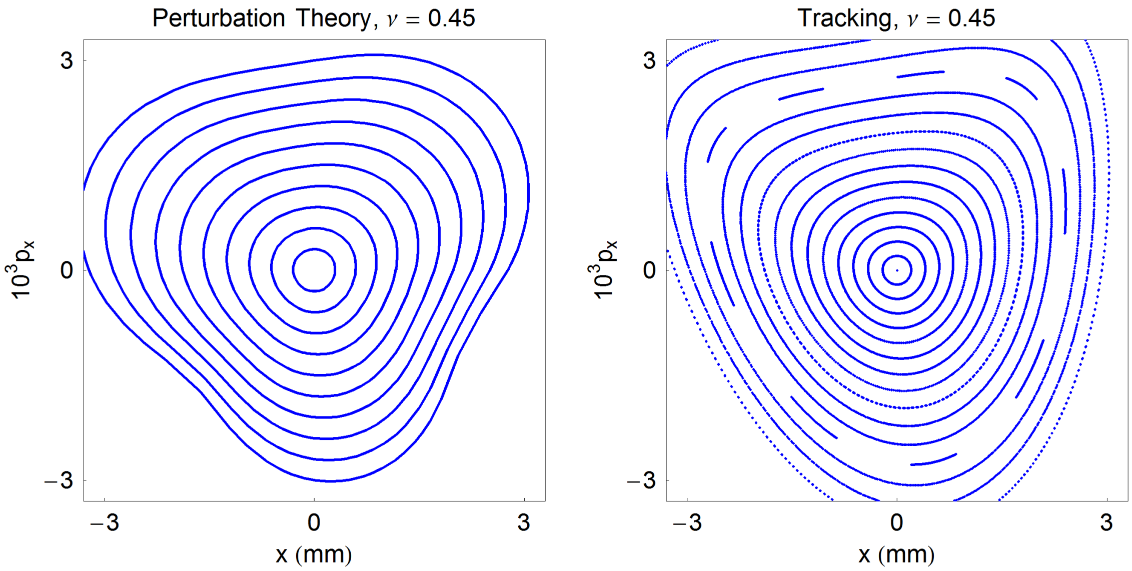
First-order perturbation theory: sextupole example



First-order perturbation theory: sextupole example



First-order perturbation theory: sextupole example



First-order perturbation theory: sextupole example

There does seem to be some resemblance between the phase space plots obtained from perturbation theory, and the plots obtained from tracking: there is the same general kind of distortion, and clearly dramatic effects around the third integer resonance.

However, perturbation theory misses many of the details revealed by tracking, and also some nonsensical features: the contours should not cross in the way they appear to do for particular tunes.

We can hope that applying perturbation theory to progressively higher order improves the situation... but that is beyond the scope of this course.

To finish this lecture, we consider in a little more detail the motion near a resonance.

We have seen how, if we are not close to a resonance, we can apply perturbation theory to find a canonical transformation to variables in which the Hamiltonian takes the form:

$$H_1 = H_0(J_1), \tag{55}$$

i.e. the Hamiltonian is purely a function of the (new) action variable.

Recall that we could *not* use a generating function to remove terms from the Hamiltonian independent of the angle variable, ϕ. But such terms simply lead to a tune shift with amplitude: and such effects are fairly innocuous. In particular, they do not distort the phase space, or limit the stability of the particle motion.

We also saw, from equation (20), that to remove a term $\tilde{V}_m \cos m\phi$ from the Hamiltonian, we need a term in the generating function given by:

$$G = \frac{1}{2\sin(\pi m\nu)} \int_s^{s+L} \tilde{V}_m \sin m \left[\phi + \psi(s') - \psi(s) - \pi\nu\right] ds'. \tag{56}$$

If $m\nu$ approaches an integer value, the generating function diverges, and we cannot hope for an accurate result. We saw this in the sextupole ($m = 3$) example, above, where we started seeing strange results close to $\nu = 1/3$.

Generally, therefore, after applying perturbation theory, we are left with a Hamiltonian of the form:

$$H = H_0(J, s) + f(J, s) \cos m\phi. \tag{57}$$

Let us assume that, by making an s-dependent canonical transformation, we can remove the s-dependence of the Hamiltonian, and put it into the form:

$$H = H_0(J) + f(J)\cos(m\phi). \tag{58}$$

For the special case,

$$H = \frac{J}{\beta(s)}, \tag{59}$$

the generating function:

$$F_2 = \phi J_1 + J_1\, 2\pi\nu\frac{s}{L} - J_1 \int_0^s \frac{ds'}{\beta(s')}, \tag{60}$$

leads to the s-independent Hamiltonian:

$$H_1 = \frac{2\pi\nu}{L} J_1, \tag{61}$$

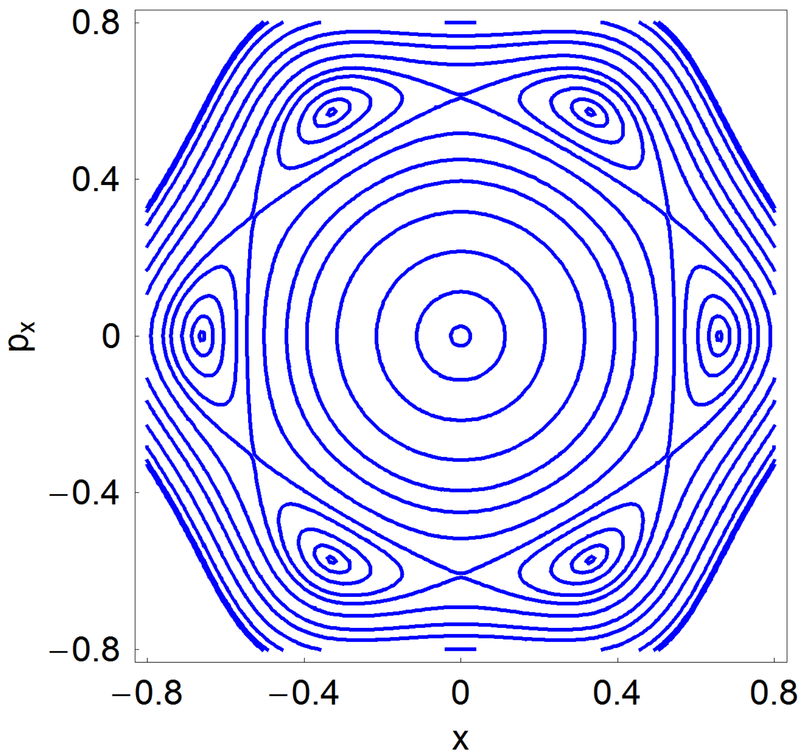
where $J_1 = J$.

Since, in the new variables, the Hamiltonian is independent of s, the Hamiltonian gives us a constant of the motion. That is, the evolution of a particle in phase space must follow a contour on which the value of H is constant.

As a specific example, consider the Hamiltonian:

$$H = 1.6J - 4J^2 + J^3\cos 6\phi. \tag{62}$$

This represents dynamics with a first-order tune shift (with respect to J), and a sixth-order resonance driving term. We can easily plot the countours of constant value for H...



The phase space clearly resembles that constructed from repeated application of a map representing a periodic section of beam line near a sixth-order resonance. Although there is a fundamental difference between the two cases (in one case the Hamiltonian is time-dependent, and in the other case it is time-independent) studying the "continuous" case can help in understanding features of the "discrete" case.

The width of the islands is related to both the tune shift with amplitude, and the strength of the driving term. Without a resonance term, the islands would vanish.

The centres of the islands represent stable fixed points. There are also unstable fixed points, where the contour lines appear to cross.

The lines passing through the unstable fixed points divide the islands from the rest of phase space: such a line is known as a separatrix.

It is possible to carry the analysis of resonant systems much further. In particular, it is possible to derive expressions for such things as the widths of the islands, in terms of the tune shifts with amplitude, and the strength of the resonant driving term. For more information, see Ruth.

In general, there will be more than one resonance present in a system. For example, sextupoles in a lattice can combine to drive resonances of any order. Perturbation theory can help to reveal the strengths of the driving terms of the different resonances.

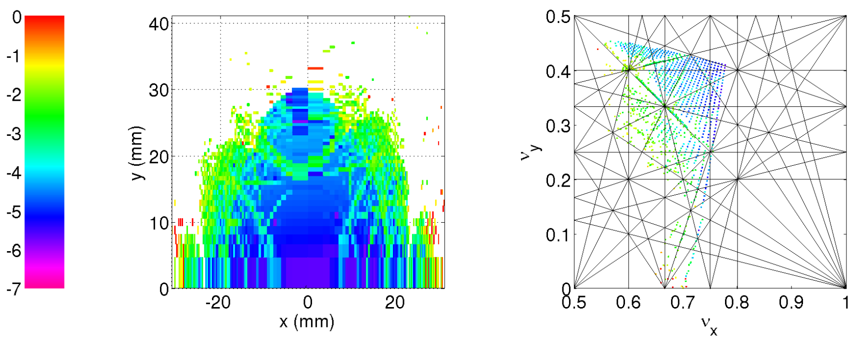
The onset of chaotic motion can be associated with two sets of resonant islands that overlap. This condition for chaotic motion is known as the Chirikov criterion.

In two degrees of freedom, a resonance is specified by integers (m, n) for which the following condition is satisfied:

$$m\nu_x + n\nu_y = \ell, \tag{63}$$

where ℓ is an integer.

Frequency map analysis (see lecture 8), based on tracking studies, can indicate the strengths of different resonances: not all resonances may be harmful.



"The KAM theorem states that if [an integrable Hamiltonian] system is subjected to a weak nonlinear perturbation, some of the invariant tori are deformed and survive, while others are destroyed. The ones that survive are those that have 'sufficiently irrational' frequencies (this is known as the non-resonance condition). This implies that the motion continues to be quasiperiodic, with the independent periods changed... The KAM theorem specifies quantitatively what level of perturbation can be applied for this to be true. An important consequence of the KAM theorem is that for a large set of initial conditions the motion remains perpetually quasiperiodic."

Wikipedia.

In applying perturbation theory, we construct a canonical transformation that puts the Hamiltonian into as simple a form as possible.

The dynamics in the new Hamiltonian are (in principle) simpler to solve. Then, the dynamics with the original Hamiltonian are obtained from the relationship between the old and new variables, defined by the canonical transformation.

Terms purely dependent on J cannot be removed from the Hamiltonian; nor can resonant terms, such as $\cos m\phi$, if the tune is close to a resonance, i.e. if $m\nu$ is close to an integer.

Some of the significant features of dynamics near a resonance can be understood in terms of a Hamiltonian in which non-resonant terms have been removed (by applying perturbation theory).

Much of the material in this lecture follows closely the report by Ruth:

R. Ruth, “Single particle dynamics in circular accelerators,” SLAC-PUB-4103 (1986).

http://www.slac.stanford.edu/pubs/slacpubs/4000/slac-pub-4103.html

This report gives a very clear explanation of the subject. It also goes some way beyond the material in this lecture, with some discussion on Hamilton-Jacobi theory, and the Chirikov criterion for the onset of chaotic motion.

Perturbation theory can also be carried out in the framework of Lie transformations. In that context, it is generally known as normal form analysis.

In the next lecture, we shall develop the theory of normal form analysis, and show how it can be applied in some simple cases.

Apply perturbation theory to an octupole perturbation. Hence, find the tune shift with amplitude generated by an octupole.