# Nonlinear Single-Particle Dynamics in High Energy Accelerators

## Part 2: Basic tools and concepts



This course consists of eight lectures:

1. Introduction – some examples of nonlinear dynamics
2. Basic mathematical tools and concepts
3. Representations of dynamical maps
4. Integrators I
5. Integrators II
6. Phase space portraits and "phenomenology"
7. Normal form analysis
8. Some numerical techniques

We have seen how nonlinear dynamics can play an important role in some diverse and common accelerator systems. Nonlinear effects have to be taken into account when designing such systems.

By making a simple analysis, we were able to compensate the most severe nonlinear effects in a bunch compressor. We were also able to develop a limited understanding of some of the effects of sextupoles in a periodic beamline (a storage ring).





Our analysis was based on representing the dynamics as a map in the form of a Taylor series, for example:

$$x \mapsto R_{11}x + R_{12}p_x + T_{111}x^2 + T_{112}xp_x + T_{122}p_x^2 + \cdots \quad (1)$$

$$p_x \mapsto R_{21}x + R_{22}p_x + T_{211}x^2 + T_{212}xp_x + T_{222}p_x^2 + \cdots \quad (2)$$

However, in several degrees of freedom, and where higher-order effects are important, Taylor series quickly become cumbersome. Also, it can be difficult to develop a real understanding of the dynamics from a set of coefficients. To make progress, we need to use more sophisticated tools.

Hamiltonian mechanics forms the basis of some very powerful methods for analysis of nonlinear systems. In this lecture, we review the basic principles of Hamiltonian mechanics in the context of accelerator beam dynamics.

In particular, we shall:

1. review Hamilton's equations;
2. discuss the significance of symplecticity;
3. derive (and solve) the nonlinear equations of motion for a drift space in an accelerator;
4. review canonical transformations, and introduce action-angle variables.

By the end of the lecture, you should be able to derive the equations of motion for a dynamical system with a given Hamiltonian; and perform canonical transformations to express relationships between different sets of canonical variables.



## Hamilton's equations

In Hamiltonian mechanics, the state of a particle is specified by giving particular values for a set of dynamical variables: the dynamical variables occur in pairs, with each pair consisting of a coordinate and a conjugate momentum, e.g. $(q, p)$.

The dynamics of the particle are described by expressing the dynamical variables as functions of an independent variable (for example, time), e.g. $q = q(t), p = p(t)$.

The dynamics are determined by the Hamiltonian, which is a function of the dynamical variables (and, possibly, the independent variable). Hamilton's equations give the general relationship between the evolution of the dynamical variables and the Hamiltonian, and provide the means for constructing the equations of motion in a particular case.

If the dynamical variables are $(q_i, p_i)$, the independent variable is $t$, and the Hamiltonian is $H(q_i, p_i; t)$, then Hamilton's equations are:

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}. \tag{3}$$

Note that the Hamiltonian must be expressed in terms of the coordinates and the conjugate momenta, and *not* in terms of the velocities.

Dynamical variables that obey Hamilton's equations are called *canonical* variables.

A simple example will probably explain how this works...





In simple cases, the Hamiltonian takes the form:

$$H = T + V, \tag{4}$$

where $T$ is the kinetic energy of the particle, and $V$ is the potential energy. Consider a particle with mass $m$ and coordinate $x$, moving in an harmonic oscillator potential, $V = \frac{1}{2}kx^2$. In this case (not in general), the momentum is $p = m\dot{x}$. Then, the Hamiltonian takes the form:

$$H = \frac{p^2}{2m} + \frac{1}{2}kx^2. \tag{5}$$

Hamilton's equations give the equations of motion for this case:

$$\frac{dx}{dt} = \frac{\partial H}{\partial p} = \frac{p}{m} \tag{6}$$

$$\frac{dp}{dt} = -\frac{\partial H}{\partial x} = -kx. \tag{7}$$

Equation (6) tells us that the momentum in this case corresponds to the usual mechanical momentum, i.e. the product of the mass and the velocity:

$$p = m\dot{x}. \tag{8}$$

Equation (7) expresses Newton's second law of motion, for a force $-kx$. Combining the two equations gives a second-order differential equation for the coordinate:

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x, \tag{9}$$

which we recognise as the usual equation of motion for a simple harmonic oscillator.





In a previous lecture course (on linear dynamics), we derived a general expression for the Hamiltonian for particle motion in an accelerator beamline. Before writing down this Hamiltonian, we should first define our variables. There are various definitions commonly used, and it is very important to be clear about the physical significance of the variables you are using.

First, we have to define a reference trajectory. This can, in principle, be any path through space; but usually it is chosen to reflect the "ideal" trajectory that we would like particles to follow. Often this is simply a sequence of straight lines, joined (through dipole fields) by arcs of circles with specified radii.

The independent variable we use is the distance $s$ along the reference trajectory. The reason for using path length, and not time, as the independent variable is that the fields along the accelerator are usually localised in space. That means it is easier to write down the Hamiltonian as a function of position than of time.

In addition to the reference trajectory, we define a reference momentum $P_0$. Again, this can, in principle, be chosen arbitrarily; but usually a beamline is designed to transport particles with a particular (average) momentum, in which case it is sensible to set the reference momentum equal to the "design" momentum.

A particle with the reference momentum $P_0$ will be travelling at a particular velocity $\beta_0 c$, for which the relativistic (Lorentz gamma) factor is $\gamma_0$:

$$P_0 = \beta_0 \gamma_0 m c, \qquad \gamma_0 = \frac{1}{\sqrt{1-\beta_0^2}}. \tag{10}$$

$m$ is the rest mass of the particle, and $c$ is the speed of light in vacuum.





The transverse coordinates $x$ and $y$ constitute two of the dynamical variables. $x$ and $y$ are simply the position of a particle with respect to the reference trajectory in a plane perpendicular to the reference trajectory at a given point $s$. The orientation of the axes must be defined at some point along the reference trajectory; the form of the Hamiltonian depends on how one chooses to "transport" the axes along the reference trajectory.

In a simple case, the reference trajectory may be defined to lie in a plane. Then, the $x$-axis can be chosen always to lie in that plane, and the $y$-axis can be chosen to lie perpendicular to the plane.

The momenta conjugate to the coordinates $x$ and $y$ are given by:

$$p_x = \frac{\gamma m \dot{x} + qA_x}{P_0}, \qquad p_y = \frac{\gamma m \dot{y} + qA_y}{P_0}, \tag{11}$$

$\dot{x}$ and $\dot{y}$ are the transverse velocities (i.e. the time derivatives of the transverse coordinates). $q$ is the electric charge of the particle, and $A_x$ and $A_y$ are the transverse components of the vector potential.

$\gamma$ is the relativistic factor for the particle (not necessarily equal to $\gamma_0$).



The accelerator Hamiltonian

The longitudinal dynamical variables are $(z, \delta)$.

The longitudinal coordinate $z$ is given by:

$$z = c(t_0 - t), \tag{12}$$

where $t_0$ is the time at which a "reference particle" travelling along the reference trajectory at speed $\beta_0 c$ is at a location $s$ along this trajectory, and $t$ is the time at which the chosen particle is in the plane perpendicular to the reference trajectory at $s$.

Note that if $t < t_0$, the chosen particle arrives at $s$ sooner than the reference particle, i.e. the chosen particle is *ahead* of the reference particle.

The longitudinal conjugate momentum $\delta$ is defined by:

$$\delta = \frac{E}{P_0 c} - \frac{1}{\beta_0} = \frac{E - E_0}{\beta_0 E_0}, \tag{13}$$

where $E$ is the kinetic energy of the particle, and $E_0$ is the kinetic energy of a particle with the reference momentum $P_0$. For brevity, we sometimes refer to $\delta$ as the "energy deviation".





Now we understand what variables we are using, we can write down the Hamiltonian. This was derived in the course on linear dynamics:

$$\begin{aligned} H &= -(1+hx)\sqrt{\left(\frac{1}{\beta_0} + \delta - \frac{q\phi}{P_0 c}\right)^2 - (p_x - a_x)^2 - (p_y - a_y)^2 - \frac{1}{\beta_0^2\gamma_0^2}} \\ &\quad -(1+hx)a_s + \frac{\delta}{\beta_0}. \end{aligned} \tag{14}$$

Here, $a_x = qA_x/P_0$, and similarly for $a_y$ and $a_s$ (the component along the reference trajectory). $\phi$ is the scalar potential.

$h$ is the curvature of the reference trajectory, assumed to lie in the $x - s$ plane:

$$h = \frac{1}{\rho}, \tag{15}$$

where $\rho$ is the local radius of curvature.

As an example, let us consider the Hamiltonian in a drift space, where $h = 0$, and there are no electric or magnetic fields (so we can take the scalar and vector potentials to be zero):

$$H = -\sqrt{\left(\frac{1}{\beta_0} + \delta\right)^2 - p_x^2 - p_y^2 - \frac{1}{\beta_0^2\gamma_0^2}} + \frac{\delta}{\beta_0}. \tag{16}$$

Note that the Hamiltonian has no dependence on the coordinates $x$, $y$ or $\delta$. This means, from Hamilton's equations, that the momenta are conserved:

$$\frac{dp_x}{ds} = -\frac{\partial H}{\partial x} = 0, \tag{17}$$

$$\frac{dp_y}{ds} = -\frac{\partial H}{\partial y} = 0, \tag{18}$$

$$\frac{d\delta}{ds} = -\frac{\partial H}{\partial z} = 0. \tag{19}$$





The equations of motion for the coordinates are also reasonably straightforward:

$$\frac{dx}{ds} = \frac{\partial H}{\partial p_x} = \frac{p_x}{p_s}, \tag{20}$$

$$\frac{dy}{ds} = \frac{\partial H}{\partial p_y} = \frac{p_y}{p_s}, \tag{21}$$

$$\frac{dz}{ds} = \frac{\partial H}{\partial \delta} = \frac{1}{\beta_0} - \frac{\frac{1}{\beta_0} + \delta}{p_s}, \tag{22}$$

where we have defined $p_s$ (not a dynamical variable!) as:

$$p_s = \sqrt{\left(\frac{1}{\beta_0} + \delta\right)^2 - p_x^2 - p_y^2 - \frac{1}{\beta_0^2\gamma_0^2}}. \tag{23}$$

Note that since $p_x$, $p_y$ and $\delta$ are constants of the motion, $p_s$ is constant.

From the above results, it is possible to write the map for a drift space in closed form. For the transverse variables, we have:

$$x \mapsto x + \frac{p_x}{p_s}\Delta s, \qquad p_x \mapsto p_x, \tag{24}$$

$$y \mapsto y + \frac{p_y}{p_s}\Delta s, \qquad p_y \mapsto p_y. \tag{25}$$

And for the longitudinal variables, we have:

$$z \mapsto z + \left( \frac{1}{\beta_0} - \frac{\frac{1}{\beta_0} + \delta}{p_s} \right) \Delta s, \quad \delta \mapsto \delta. \tag{26}$$

The value of $p_s$ (a constant of the motion) is given by (23):

$$p_s = \sqrt{\left( \frac{1}{\beta_0} + \delta \right)^2 - p_x^2 - p_y^2 - \frac{1}{\beta_0^2 \gamma_0^2}}.$$





We see that the map has a nonlinear dependence on the momenta $p_x$, $p_y$ and $\delta$. However, the nonlinear effects only become significant when the values of the momenta become very large. To illustrate this, consider the case $p_y = \delta = 0$. Then:

$$p_s = \sqrt{1 - p_x^2}. \tag{27}$$

Note that in this case:

$$p_x = \frac{\gamma_0 m \dot{x}}{P_0}, \tag{28}$$

so the maximum value of $p_x$ is:

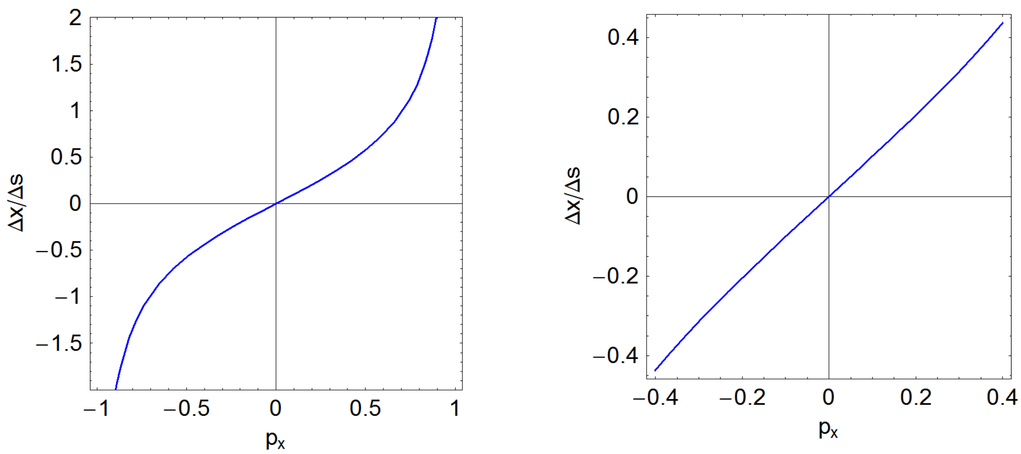
$$\lim_{\dot{x} \to \beta_0 c} p_x = 1. \tag{29}$$

As we might expect $p_x$ has a maximum value of 1, and this occurs when the particle is travelling perpendicular to the reference trajectory.

Let us now plot

$$\frac{\Delta x}{\Delta s} = \frac{p_x}{\sqrt{1 - p_x^2}}. \qquad (30)$$

We see that there is a significant deviation from linearity when $p_x$ is larger than 0.1.








In the case that $p_x = p_y = 0$, the particle is travelling parallel to the reference trajectory. Then, the Hamiltonian becomes:

$$H = \frac{\delta}{\beta_0} - \sqrt{\left(\frac{1}{\beta_0} + \delta\right)^2 - \frac{1}{\beta_0^2 \gamma_0^2}}. \qquad (31)$$

It follows that the equation of motion for the longitudinal coordinate is:

$$\frac{dz}{ds} = \frac{\partial H}{\partial \delta} = \frac{1}{\beta_0} - \frac{\frac{1}{\beta_0} + \delta}{\sqrt{\left(\frac{1}{\beta_0} + \delta\right)^2 - \frac{1}{\beta_0^2 \gamma_0^2}}}. \qquad (32)$$

Since, from equation (13), we have:

$$\delta = \frac{E - E_0}{\beta_0 E_0} = \frac{\gamma - \gamma_0}{\beta_0 \gamma_0}, \qquad \therefore \qquad \frac{1}{\beta_0} + \delta = \frac{\gamma}{\beta_0 \gamma_0}, \qquad (33)$$

we find that:

$$\frac{dz}{ds} = \frac{1}{\beta_0} - \frac{1}{\beta}, \qquad (34)$$

which is consistent with our interpretation of $z$, equation (12).

Using a Hamiltonian approach, even the map for a drift space is rather complicated. It is possible to do the dynamics using different variables, that simplify the map. For example, instead of using $p_x$ and $p_y$, we could define:

$$x' = \frac{dx}{ds}, \quad \text{and} \quad y' = \frac{dy}{ds}. \tag{35}$$

Then, the (transverse) map for a drift space would simply be:

$$x \mapsto x + x' \Delta s, \qquad x' \mapsto x', \tag{36}$$

$$y \mapsto y + y' \Delta s, \qquad y' \mapsto y', \tag{37}$$

with no dependence at all on the energy deviation.

This looks much simpler – why do we bother with the Hamiltonian? There are three reasons...





First, Hamiltonian mechanics provides a highly systematic framework for constructing the equations of motion for a relativistic particle in even quite complicated electromagnetic fields.

This is an important feature, but of course not unique to Hamiltonian mechanics.

The main reason for using canonical variables, is that for a particle moving through an electromagnetic field, neglecting radiation and interactions with other particles, the map expressed in these variables must be symplectic. That is, the volumes of small elements in phase space must be conserved.

Conserved quantities provide a powerful tool for verifying the accuracy of analytical and computational calculations. If phase space volumes are not conserved in your calculations, you know something must be wrong. (Though, of course, it does not mean you have got the calculations right just because phase space volumes *are* conserved.)

Furthermore, some effects (such as radiation damping in storage rings) can conveniently be quantified in terms of deviations from symplecticity. If the maps you are using before including these effects are not symplectic, you have to work somewhat harder to calculate how large these effects are.





The final reason for using Hamiltonian mechanics extensively for accelerator dynamics, is that Hamilton's equations lead to some powerful analytical and numerical techniques for solving the equations of motion while retaining important features, such as symplecticity.

Much of the rest of this course will be devoted to exploring these techniques. But, since the idea of symplecticity is an important motivation for the approach we will take, let us start by considering it in a little more detail.

Let $\vec{x}$ be a vector of phase space variables. We will show that if the values of the phase space variables at position $s + \Delta s$ on the reference trajectory are given by $\vec{X} = \vec{X}\left(\vec{x}(s); \Delta s\right)$, then:

$$J^{\mathsf{T}} \cdot S \cdot J = S, \tag{38}$$

where $J$ is the Jacobian of the transformation from $s$ to $s + \Delta s$, i.e.:

$$J_{ij} = \frac{\partial X_i}{\partial x_j}, \tag{39}$$

and $S$ is a block-diagonal matrix constructed from $2 \times 2$ antisymmetric matrices $S_2$:

$$S_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{40}$$

Any matrix that satisfies equation (38) is said to be a *symplectic* matrix.





To prove equation (38), first note that with $\vec{x}$ given by:

$$\vec{x} = (x, p_x, y, p_y, z, \delta)\,, \tag{41}$$

Hamilton's equations may be written:

$$\frac{d\vec{x}}{ds} = S \cdot \frac{\partial H}{\partial \vec{x}}. \tag{42}$$

We will assume that we work inside a given accelerator element, where the Hamiltonian $H$ is independent of $s$. At some position in the element, the variables can be expressed as functions of the variables at some earlier position, i.e.:

$$\vec{X} = \vec{X}\left(\vec{x}(s); \Delta s\right). \tag{43}$$

The changes in the variables with respect to motion along the reference trajectory are related by the Jacobian $J$:

$$\frac{d\vec{X}}{ds} = \frac{\partial \vec{X}}{\partial \vec{x}} \cdot \frac{d\vec{x}}{ds} = J \cdot \frac{d\vec{x}}{ds} = J \cdot S \cdot \frac{\partial H}{\partial \vec{x}}. \tag{44}$$

Now, since:

$$\frac{\partial H}{\partial \vec{x}} = J^\mathsf{T} \cdot \frac{\partial H}{\partial \vec{X}}, \tag{45}$$

we find that:

$$\frac{d\vec{X}}{ds} = J \cdot S \cdot J^\mathsf{T} \cdot \frac{\partial H}{\partial \vec{X}}. \tag{46}$$

But the evolution of the variables $\vec{X}$ must be determined by Hamilton's equations, with the same Hamiltonian that governs the evolution of the variables $\vec{x}$:

$$\frac{d\vec{X}}{ds} = S \cdot \frac{\partial H}{\partial \vec{X}}. \tag{47}$$

Comparing equations (46) and (47), it is clear that:

$$J \cdot S \cdot J^\mathsf{T} = S. \tag{48}$$





Using the properties:

$$S^{-1} = S^\mathsf{T} = -S, \quad \text{and} \quad \left(J^{-1}\right)^\mathsf{T} = \left(J^\mathsf{T}\right)^{-1}, \tag{49}$$

equation (48) may be rewritten as (38):

$$J^\mathsf{T} \cdot S \cdot J = S.$$

This means that for a system governed by Hamilton's equations, the Jacobian of the map from $s$ to $s + \Delta s$ must be a symplectic matrix. For short, we say that the map must be symplectic.

Note that for a linear map, the Jacobian is simply a matrix of numbers. For a nonlinear map, the Jacobian will be a function of the phase space variables $\vec{x}$; but equation (38) must still hold.

Since the determinant of $S$ is unity:

$$|S| = 1, \tag{50}$$

it follows from (38) that if $J$ is the Jacobian of a symplectic map:

$$|J|^2 = 1, \quad \text{i.e.} \quad |J| = \pm 1. \tag{51}$$

Therefore, for a map to by symplectic, it is a necessary (but not sufficient) condition for the Jacobian to have determinant $\pm 1$. It follows immediately from this that symplectic maps preserve volumes in phase space:

$$\int \ldots \int d\vec{X} = \int \ldots \int |J| \, d\vec{x} = \pm \int \ldots \int d\vec{x}. \tag{52}$$

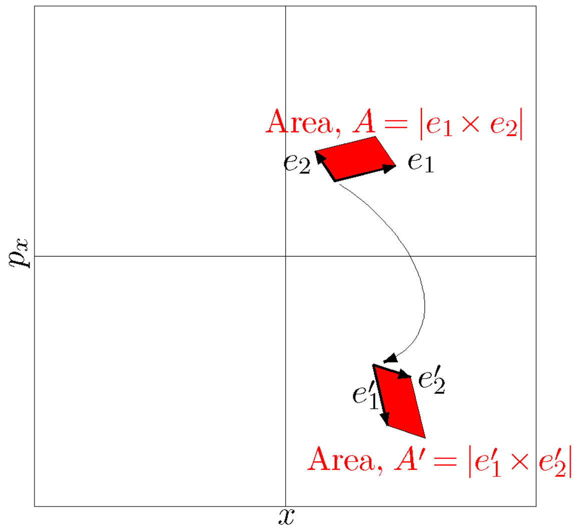




In the context of Hamilton mechanics, equation (52) is called Liouville's theorem. In accelerator beam dynamics, it tells us that as a bunch of particles is transported along a beamline, then neglecting radiation and interactions between the particles, the volume of phase space occupied by the particles remains constant. (Of course, one needs to be a bit careful here, and we ought to define carefully the boundaries of the volume occupied by a bunch of particles...)

Side note: the total volume in phase space is one of a number of invariants of Hamiltonian systems, known as Poincare invariants. The others are not so easily expressed as the volume of an element in phase space, and since we do not need them in this course, we do not discuss them further.

Liouville's theorem is easiest to visualise in one degree of freedom, with a linear map...




...but the theorem generalises to more degrees of freedom, and nonlinear maps.





As an example of a symplectic map, consider again the case of a drift space. To simplify things further, let us consider only the transverse motion. The map can be written:

$$X = x + \frac{p_x s}{\sqrt{1 - p_x^2}} \tag{53}$$

$$P_X = p_x. \tag{54}$$

The Jacobian is:

$$J = \begin{pmatrix} \frac{\partial X}{\partial x} & \frac{\partial X}{\partial p_x} \\ \frac{\partial P_X}{\partial x} & \frac{\partial P_X}{\partial p_x} \end{pmatrix} = \begin{pmatrix} 1 & \frac{s}{(1-p_x^2)^{3/2}} \\ 0 & 1 \end{pmatrix}. \tag{55}$$

Note that the Jacobian is a function of the dynamical variables. Nonetheless, we can still work out the matrix product with $S$; and we find, as expected, that:

$$J^{\mathsf{T}} \cdot S \cdot J = S.$$

The case in three degrees of freedom starts to look more complicated, but we still find that the map is symplectic.

In our discussion of symplecticity, we treated a map as a transformation from one point along the reference trajectory to another. However, it should be clear from the discussion, that if *any* new set of variables $\vec{X}$ is derived from an existing set of canonical variables $\vec{x}$ by a symplectic transformation, i.e.

$$\vec{X} = \vec{X}(\vec{x}), \quad \text{where} \quad \frac{\partial \vec{X}}{\partial \vec{x}} = J, \quad \text{and} \quad J^{\mathsf{T}} \cdot S \cdot J = S, \tag{56}$$

then the new set of variables obeys Hamilton's equations:

$$\frac{d\vec{X}}{ds} = S \cdot \frac{\partial H}{\partial \vec{X}}. \tag{57}$$

That is, the new variables $\vec{X}$ are canonical variables, just like the variables $\vec{x}$.

A transformation from one set of canonical variables to another is called a *canonical transformation*.





Sometimes, it is convenient to work with dynamical variables other than the "cartesian" variables $(x, p_x, y, p_y, z, \delta)$. This is particularly true for nonlinear dynamics, where we often use "action-angle" variables.

The action-angle variables $(J_x, \phi_x)$ for the horizontal motion are defined by:

$$2J_x = \gamma_x x^2 + 2\alpha_x x p_x + \beta_x p_x^2, \tag{58}$$

$$\tan \phi_x = -\alpha_x - \beta_x \frac{p_x}{x}. \tag{59}$$

Here, $\alpha_x$, $\beta_x$ and $\gamma_x$ are the usual Twiss parameters, defined for linear motion.

It can be shown (an exercise for the student!) that the Jacobian of the transformation is symplectic: therefore, $(\phi_x, J_x)$ are canonical variables. (Note that the angle $\phi_x$ is the coordinate, and the action $J_x$ is the conjugate momentum).

Action-angle variables are very useful for linear dynamics. In that case, we know that the betatron action is constant, and that the rate of increase of betatron phase is given by $1/\beta_x$:

$$\frac{d\phi_x}{ds} = \frac{1}{\beta_x}, \tag{60}$$

$$\frac{dJ_x}{ds} = 0. \tag{61}$$

Since action-angle variables are canonical variables, it should be possible to obtain these equations of motion from a suitable Hamiltonian. In fact, an appropriate Hamiltonian is given by:

$$H = \frac{J_x}{\beta_x}. \tag{62}$$

It turns out that action-angle variables are also useful in nonlinear dynamics, and we shall make extensive use of them in this course.





There is a useful recipe (or rather, set of recipes) for constructing canonical transformations. The technique is based on *generating functions*.

Many standard texts on classical mechanics give a full discussion of generating functions, including a derivation of the main formulae. See, for example, Goldstein.

Here, we simply present the results, and give an example. Since the results are general (i.e. not specific to accelerator beam dynamics) we revert to a general notation in which the coordinates are denoted $q_i$, the conjugate momenta are $p_i$, and the independent variable is $t$.

Let us define a function $F_1$ of the "old" coordinates $q_i$ and the "new" coordinates $Q_i$. In general, $F_1$ may also be a function of the independent variable, $t$:

$$F_1 = F_1(q_i, Q_i, t). \tag{63}$$

Then, $F_1$ generates a canonical transformation, in which the relationships between the old and new variables is given by:

$$p_i = \frac{\partial F_1}{q_i}, \qquad P_i = -\frac{\partial F_1}{Q_i}. \tag{64}$$

Here, $p_i$ are the momenta conjugate to $q_i$, and $P_i$ are the momenta conjugate to $Q_i$. The Hamiltonian for the new variables is given by:

$$K = H + \frac{\partial F_1}{\partial t}. \tag{65}$$





Canonical transformations can be useful in simplifying the equations of motion. For example, suppose that we have a system in one degree of freedom, whose dynamics are governed by the Hamiltonian:

$$H = p^2 - 6pq^2 + q^2 + 9q^4. \tag{66}$$

The equations of motion are nonlinear, and difficult to solve. However, let us make a canonical transformation to new variables, using the generating function:

$$F_1(q, Q) = qQ + q^3. \tag{67}$$

Using equations (64), we find:

$$p = \frac{\partial F_1}{\partial q} = Q + 3q^2, \quad \therefore \quad Q = p - 3q^2, \tag{68}$$

and:

$$P = -\frac{\partial F_1}{\partial Q} = -q. \tag{69}$$

The new variables expressed in terms of the old variables are:

$$Q = p - 3q^2, \quad \text{and} \quad P = -q. \tag{70}$$

The old variables expressed in terms of the new variables are:

$$q = -P, \quad \text{and} \quad p = Q + 3P^2. \tag{71}$$

Using (65), (66) and (71) the new Hamiltonian $K$, which gives the equations of motion expressed in terms of the new variables, is:

$$K = H + \frac{\partial F_1}{\partial t} = p^2 - 6pq^2 + q^2 + 9q^4 = P^2 + Q^2. \tag{72}$$

In terms of the new variables, the equations of motion are simply those for an harmonic oscillator. We can easily solve the equations of motion in the new variables, then transform back to the old variables, using (71).

Note: if you thought that the original Hamiltonian (66) was contrived to allow an easy solution using a relatively simple generating function, you would be right... In general, it is not easy to solve problems in this way.





The generating function we introduced above, $F_1(q_i, Q_i, t)$, was a function of the old coordinates and the new coordinates. But of course, if we are going to mix up old and new variables, there are four different ways to do it. Each of these leads to a different "kind" of generating function, each with its own set of relationships between the old and new variables.

We have already met $F_1$, a "generating function of the first kind":

$$F_1 = F_1(q_i, Q_i, t), \tag{73}$$

$$p_i = \frac{\partial F_1}{\partial q_i}, \qquad P_i = -\frac{\partial F_1}{\partial Q_i}, \tag{74}$$

$$K = H + \frac{\partial F_1}{\partial t}. \tag{75}$$

$F_2$ is a "generating function of the second kind":

$$F_2 = F_2(q_i, P_i, t), \tag{76}$$

$$p_i = \frac{\partial F_2}{\partial q_i}, \qquad Q_i = \frac{\partial F_2}{\partial P_i}, \tag{77}$$

$$K = H + \frac{\partial F_2}{\partial t}. \tag{78}$$

It is possible to express the identity transformation in terms of a generating function of the second kind:

$$F_2(q_i, P_i) = \sum_i q_i P_i. \tag{79}$$





$F_3$ is a "generating function of the third kind":

$$F_3 = F_3(p_i, Q_i, t), \tag{80}$$

$$q_i = -\frac{\partial F_3}{\partial p_i}, \qquad P_i = -\frac{\partial F_3}{\partial Q_i}, \tag{81}$$

$$K = H + \frac{\partial F_3}{\partial t}. \tag{82}$$

And finally, $F_4$ is a "generating function of the fourth kind":

$$F_4 = F_4(p_i, P_i, t), \tag{83}$$

$$q_i = -\frac{\partial F_4}{\partial p_i}, \qquad Q_i = \frac{\partial F_4}{\partial P_i}, \tag{84}$$

$$K = H + \frac{\partial F_4}{\partial t}. \tag{85}$$

To finish this lecture, we note that since action-angle variables $(\phi_x, J_x)$ are obtained from cartesian variables $(x, p_x)$ by a canonical transformation, we would expect that we can write down a generating function for the transformation (although note that we have *not* shown that every canonical transformation can be obtained from some generating function).

We can construct the canonical transformation from cartesian to action-angle variables using a generating function of the first kind:

$$F_1(x, \phi_x) = -\frac{x^2}{2\beta_x}(\tan\phi_x + \alpha_x)\,. \tag{86}$$



## Summary

- The equations of motion for a particle moving through electromagnetic fields in an accelerator beamline (neglecting radiation and interactions between particles) can be derived from Hamilton's equations, with an appropriate Hamiltonian.
- Expressed in canonical variables, the transformation representing motion of a particle from one point along a beamline to another is symplectic (that is, the Jacobian of the transformation is a symplectic matrix).
- A symplectic transformation from one set of variables to another is called a canonical transformation. Sometimes, canonical transformations provide a way to simplify the equations of motion.
- An example of a canonical transformation is provided by the relationships between action-angle variables and the usual cartesian variables. Action-angle variables are widely used in accelerator beam dynamics.
- Canonical transformations can be constructed using generating functions.

1. Write down the Hamiltonian for (a) a dipole magnet, and (b) a quadrupole magnet. Are the dynamics linear or nonlinear in each case?

2. Show that the transformation from cartesian to action-angle variables is canonical.

3. Show that the transformation from cartesian to action-angle variables may be obtained from the generating function (86):

$$F_1(x, \phi_x) = -\frac{x^2}{2\beta_x}(\tan\phi_x + \alpha_x)\,.$$