

Cockcroft Lectures: Linear Dynamics

Problem Set 9

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1. In one degree of freedom, the beam distribution matrix (the Σ matrix) is a 2×2 symmetric matrix, which is parameterised in terms of the Twiss parameters α_x , β_x , γ_x , and the emittance ϵ_x :

$$\Sigma = \begin{pmatrix} \langle x^2 \rangle & \langle xp_x \rangle \\ \langle xp_x \rangle & \langle p_x^2 \rangle \end{pmatrix} = \begin{pmatrix} \beta_x & -\alpha_x \\ -\alpha_x & \gamma_x \end{pmatrix} \epsilon_x \quad (1)$$

The matrix Σ has three independent components. Using the parameterisation in terms of the Twiss parameters and the emittance, there are apparently $3 + 1 = 4$ variables. However, the condition:

$$\beta_x \gamma_x - \alpha_x^2 = 1 \quad (2)$$

imposes one constraint, allowing us to eliminate one of the Twiss parameters. The number of independent variables in the Twiss/emittance parameterisation is therefore equal to the number of independent components in the beam distribution matrix, Σ .

- (a) In three degrees of freedom, how many independent components are there of the beam distribution matrix, Σ ?
 - (b) In three degrees of freedom, the “generalised Twiss parameters” are the components of the 6×6 symmetric matrices B^k . How many generalised Twiss parameters are there in three degrees of freedom? How many emittances are there?
 - (c) How many relationships must there be between the generalised Twiss parameters in three degrees of freedom (analogous to the relationship $\beta_x \gamma_x - \alpha_x^2 = 1$ in one degree of freedom)?
2. Assume that the horizontal coordinate of a particle in a bunch is given by:

$$x = \sqrt{2\beta_x J_x} \cos \phi_x + \eta_x \delta \quad (3)$$

where the angle variables ϕ_x are uncorrelated with any other variables, and η_x is the dispersion. Assume also that the action variables J_x are uncorrelated with the energy deviations δ .

- (a) Using equation (3), find a relationship between the moments $\langle x\delta \rangle$ and $\langle \delta^2 \rangle$.
- (b) Write the moments $\langle x\delta \rangle$ and $\langle \delta^2 \rangle$ in terms of generalised Twiss parameters β_{ij}^k , and the emittances ϵ_k .
- (c) Write down an expression for the dispersion η_x in terms of generalised Twiss parameters.