# Cockcroft Lectures: Linear Dynamics

# Problem Set 6

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1. (a) Find a constraint on the values of $\alpha$, $\beta$ and $\gamma$, so that the matrix:

$$R = I\,\cos\mu + S \cdot A\,\sin\mu \tag{1}$$

is symplectic for any value of $\mu$, where $I$ is the $2 \times 2$ identity matrix, and the matrices $S$ and $A$ are given by:

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad A = \begin{pmatrix} \gamma & \alpha \\ \alpha & \beta \end{pmatrix} \tag{2}$$

Show that, with this constraint, $A$ is symplectic.

(b) Find a constraint on the values of $\tilde{\alpha}$, $\tilde{\beta}$ and $\tilde{\gamma}$, so that the matrix:

$$\tilde{R} = I\,\cosh\mu + S \cdot \tilde{A}\,\sinh\mu \tag{3}$$

is symplectic for any value of $\mu$, where:

$$\tilde{A} = \begin{pmatrix} \tilde{\gamma} & \tilde{\alpha} \\ \tilde{\alpha} & \tilde{\beta} \end{pmatrix} \tag{4}$$

Show that, with this constraint, $\tilde{A}$ satisfies:

$$\tilde{A}^{\mathrm{T}} \cdot S \cdot \tilde{A} = -S \tag{5}$$

(c) Show that $J$, defined by:

$$J = \frac{1}{2} \begin{pmatrix} x & p_x \end{pmatrix} \cdot A \cdot \begin{pmatrix} x \\ p_x \end{pmatrix} \tag{6}$$

is an invariant under the transformation:

$$\begin{pmatrix} x \\ p_x \end{pmatrix} \to R \cdot \begin{pmatrix} x \\ p_x \end{pmatrix} \tag{7}$$

and that $\tilde{J}$, defined by:

$$\tilde{J} = \frac{1}{2} \begin{pmatrix} x & p_x \end{pmatrix} \cdot \tilde{A} \cdot \begin{pmatrix} x \\ p_x \end{pmatrix} \tag{8}$$

is an invariant under:

$$\begin{pmatrix} x \\ p_x \end{pmatrix} \to \tilde{R} \cdot \begin{pmatrix} x \\ p_x \end{pmatrix} \tag{9}$$

(d) Using the appropriate invariants of a symplectic transfer matrix $R$, sketch the phase-space diagrams produced by applying the transfer matrix repeatedly to a particle in the cases (i) $\mathrm{Tr}\, R < 2$, and (ii) $\mathrm{Tr}\, R > 2$, where $\mathrm{Tr}\, R$ is the trace of $R$. Comment on the stability of the motion in each case.

2. Using the generating function:

$$F_1 = F_1(x, \phi_x) = -\frac{x^2}{2\beta_x}(\tan\phi_x + \alpha_x) \tag{10}$$

relate the "new" canonical variables $(\phi_x, J_x)$ to the "old" canonical variables $(x, p_x)$.

3. Find a relationship between the quadrupole focal length $f_0$ and the drift length $L$ in a FODO cell with phase advance 90°. Find the maximum and minimum values of the beta functions in such a cell, as functions of the focal length of the quadrupoles.