Linear Dynamics, Lecture 5

Three Loose Ends: Edge Focusing; Chromaticity; Beam Rigidity.

Andy Wolski
University of Liverpool, and the Cockcroft Institute, Daresbury, UK.

November, 2012

What we Learned in the Previous Lecture

In the previous lecture, we considered the dynamics of a particle moving inside various kinds of electromagnetic field. Specifically, we looked at the cases of dipole, normal and skew quadrupole fields, TM_{010} cavities, and solenoids.

In each case, we solved the equations of motion in the *paraxial approximation*. This involves constructing an approximate Hamiltonian by expanding the exact Hamiltonian to second order in the dynamical variables.

The Hamiltonian for the fields we considered can be solved in the paraxial approximation to yield equations of motion that are linear in the dynamical variables. The solutions to the equations of motion may be expressed in terms of transfer matrices.

Part I (Lectures 1 – 5): Dynamics of a relativistic charged particle in the electromagnetic field of an accelerator beamline.

1. Review of Hamiltonian mechanics

2. The accelerator Hamiltonian in a straight coordinate system

3. The Hamiltonian for a relativistic particle in a general electromagnetic field using accelerator coordinates

4. Dynamical maps for linear elements

5. Three loose ends: edge focusing; chromaticity; beam rigidity

Fringe Fields

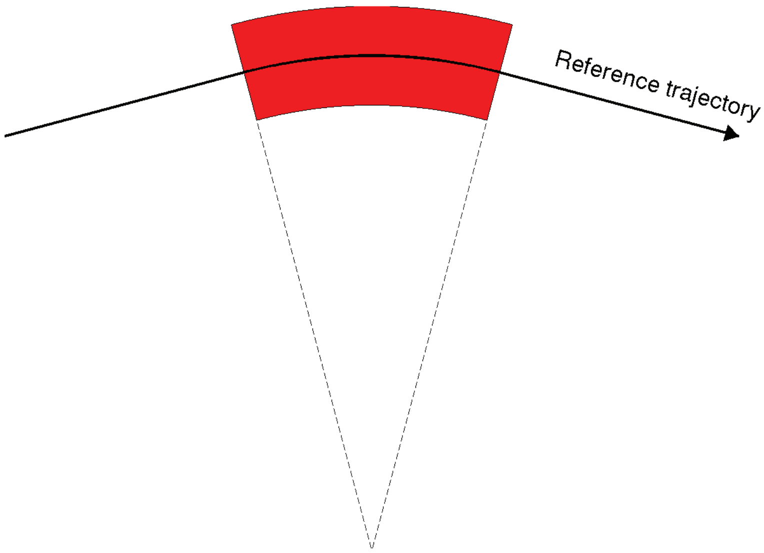
We did not so far consider effects that may occur when a particle moves from a region with one kind of field, to a region with another kind of field; e.g. when a particle moves from a field-free region (i.e. a "drift space") into a dipole field.

Since the electromagnetic fields obey Maxwell's equations, they must vary smoothly and continuously in space. In other words, the transition from a drift space to a dipole cannot be abrupt. There must be a region of space that is neither drift space nor dipole field, but something else (usually something more complicated). This region of space is generally referred to as the "fringe field".

It turns out that fringe fields have important effects. However, their exact description is complicated, and dependent on details of the magnet design. In this lecture, we shall consider the effects of fringe fields using very simple approximations.

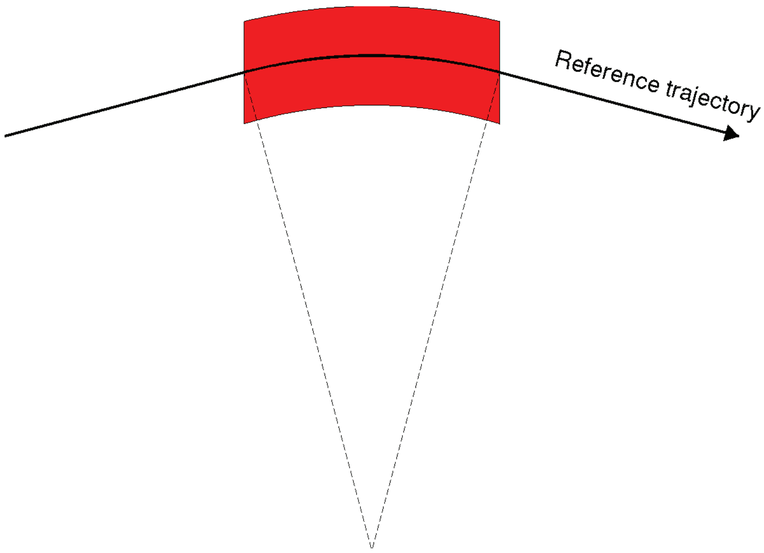
Dipole Pole Faces Normal to the Reference Trajectory

From the point of view of the dynamics, the simplest dipole field to consider is one that is completely independent of x and y. In this case, the pole faces at the entrance and the exit are perpendicular to the reference trajectory.

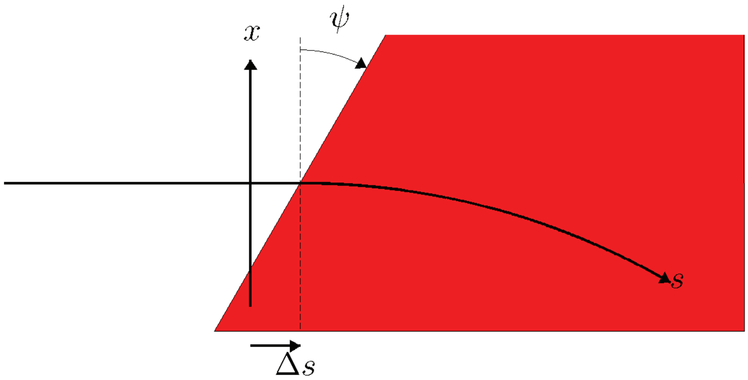


Dipole Pole Faces Angled to the Reference Trajectory

However, magnet engineers do not like to design or build this kind of magnet. From their point of view, the simplest kind of magnet has a rectangular footprint, with the entrance and exit pole faces parallel to each other.



Consider the magnetic field close to the entrance pole face of a dipole. We consider an arbitrary rotation of the pole face through angle ψ from the normal to the reference trajectory.



The effect of the fringe field (including the pole face rotation) is contained within the region $0 < s < \Delta s$, and is represented by the magnetic field:

$$B_x = -B_0 \frac{y}{\Delta s} \tan\psi, \qquad B_y = B_0 \left(\frac{s}{\Delta s} - \frac{x}{\Delta s} \tan\psi \right), \qquad B_s = B_0 \frac{y}{\Delta s}. \tag{1}$$

Note that without the rotation of the pole face, the fringe field is written:

$$B_x = 0, \qquad B_y = B_0 \frac{s}{\Delta s}, \qquad B_s = B_0 \frac{y}{\Delta s}. \tag{2}$$

The vertical field increases linearly from zero at $s = 0$ to the full dipole field B_0 at $s = \Delta s$; there is a longitudinal component to the field, dependent on the vertical coordinate, required by Maxwell's equations.

The dipole fringe field with pole face rotation (1) is obtained from the fringe field without any pole face rotation (2) by a simple rotation, and rescaling the gradient so that the field still increases from zero to B_0 from $s = 0$ to $s = \Delta s$.

The dipole fringe field (1) may be obtained from the vector potential:

$$A_x = \frac{1}{2}\frac{B_0}{\Delta s}\left(s^2 - y^2\right), \qquad A_y = 0, \qquad A_s = -\frac{1}{2}\frac{B_0}{\Delta s}\left(y^2 - x^2\right)\tan\psi. \tag{3}$$

We need to be careful in the transition from the field free region into the fringe field, and from the fringe field into the main dipole field. This is because the vector potential has a horizontal component in the fringe field that does not appear in the field free region or the dipole field. Associated with any change in the vector potential is a change in the canonical momentum:

$$\Delta p_x = \frac{q}{P_0}\Delta A_x \tag{4}$$

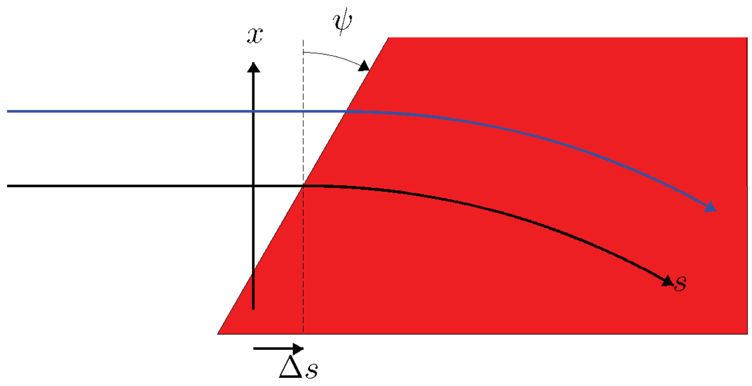
In this particular case, the change in the vector potential is second order in the coordinate y, so there are no linear effects, only (possible) nonlinear effects.

The horizontal component of the vector potential in a dipole fringe field (3) leads to nonlinear (higher-order) effects. However, the longitudinal component has the form of the potential in quadrupole field, with normalised quadrupole strength:

$$k_1 = -\frac{q}{P_0}\frac{B_0}{\Delta s}\tan\psi \tag{5}$$

The vertical focusing comes from the longtudinal component of the magnetic field that is required by Maxwell's equations, and varies linearly with the vertical coordinate. If the pole face is rotated, the "longitudinal" field now has some horizontal component, and a particle travelling parallel to the reference trajectory receives a vertical kick proportional to the vertical position of the particle.

The horizontal focusing in the fringe field is really a geometric effect.



A particle travelling parallel to the reference trajectory but with some positive horizontal offset does not see the main bending field of the dipole until some time later than a particle travelling exactly along the reference trajectory. The result is that the first particle is no longer travelling parallel to the reference trajectory: there is an effective deflection proportional to the horizontal offset of the particle.

In the absence of other information, we usually consider the fringe field of a dipole in the limit $\Delta s \to 0$. In this case, the transfer matrix can be derived directly from that for a "thin" quadrupole (a quadrupole in the limit of zero length and fixed integrated gradient):

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -K_1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & K_1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{6}$$

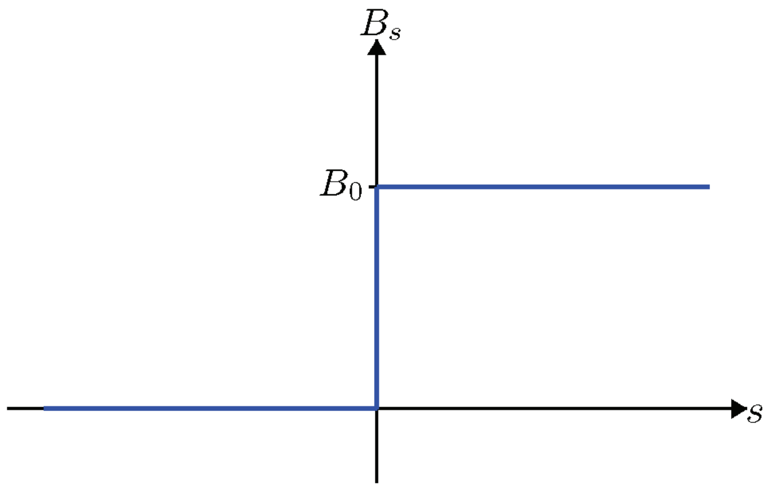
where

$$K_1 = -\frac{q}{P_0} B_0 \tan\psi \tag{7}$$

Note that for q, B_0 and ψ all positive, the fringe field is horizontally defocusing and vertically focusing.

As for the dipole, the fringe field for a solenoid is usually treated in the "hard edge" approximation, in which the extent of the fringe field approaches zero. In this case, the longitudinal field at the entrance to the solenoid is a step function:

$$B_s = B_0 \Theta(s) \tag{8}$$



To satisfy Maxwell's equations, the horizontal and vertical fields must be Dirac delta functions:

$$B_s = B_0 \Theta(s), \quad B_x = -\frac{1}{2} B_0 \delta(s) x, \quad B_y = -\frac{1}{2} B_0 \delta(s) y \tag{9}$$

where the delta function is given by:

$$\delta(s) = \frac{d\Theta}{ds} \tag{10}$$

and satisfies:

$$\delta(s) = 0 \text{ for } s \neq 0 \tag{11}$$

and:

$$\int_{-\infty}^{\infty} \delta(s) ds = 1 \tag{12}$$

The solenoid fringe field (9) can be obtained from the vector potential:

$$A_x = -\frac{1}{2}B_0\Theta(s)y \tag{13}$$

$$A_y = \frac{1}{2}B_0\Theta(s)x \tag{14}$$

$$A_s = 0 \tag{15}$$

with the usual relation:

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{16}$$

Using the usual accelerator Hamiltonian (in straight coordinates):

$$H = \frac{\delta}{\beta_0} - \sqrt{\left(\frac{1}{\beta_0} + \delta\right)^2 - (p_x - a_x)^2 - (p_y - a_y)^2 - \frac{1}{\beta_0^2\gamma_0^2}} - a_s \tag{17}$$

and making the paraxial approximation, we construct the Hamiltonian for the solenoid fringe field to second order in the dynamical variables:

$$H_2 = \frac{1}{2}(p_x + k_s\Theta(s)y)^2 + \frac{1}{2}(p_y - k_s\Theta(s)x)^2 + \frac{1}{2\beta_0^2\gamma_0^2}\delta^2 \tag{18}$$

The equations of motion for the horizontal variables are:

$$\frac{dx}{ds} = \frac{\partial H}{\partial p_x} = p_x + k_s \Theta(s) y \qquad (19)$$

$$\frac{dp_x}{ds} = -\frac{\partial H}{\partial x} = k_s \Theta(s) \left(p_y - k_s \Theta(s) x \right) \qquad (20)$$

and similarly in the vertical plane:

$$\frac{dy}{ds} = \frac{\partial H}{\partial p_y} = p_y - k_s \Theta(s) y \qquad (21)$$

$$\frac{dp_y}{ds} = -\frac{\partial H}{\partial y} = -k_s \Theta(s) \left(p_x + k_s \Theta(s) x \right) \qquad (22)$$

Note that in each of the above equations, the quantity on the right hand side is finite. This implies that, if we consider an infinitesimal step Δs across the fringe field at $s = 0$, we must have in the limit $\Delta s \to 0$:

$$\Delta x \to 0 \qquad \Delta y \to 0 \qquad (23)$$

$$\Delta p_x \to 0 \qquad \Delta p_y \to 0 \qquad (24)$$

The results (23) for Δx and Δy are as expected, since the trajectory of the particle must be continuous across the fringe field. But the results (24) for Δp_x and Δp_y are surprising, since we expect the fringe field to cause a non-zero kick: although the trajectory is continuous, in the approximation we have used, it need not be smooth. To see this, consider Newton's equation of motion with the Lorentz force:

$$\frac{d\bar{p}_x}{dt} = q \left(\dot{y} B_s - \dot{s} B_y \right) \qquad (25)$$

where $\bar{p}_x$ is the horizontal component of the *mechanical* momentum. Using the field (9), we have:

$$\frac{d\bar{p}_x}{dt} = q \left(\dot{y} B_0 \Theta(s) + \frac{1}{2} B_0 \dot{s} \delta(s) y \right) \qquad (26)$$

Using $\dot{s} \approx c$ and Equation (12) for the integral of the delta function, it follows that for an infinitesimal step Δt across the fringe field in the limit $\Delta t \to 0$:

$$\Delta \bar{p}_x \approx \frac{1}{2} q B_0 y \qquad (27)$$

We find from Newton's equation, combined with the Lorentz force for the solenoid fringe field (9), that making an infinitesimal step across the fringe field results in a non-zero change in transverse momentum:

$$\Delta \bar{p}_x \approx \frac{1}{2} q B_0 y \tag{28}$$

However, there is also a non-zero change in the horizontal component of the vector potential (15):

$$\Delta A_x = -\frac{1}{2} B_0 y \tag{29}$$

Recall that the canonical momentum is the mechanical momentum plus the product of the charge and the vector potential. Taking into account the normalisation with respect to the reference momentum P_0:

$$p_x \approx \frac{\bar{p}_x}{P_0} + \frac{q}{P_0} A_x \tag{30}$$

It follows from Equation (30) that, in making an infinitesimal step across the solenoid fringe field, the total change in the canonical momentum is:

$$\Delta p_x \approx \frac{1}{P_0} \Delta \bar{p}_x + \frac{q}{P_0} \Delta A_x = 0 \tag{31}$$

where the last step follows from (28) and (29). In other words, the change in the vector potential cancels the change in the mechanical momentum, so the change in the canonical momentum is (close to) zero. This is consistent with the result (24) that we found from Hamilton's equations. We have shown this is the case for the horizontal motion at the entrance to the solenoid: exactly the same effect happens for the vertical motion at the entrance, and for the horizontal and vertical motion at the exit of the solenoid.

From Equations (23) and (24), the transfer matrix for a solenoid fringe field (entrance and exit) is simply the identity matrix:

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{32}$$

It is important to note that this is only the case when the *canonical* momentum p_x is used as a dynamical variable. Sometimes, the quantity x' is used to describe the motion instead of p_x: in that case, we do not get a cancellation between the mechanical kick and the change in the vector potential, so there is a non-zero change in x' across the fringe field, and the transfer matrix will be different from the identity.

Quadrupoles also have fringe fields. However, it can be shown that the effects are higher-order (and are similar to the effects of sextupoles), and generally quite weak. Nonlinear effects are beyond the scope of this course, so we do not consider quadrupole fringe fields any further. Similarly, for purposes of linear dynamics, the fringe fields of combined function bends have the same effect (to first order) as the fringe fields of bending magnets without quadrupole component, but with the same strength dipole field.

We have now completed our derivation of transfer matrices for "linear" elements in accelerator beamlines. These matrices were obtained from the Hamiltonian for the corresponding electromagnetic field in the paraxial approximation, and are valid for small values of the dynamical variables.

You should now be able to write a simple tracking code to calculate the transport of charged particles along a linear beamline, including the effects of drift spaces, bending magnets (with and without quadrupole gradient), dipole fringe fields for arbitrary pole-face rotations, normal and skew quadrupoles, TM_{010} RF cavities, and solenoids. For convenient reference, the transfer matrices for all these elements are given in an appendix at the end of this lecture.

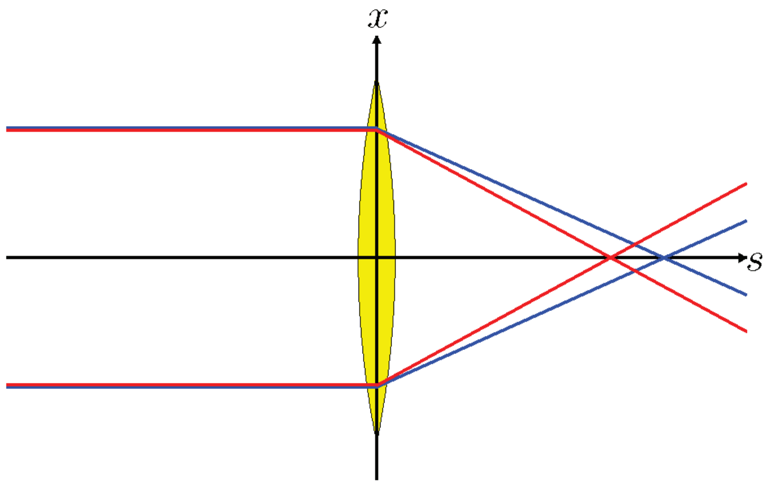
In the transfer matrix for a dipole, we saw terms associated with the *dispersion*, i.e. the change in trajectory of a particle with respect to the energy deviation δ. Recall that the transfer matrix is:

$$R = \begin{pmatrix} \cos\omega L & \frac{\sin\omega L}{\omega} & 0 & 0 & 0 & \frac{1-\cos\omega L}{\omega\beta_0} \\ -\omega\sin\omega L & \cos\omega L & 0 & 0 & 0 & \frac{\sin\omega L}{\beta_0} \\ 0 & 0 & 1 & L & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -\frac{\sin\omega L}{\beta_0} & -\frac{1-\cos\omega L}{\omega\beta_0} & 0 & 0 & 1 & \frac{L}{\beta_0^2\gamma_0^2} - \frac{\omega L - \sin\omega L}{\omega\beta_0^2} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (33)$$

where L is the length of the dipole, and $\omega = k_0$ is the dipole field strength normalised with respect to the reference momentum. The *dispersion* terms are R_{16} and R_{26}.

Dispersion is an example of an energy-dependent transverse effect: it may be viewed as a coupling between the longitudinal and transverse planes. Quadrupoles also have energy-dependent transverse effects. This may be seen very easily from the physical effect of a quadrupole. Consider a set of particles entering a quadrupole parallel to the reference trajectory. The higher-energy particles are deflected less strongly by the quadrupole field than the lower-energy particles, so the focal length of the quadrupole is dependent on the energy.

Chromaticity is the variation in *focusing strength* of a quadrupole (or other "linear" element) with the energy of the particle. The higher the energy of the particle the longer the focal length.



Recall the transfer matrix for a quadrupole:

$$R = \begin{pmatrix} \cos\omega L & \frac{\sin\omega L}{\omega} & 0 & 0 & 0 & 0 \\ -\omega\sin\omega L & \cos\omega L & 0 & 0 & 0 & 0 \\ 0 & 0 & \cosh\omega L & \frac{\sinh\omega L}{\omega} & 0 & 0 \\ 0 & 0 & \omega\sinh\omega L & \cosh\omega L & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \frac{L}{\beta_0^2\gamma_0^2} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad (34)$$

where

$$\omega = \sqrt{k_1} \qquad (35)$$

and k_1 is the quadrupole gradient, normalised by the reference momentum P_0.

Where is the chromaticity in the transfer matrix?

We lost the chromaticity in a quadrupole when we made the paraxial approximation in the Hamiltonian. More strictly, we lost the chromaticity when we made the paraxial approximation *including the longitudinal variables*. The full Hamiltonian for a quadrupole is:

$$H = \frac{\delta}{\beta_0} - \sqrt{\left(\frac{1}{\beta_0} + \delta\right)^2 - p_x^2 - p_y^2 - \frac{1}{\beta_0^2\gamma_0^2}} + \frac{1}{2}k_1\left(x^2 - y^2\right) \qquad (36)$$

Expanding the Hamiltonian (36) to second order in all the dynamical variables (making the paraxial approximation) we constructed the Hamiltonian:

$$H_2 = \frac{1}{2}p_x^2 + \frac{1}{2}p_y^2 + \frac{1}{2}k_1 x^2 - \frac{1}{2}k_1 y^2 + \frac{1}{2\beta_0^2\gamma_0^2}\delta^2 \qquad (37)$$

In the second-order Hamiltonian (37) there are no terms that include transverse and longitudinal variables together. Thus, there are no energy-dependent transverse effects - so there is no chromaticity.

Chromaticity is an extremely important effect: in practice, the behaviour of a beam in a lattice is strongly dependent on the chromaticity. Is there anything we can do to restore the chromatic effects in the equations of motion?

The answer is "yes", but we have to pay a price. We first of all note that in the full Hamiltonian (36), the longitudinal coordinate z does not appear at all. Consequently, from Hamilton's equations, the energy deviation δ is constant. We can therefore expand the Hamiltonian to second order in the *transverse* variables, while keeping the full dependence on δ:

$$H_2 = \frac{\delta}{\beta_0} - D + \frac{p_x^2}{2D} + \frac{p_y^2}{2D} + \frac{1}{2}k_1 x^2 - \frac{1}{2}k_1 y^2 \tag{38}$$

where the constant D is given by:

$$D = \sqrt{1 + \frac{2\delta}{\beta_0} + \delta^2} \tag{39}$$

Since δ, and hence D are constants, the Hamiltonian (38) leads to equations of motion that can be solved for the transverse variables. Expressed as a 4×4 transfer matrix, the solutions to the equations of motion are:

$$\vec{x}(s = L) = \tilde{R} \cdot \vec{x}(s = 0) \tag{40}$$

where:

$$\vec{x} = \begin{pmatrix} x \\ p_x \\ y \\ p_y \end{pmatrix} \qquad \tilde{R} = \begin{pmatrix} \cos\tilde{\omega}L & \frac{\sin\tilde{\omega}L}{D\tilde{\omega}} & 0 & 0 \\ -D\tilde{\omega}\sin\tilde{\omega}L & \cos\tilde{\omega}L & 0 & 0 \\ 0 & 0 & \cosh\tilde{\omega}L & \frac{\sinh\tilde{\omega}L}{D\tilde{\omega}} \\ 0 & 0 & D\tilde{\omega}\sinh\tilde{\omega}L & \cosh\tilde{\omega}L \end{pmatrix} \tag{41}$$

and:

$$\tilde{\omega} = \sqrt{\frac{k_1}{D}} \tag{42}$$

The longitudinal equation of motion from Hamilton's equations with the Hamiltonian (38) is now a little more complicated than before:

$$\frac{dz}{ds} = \frac{1}{\beta_0} - \frac{1}{2}\left(\frac{\beta_0}{D}\right)^3 (1+\beta_0\delta)\left(\frac{2D^2}{\beta_0^4} + x^2 + y^2\right) \qquad (43)$$

However, with the known solutions for $x(s)$ and $y(s)$ from (41), we can solve the equation of motion (43). The result is not especially enlightening, and we do not give it explicitly here.

The chromatic effects in other "linear" components can be treated in the same way as for a quadrupole as long as the energy deviation δ is constant (which is not generally the case for an RF cavity). The result is a map that is linear in the transverse variables, but has a nonlinear dependence on the energy deviation. For this reason, chromaticity is formally a "nonlinear" rather than a linear effect (in contrast to the dispersion, which does appear in the 6×6 linear transfer matrix for a bending magnet).

Sometimes people talk about "linear chromaticity". By this, they generally mean the linear part of the dependence of the *focusing* strength (e.g. ω in the quadrupole map) on the energy deviation. This is not the same thing as a linear dependence of the transformation of the dynamical variables (x, p_x, etc.) on the energy deviation.

We have frequently used the *reference momentum* P_0 to normalise quantities in the linear dynamics we have studied so far. For example, the normalised field strength of a dipole magnet is given by:

$$k_0 = \frac{q}{P_0} B_0 \tag{44}$$

The quantity that appears in the equation of motion in a dipole is k_0, rather than the absolute field strength B_0.

Beam Rigidity

Consider a particle of charge q and mechanical momentum P_0 moving in a plane perpendicular to a uniform field B. The magnitude of the Lorentz force acting on the particle is:

$$F = q\beta_0 cB \tag{45}$$

Since the force is always perpendicular to the instantaneous velocity, the particle must be following a circular trajectory. The centripetal force is provided by the Lorentz force, and is given (for general circular motion) by:

$$F = \frac{P_0 \beta_0 c}{\rho} \tag{46}$$

where ρ is the radius of the particle's trajectory. Equating the Lorentz force (45) and the centripetal force (46), we find:

$$B\rho = \frac{P_0}{q} \tag{47}$$

A particle of charge q moving in a uniform field B with momentum P_0 follows a circular trajectory with radius ρ given by (47):

$$B\rho = \frac{P_0}{q} \tag{48}$$

In other words, the product of the field and the radius is a function only of the particle momentum and charge. We also notice that the particular combination P_0/q is exactly that which appears in the normalisation of many physical quantities in beam dynamics.

The quantity $B\rho$, called the *beam rigidity*, is often used instead of P_0/q. Note that the beam rigidity does not refer to a specific field strength or radius of curvature of the trajectory: it should be thought of as simply another way of writing the reference momentum P_0.

The beam rigidity is useful because it gives a more intuitive way of writing many formulae than using the reference momentum. For example, the curvature $1/\rho_0$ of the trajectory of a particle with the reference momentum P_0 in a field of strength B_0 is:

$$\frac{1}{\rho_0} = \frac{qB_0}{P_0} = \frac{B_0}{B\rho} \tag{49}$$

For a high-energy particle ($E_0 \gg m_0c^2$), the beam rigidity can be conveniently calculated from:

$$B\rho[\mathrm{Tm}] \approx \frac{E_0[\mathrm{eV}]}{c[\mathrm{m/s}]} \tag{50}$$

where the energy E_0 is expressed in electron-volts, the speed of light is expressed in meters per second, and the beam rigidity $B\rho$ is given in tesla-meters.

The entrance and exit faces of dipoles are often made parallel so that the beam does not enter or exit the magnet normal to the face. The effect of these *pole face rotations* is to introduce an additional focusing, that acts in both the horizontal plane (because of the geometry of the magnet) and the vertical plane (because of the fringe field of the magnet).

The fringe field of a solenoid results in a kick in the mechanical momenta. However, in canonical coordinates, this kick is cancelled by the change in the magnetic vector potential when entering or exiting the solenoid. Thus, the transfer matrix for a solenoid fringe field in canonical coordinates is the identity.

Quadrupoles have chromatic effects which, strictly speaking, are nonlinear; but ought not be ignored.

The *beam rigidity* is often used as an alternative to the reference momentum.

Appendix: Transfer Matrices

The transfer matrix for a drift space of length L is:

$$R = \begin{pmatrix} 1 & L & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & L & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \frac{L}{\beta_0^2\gamma_0^2} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{51}$$

The transfer matrix for a dipole of length L and vertical field strength B_0 is:

$$R = \begin{pmatrix} \cos\omega L & \frac{\sin\omega L}{\omega} & 0 & 0 & 0 & \frac{1-\cos\omega L}{\omega\beta_0} \\ -\omega\sin\omega L & \cos\omega L & 0 & 0 & 0 & \frac{\sin\omega L}{\beta_0} \\ 0 & 0 & 1 & L & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -\frac{\sin\omega L}{\beta_0} & -\frac{1-\cos\omega L}{\omega\beta_0} & 0 & 0 & 1 & \frac{L}{\beta_0^2\gamma_0^2} - \frac{\omega L - \sin\omega L}{\omega\beta_0^2} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{52}$$

where $\omega = k_0 = \frac{q}{P_0}B_0$.

The transfer matrix for a combined-function dipole of length L and vertical field strength B_0 is:

$$R = \begin{pmatrix} \cos\omega_x L & \frac{\sin\omega_x L}{\omega_x} & 0 & 0 & 0 & \frac{k_0}{\beta_0}\frac{(1-\cos\omega_x L)}{\omega_x^2} \\ -\omega_x\sin\omega_x L & \cos\omega_x L & 0 & 0 & 0 & \frac{k_0}{\beta_0}\frac{\sin\omega_x L}{\omega_x} \\ 0 & 0 & \cosh\omega_y L & \frac{\sinh\omega_y L}{\omega_y} & 0 & 0 \\ 0 & 0 & \omega_y\sinh\omega_y L & \cosh\omega_y L & 0 & 0 \\ -\frac{k_0}{\beta_0}\frac{\sin\omega_x L}{\omega_x} & -\frac{k_0}{\beta_0}\frac{(1-\cos\omega_x L)}{\omega_x^2} & 0 & 0 & 1 & \frac{L}{\beta_0^2\gamma_0^2} - \frac{k_0^2}{\beta_0^2}\frac{(\omega_x L - \sin\omega_x L)}{\omega_x^3} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{53}$$

where:

$$\omega_x = \sqrt{k_0^2 + k_1}, \qquad \omega_y = \sqrt{k_1} \tag{54}$$

The field is:

$$B_x = b_2\frac{y}{r_0}, \qquad B_y = b_1 + b_2\frac{x}{r_0}, \qquad B_s = 0 \tag{55}$$

and the normalised field strengths are:

$$k_0 = \frac{q}{P_0}b_1, \qquad k_1 = \frac{q}{P_0}\frac{b_2}{r_0} \tag{56}$$

In the "hard edge" approximation, the transfer matrix for a dipole fringe field (exit and entrance) is:

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -K_1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & K_1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (57)$$

where

$$K_1 = -\frac{q}{P_0} B_0 \tan\psi \quad (58)$$

B_0 is the dipole field strength, and ψ is the rotation angle of the pole face. A dipole with a "rectangular" footprint has parallel pole faces, with the rotation angle ψ positive for each pole face.

The transfer matrix for a normal quadrupole of length L is:

$$R = \begin{pmatrix} \cos\omega L & \frac{\sin\omega L}{\omega} & 0 & 0 & 0 & 0 \\ -\omega\sin\omega L & \cos\omega L & 0 & 0 & 0 & 0 \\ 0 & 0 & \cosh\omega L & \frac{\sinh\omega L}{\omega} & 0 & 0 \\ 0 & 0 & \omega\sinh\omega L & \cosh\omega L & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \frac{L}{\beta_0^2\gamma_0^2} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (59)$$

where

$$\omega = \sqrt{k_1}, \qquad k_1 = \frac{q}{P_0}\frac{b_2}{r_0} \quad (60)$$

and the quadrupole field is:

$$B_x = b_2\frac{y}{r_0}, \qquad B_y = b_2\frac{x}{r_0}, \qquad B_s = 0 \quad (61)$$

The transfer matrix for a skew quadrupole of length L is:

$$\begin{pmatrix} \frac{1}{2}(\cos\omega L + \cosh\omega L) & \frac{1}{2\omega}(\sin\omega L + \sinh\omega L) & \frac{1}{2}(\cos\omega L - \cosh\omega L) & \frac{1}{2\omega}(\sin\omega L - \sinh\omega L) & 0 & 0 \\ -\frac{1}{2}\omega(\sin\omega L - \sinh\omega L) & \frac{1}{2}(\cos\omega L + \cosh\omega L) & -\frac{1}{2}\omega(\sin\omega L + \sinh\omega L) & \frac{1}{2}(\cos\omega L - \cosh\omega L) & 0 & 0 \\ \frac{1}{2}(\cos\omega L - \cosh\omega L) & \frac{1}{2\omega}(\sin\omega L - \sinh\omega L) & \frac{1}{2}(\cos\omega L + \cosh\omega L) & \frac{1}{2\omega}(\sin\omega L + \sinh\omega L) & 0 & 0 \\ -\frac{\omega}{2}(\sin\omega L + \sinh\omega L) & \frac{1}{2}(\cos\omega L - \cosh\omega L) & -\frac{\omega}{2}(\sin\omega L - \sinh\omega L) & \frac{1}{2}(\cos\omega L + \cosh\omega L) & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \frac{L}{\beta_0^2\gamma_0^2} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{62}$$

where

$$\omega = \sqrt{k_{1s}}, \qquad k_{1s} = \frac{q}{P_0}\frac{a_2}{r_0} \tag{63}$$

and the quadrupole field is:

$$B_x = a_2\frac{x}{r_0}, \qquad B_y = -a_2\frac{y}{r_0}, \qquad B_s = 0 \tag{64}$$

Appendix: Transfer Matrices

The transfer matrix for a solenoid of length L is:

$$R = \begin{pmatrix} \cos^2\omega L & \frac{\sin 2\omega L}{2\omega} & \frac{1}{2}\sin 2\omega L & \frac{\sin^2\omega L}{\omega} & 0 & 0 \\ \frac{\omega}{2}\sin 2\omega L & \cos^2\omega L & -\omega\sin^2\omega L & \frac{1}{2}\sin 2\omega L & 0 & 0 \\ -\frac{1}{2}\sin 2\omega L & -\frac{\sin^2\omega L}{\omega} & \cos^2\omega L & \frac{\sin 2\omega L}{2\omega} & 0 & 0 \\ \omega\sin^2\omega L & -\frac{1}{2}\sin 2\omega L & -\frac{\omega}{2}\sin 2\omega L & \cos^2\omega L & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \frac{L}{\beta_0^2\gamma_0^2} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{65}$$

where

$$\omega = k_s = \frac{1}{2}\frac{q}{P_0}B_0 \tag{66}$$

and B_0 is the solenoid field strength. In the "hard edge" approximation, the transfer matrix for the fringe field of a solenoid is the identity.

The transfer matrix for a TM_{010} RF cavity of length $L = \pi/k$ (where $k = 2\pi f_{RF}/c$ for RF frequency f_{RF}) is:

$$R = \begin{pmatrix} \cos\psi_\perp & \frac{L}{\psi_\perp}\sin\psi_\perp & 0 & 0 & 0 & 0 \\ -\frac{\psi_\perp}{L}\sin\psi_\perp & \cos\psi_\perp & 0 & 0 & 0 & 0 \\ 0 & 0 & \cos\psi_\perp & \frac{L}{\psi_\perp}\sin\psi_\perp & 0 & 0 \\ 0 & 0 & -\frac{\psi_\perp}{L}\sin\psi_\perp & \cos\psi_\perp & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos\psi_\parallel & \frac{1}{\beta_0^2\gamma_0^2}\frac{L}{\psi_\parallel}\sin\psi_\parallel \\ 0 & 0 & 0 & 0 & -\beta_0^2\gamma_0^2\frac{\psi_\parallel}{L}\sin\psi_\parallel & \cos\psi_\parallel \end{pmatrix} \tag{67}$$

where:

$$\psi_\perp = \sqrt{\frac{\pi\alpha\cos\phi_0}{2}} \qquad \psi_\parallel = \frac{\sqrt{\pi\alpha\cos\phi_0}}{\gamma_0\beta_0} \tag{68}$$

and, for "RF voltage" $\hat{V}$ (which includes the transit time factor):

$$\alpha = \frac{q\hat{V}}{P_0 c} \tag{69}$$