Linear Dynamics, Lecture 2

The Accelerator Hamiltonian in a Straight Coordinate System

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What we Learned in the Previous Lecture

In the previous lecture, we saw how the dynamics of a conservative system could be derived from an appropriate Hamiltonian.

The Hamiltonian is an expression containing the coordinates and conjugate momenta (the canonical dynamical variables).

Using the Hamiltonian in Hamilton's equations gives the equations of motion for the system. These are first-order simultaneous differential equations that one must solve to find explicit expressions for the coordinates and momenta as functions of the independent variable (usually, the time t).

We looked at a number of examples, including the Hamiltonian for a non-relativistic particle moving through an electromagnetic field.

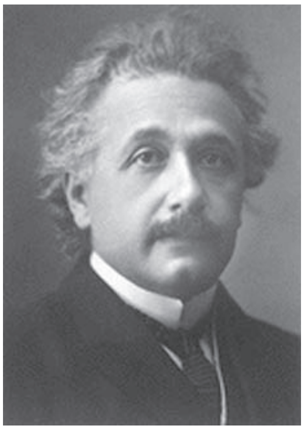
Course Outline

Part I (Lectures 1 – 5): Dynamics of a relativistic charged particle in the electromagnetic field of an accelerator beamline.

1. Review of Hamiltonian mechanics
2. The accelerator Hamiltonian in a straight coordinate system
3. The Hamiltonian for a relativistic particle in a general electromagnetic field using accelerator coordinates
4. Dynamical maps for linear elements
5. Three loose ends: edge focusing; chromaticity; beam rigidity.

Goals of This Lecture

In this lecture, we study the Hamiltonian for a relativistic particle moving through an electromagnetic field in a straight coordinate system. We shall use canonical transformations to express the Hamiltonian in terms of dynamical variables that are convenient for accelerator physics.



Albert Einstein, 1879-1955

Einstein's equation in Special Relativity relating the energy E and momentum $\bar{\mathbf{p}}$ of a particle is:

$$E^2 = \bar{\mathbf{p}}^2 c^2 + m^2 c^4 \tag{1}$$

where m is the rest mass. Note that $\bar{\mathbf{p}}$ in this equation is the *mechanical* momentum (indicated by the bar), not the conjugate (canonical) momentum.

We saw in Lecture 1 that the Hamiltonian often took the form:

$$H = T + V \tag{2}$$

where T is the kinetic energy, and V is the potential energy; i.e. the Hamiltonian is often the total energy of the system, expressed in canonical variables.

Therefore, using Einstein's equation (1), we write down for our relativistic Hamiltonian:

$$H = \sqrt{\mathbf{p}^2 c^2 + m^2 c^4} \tag{3}$$

where, in the absence of an electromagnetic field, the conjugate momentum $\mathbf{p}$ is equal to the mechanical momentum $\bar{\mathbf{p}}$.

What equations of motion does the Hamiltonian (3) lead to? Using Hamilton's equations:

$$\frac{dx_i}{dt} = \frac{\partial H}{\partial p_i} = \frac{cp_i}{\sqrt{\mathbf{p}^2 + m^2c^2}} \tag{4}$$

and:

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial x_i} = 0 \tag{5}$$

Equation (5) simply expresses the conservation of momentum: there are no forces acting on the particle, because we have not yet introduced any electromagnetic field.

Equation (4) is equally interesting. Rearranging, we find:

$$\mathbf{p} = \frac{m\dot{\mathbf{x}}}{\sqrt{1 - \dot{\mathbf{x}}^2/c^2}} \tag{6}$$

where, as usual, $\dot{\mathbf{x}} = \frac{d\mathbf{x}}{dt}$.

To summarise, the Hamiltonian (3):

$$H = \sqrt{\mathbf{p}^2c^2 + m^2c^4} \tag{7}$$

leads to the conservation of momentum (4):

$$\dot{\mathbf{p}} = 0 \tag{8}$$

and an expression for the relativistic momentum (6):

$$\mathbf{p} = \boldsymbol{\beta}\gamma mc \tag{9}$$

where:

$$\boldsymbol{\beta} = \frac{\dot{\mathbf{x}}}{c} \qquad \gamma = \frac{1}{\sqrt{1 - \boldsymbol{\beta}^2}} \tag{10}$$

Finally, substituting (9) and (10) back into the expression for the Hamiltonian (7), and identifying the energy E of the particle with the Hamiltonian, we find:

$$E = \gamma mc^2 \tag{11}$$

Eqs. (9) and (11) are as expected from Special Relativity.

What about the electromagnetic field? For the nonrelativistic case, we found that the Lorentz force equation followed from the Hamiltonian if the potential energy was:

$$V = q\phi \tag{12}$$

and the *conjugate* or *canonical* momentum was:

$$\mathbf{p} = m\dot{\mathbf{x}} + q\mathbf{A} \tag{13}$$

so that the non-relativistic Hamiltonian took the form:

$$H = \frac{(\mathbf{p} - q\mathbf{A})^2}{2m} + q\phi \tag{14}$$

This suggests that for the relativistic case, the Hamiltonian should be:

$$H = \sqrt{(\mathbf{p} - q\mathbf{A})^2 c^2 + m^2 c^4} + q\phi \tag{15}$$

Our Hamiltonian for relativistic particles in an electromagnetic field is (15):

$$H = \sqrt{(\mathbf{p} - q\mathbf{A})^2 c^2 + m^2 c^4} + q\phi \tag{16}$$

What are the equations of motion that follow from this Hamiltonian? Hamilton's first equation gives:

$$\frac{dx}{dt} = \frac{\partial H}{\partial p_x} = \frac{c\,(p_x - qA_x)}{\sqrt{(\mathbf{p} - q\mathbf{A})^2 + m^2c^2}} \tag{17}$$

Rearranging gives:

$$\mathbf{p} - q\mathbf{A} = \boldsymbol{\beta}\gamma mc \tag{18}$$

In other words, the *canonical* momentum is given by:

$$\mathbf{p} = \boldsymbol{\beta}\gamma mc + q\mathbf{A} \tag{19}$$

The Hamiltonian (15) is:

$$H = \sqrt{(\mathbf{p} - q\mathbf{A})^2 c^2 + m^2 c^4} + q\phi \tag{20}$$

Hamilton's second equation gives:

$$\begin{aligned} \frac{dp_x}{dt} &= -\frac{\partial H}{\partial x} \qquad (21) \\ &= \frac{qc}{\sqrt{(\mathbf{p} - q\mathbf{A})^2 + m^2c^2}} \times \\ & \left[(p_x - qA_x)\frac{\partial A_x}{\partial x} + (p_y - qA_y)\frac{\partial A_y}{\partial x} + (p_z - qA_z)\frac{\partial A_z}{\partial x} \right] \\ & -q\frac{\partial \phi}{\partial x} \qquad (22) \end{aligned}$$

This looks a bit frightening, but with the help of the expression (18) for the canonical momentum, we find that:

$$\frac{dp_x}{dt} = q\left(\dot{x}\frac{\partial A_x}{\partial x} + \dot{y}\frac{\partial A_y}{\partial x} + \dot{z}\frac{\partial A_z}{\partial x} \right) - q\frac{\partial \phi}{\partial x} \tag{23}$$

The equation of motion for the canonical momentum is (23):

$$\frac{dp_x}{dt} = q\left(\dot{x}\frac{\partial A_x}{\partial x} + \dot{y}\frac{\partial A_y}{\partial x} + \dot{z}\frac{\partial A_z}{\partial x} \right) - q\frac{\partial \phi}{\partial x} \tag{24}$$

This has exactly the same form as for the non-relativistic case that we looked at in Lecture 1. So we can immediately write down the solution:

$$\frac{d}{dt}(\mathbf{p} - q\mathbf{A}) = q(\mathbf{E} + \dot{\mathbf{x}} \times \mathbf{B}) \tag{25}$$

where the electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$ are defined as usual:

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} \qquad \mathbf{B} = \nabla \times \mathbf{A} \tag{26}$$

Recalling the expression for the canonical momentum (18) in the relativistic case, we have:

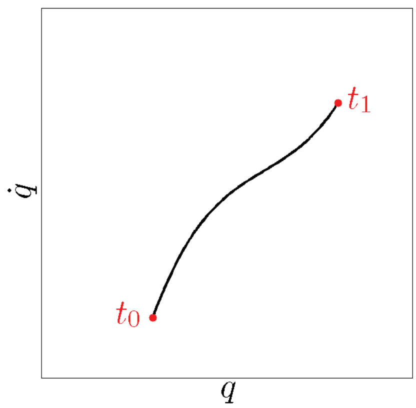
$$\frac{d}{dt}\boldsymbol{\beta}\gamma mc = q(\mathbf{E} + \dot{\mathbf{x}} \times \mathbf{B}) \tag{27}$$

We now have a Hamiltonian (15) that describes the motion of a relativistic charged particle in a general magnetic field:

$$H = \sqrt{(\mathbf{p} - q\mathbf{A})^2 c^2 + m^2 c^4} + q\phi \tag{28}$$

In an accelerator, the magnets, RF cavities and other components are at defined locations along the reference trajectory: we know the longitudinal position at which a particle arrives at a magnet, but we don't necessarily know the time at which it arrives. This means it is more convenient to work with the path length s along the reference trajectory as the independent variable, than the time t.

A change in independent variable from time t to path length s may be accomplished with recourse to the principle of least action, that we saw in Lecture 1.

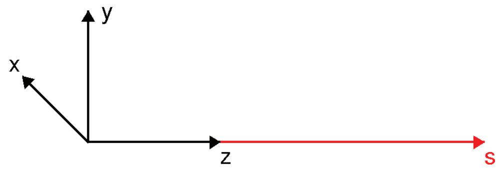


Recall the Principle of Least Action: the Euler-Lagrange equations define a path in a plot of $\dot{q}$ vs q for which the action S is a minimum:

$$\delta S = \delta \left[\int_{t_0}^{t_1} L dt \right] = 0 \tag{29}$$

Now write the action in terms of the Hamiltonian:

$$S = \int_{t_0}^{t_1} (p_x \dot{x} + p_y \dot{y} + p_z \dot{z} - H)\, dt \qquad (30)$$



Let us choose our coordinates so that the z axis defines the reference trajectory. Changing the variable of integration from time t to path length z, the action becomes:

$$S = \int_{z_0}^{z_1} \left(p_x x' + p_y y' + p_z - Ht' \right) dz \qquad (31)$$

where the prime denotes the derivative with respect to z.

The action with time as the independent variable is (30):

$$S = \int_{t_0}^{t_1} (p_x \dot{x} + p_y \dot{y} + p_z \dot{z} - H)\, dt \qquad (32)$$

and the action with path length as the independent variable is (31):

$$S = \int_{z_0}^{z_1} \left(p_x x' + p_y y' - Ht' + p_z \right) dz \qquad (33)$$

Comparing equations (32) and (33), we see that to describe the motion in Hamiltonian mechanics with path length z as the independent variable, we should take as our canonical variables:

$$(x, p_x), \quad (y, p_y), \quad (-t, H) \qquad (34)$$

and use for the Hamiltonian:

$$H_1 = -p_z \qquad (35)$$

Identifying the Hamiltonian H with the energy E, we can rearrange equation (15) to express p_z as:

$$p_z = \sqrt{\frac{(E - q\phi)^2}{c^2} - m^2c^2 - (p_x - qA_x)^2 - (p_y - qA_y)^2} + qA_z \quad (36)$$

Therefore, in the new variables, our Hamiltonian is:

$$H_1 = -\sqrt{\frac{(E - q\phi)^2}{c^2} - m^2c^2 - (p_x - qA_x)^2 - (p_y - qA_y)^2} - qA_z \quad (37)$$

where E is the total energy of the particle, and is now a canonical momentum variable conjugate to $-t$.

It is useful to work with variables whose values remain small as the particle moves through the accelerator: this enables us to make some useful approximations. To do this, we introduce the *reference momentum* P_0. In principle, P_0 can be chosen to have any value you wish; but you would be wise to choose a value close to the nominal momentum of particles in your accelerator.

It is easy to see that if we make the substitutions:

$$p_i \to \tilde{p}_i = \frac{p_i}{P_0} \quad (38)$$

then Hamilton's equations remain unchanged as long as we simultaneously make the substitution:

$$H_1 \to \tilde{H} = \frac{H}{P_0} \quad (39)$$

In terms of the *normalised momenta* (38), the Hamiltonian is:

$$\tilde{H} = -\sqrt{\frac{(E - q\phi)^2}{P_0^2 c^2} - \frac{m^2 c^2}{P_0^2} - (\tilde{p}_x - a_x)^2 - (\tilde{p}_y - a_y)^2} - a_z \quad (40)$$

where the normalised vector potential is defined by:

$$\mathbf{a} = q\frac{\mathbf{A}}{P_0} \quad (41)$$

The transverse normalised momenta $\tilde{p}_x$ and $\tilde{p}_y$ should now be small, but the longitudinal normalised momentum E/P_0 will in general be close to the speed of light, c. Therefore, we make a canonical transformation, using a generating function of the second kind:

$$F_2(x, P_x, y, P_y, -t, \delta, z) = xP_x + yP_y + \left(\frac{z}{\beta_0} - ct\right)\left(\frac{1}{\beta_0} + \delta\right) \quad (42)$$

where P_x, P_y and δ are our new momentum variables, and β_0 is the normalised velocity of a particle with the reference momentum P_0. Using the equations:

$$\tilde{p}_i = \frac{\partial F_2}{\partial q_i} \qquad Q_i = \frac{\partial F_2}{\partial P_i} \qquad K = \tilde{H} + \frac{\partial F_2}{\partial z} \quad (43)$$

we find that the transverse variables are unchanged:

$$\tilde{p}_x = P_x, \qquad X = x \quad (44)$$

$$\tilde{p}_y = P_y, \qquad Y = y \quad (45)$$

The old and new longitudinal variables are related by:

$$\frac{E}{P_0} = c\left(\frac{1}{\beta_0} + \delta\right), \qquad Z = \frac{z}{\beta_0} - ct \tag{46}$$

and the new Hamiltonian (dropping a constant term) is:

$$K = \frac{\delta}{\beta_0} - \sqrt{\left(\frac{1}{\beta_0} + \delta - \frac{q\phi}{P_0 c}\right)^2 - (P_x - a_x)^2 - (P_y - a_y)^2 - \frac{m^2c^2}{P_0^2}} - a_z \tag{47}$$

The new dynamical variable δ is given by:

$$\delta = \frac{E}{P_0 c} - \frac{1}{\beta_0} \tag{48}$$

For a relativistic particle with the reference momentum P_0, δ will be zero. δ is generally called the "energy deviation".

We have made a series of transformations. Let us tidy up the notation, and rewrite:

$$K \to H, \qquad P_i \to p_i, \qquad z \to s, \qquad Z \to z \tag{49}$$

Then the Hamiltonian for a relativistic particle in an electromagnetic field, using the distance along a straight reference trajectory as the independent variable is:

$$H = \frac{\delta}{\beta_0} - \sqrt{\left(\frac{1}{\beta_0} + \delta - \frac{q\phi}{P_0 c}\right)^2 - (p_x - a_x)^2 - (p_y - a_y)^2 - \frac{m^2c^2}{P_0^2}} - a_z \tag{50}$$

Since $\frac{mc}{P_0} = 1/\gamma_0\beta_0$, where $\gamma_0 = 1/\sqrt{1-\beta_0^2}$ we can write:

$$H = \frac{\delta}{\beta_0} - \sqrt{\left(\frac{1}{\beta_0} + \delta - \frac{q\phi}{P_0 c}\right)^2 - (p_x - a_x)^2 - (p_y - a_y)^2 - \frac{1}{\beta_0^2\gamma_0^2}} - a_z \tag{51}$$

Let us remind ourselves of a few definitions. The following are physical constants: q is the charge of the particle; m is the rest mass of the particle; c is the speed of light. The *reference momentum* P_0 can be chosen freely, but should have a value close to the nominal momentum of particles in the accelerator. β_0 is the normalised velocity of a particle moving with the reference momentum. The dynamical variables are:

$$(x, p_x), \qquad (y, p_y), \qquad (z, \delta) \tag{52}$$

The energy deviation is defined by (48):

$$\delta = \frac{E}{P_0 c} - \frac{1}{\beta_0} \tag{53}$$

Finally, the electromagnetic potential functions are:

$$\phi(x, y, z; s), \qquad \mathbf{a}(x, y, z; s) = \frac{q}{P_0}\mathbf{A}(x, y, z; s) \tag{54}$$

The physical meaning of the transverse coordinates x and y is clear enough: x and y are simply the coordinates of the particle in a Cartesian coordinate system. The energy deviation is given by (48):

$$\delta = \frac{E}{P_0 c} - \frac{1}{\beta_0} \tag{55}$$

Using Hamilton's equations with the Hamiltonian (51), we can derive the equation of motion for the longitudinal coordinate z. In a field-free region:

$$H = \frac{\delta}{\beta_0} - \sqrt{\left(\frac{1}{\beta_0} + \delta\right)^2 - p_x^2 - p_y^2 - \frac{1}{\beta_0^2 \gamma_0^2}} \tag{56}$$

It follows from Hamilton's equations:

$$\frac{dz}{ds} = \frac{\partial H}{\partial \delta} = \frac{1}{\beta_0} - \frac{\frac{1}{\beta_0} + \delta}{\sqrt{\left(\frac{1}{\beta_0} + \delta\right)^2 - p_x^2 - p_y^2 - \frac{1}{\beta_0^2 \gamma_0^2}}} \tag{57}$$

For the special case $p_x = p_y = 0$, and using:

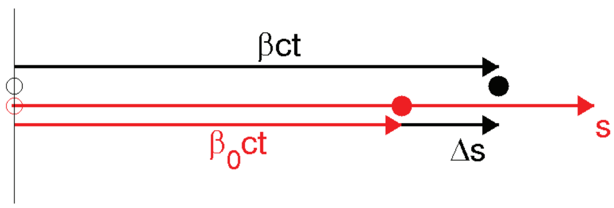
$$\frac{1}{\beta_0} + \delta = \frac{E}{P_0 c} = \frac{\gamma}{\gamma_0 \beta_0} \tag{58}$$

we find:

$$\frac{dz}{ds} = \frac{1}{\beta_0} - \frac{1}{\beta} \tag{59}$$

Therefore:

$$\frac{d}{ds}\beta z = \frac{\beta}{\beta_0} - 1 \tag{60}$$



Now consider two particles moving along the reference trajectory; one (the reference particle), with speed $\beta_0 c$, and the other with speed βc. The rate of change of distance Δs between them is:

$$\frac{d}{ds}\Delta s = \frac{\beta ct - \beta_0 ct}{\beta_0 ct} = \frac{\beta}{\beta_0} - 1 \tag{61}$$

Comparing (60) and (61), we see that in a field-free region, for a particle moving along the reference trajectory, the rate of change of βz is equal to the rate of change of the distance of the particle from the reference particle. We can think of βz as the distance that the particle is *ahead* of the reference particle.

Staying in a field-free region, from the Hamiltonian (56) we use Hamilton's equations:

$$\frac{dx}{ds} = \frac{\partial H}{\partial p_x} \qquad \frac{dy}{ds} = \frac{\partial H}{\partial p_y} \tag{62}$$

to find:

$$p_x = D\frac{x'}{\sqrt{1+x'^2+y'^2}} \approx x' \tag{63}$$

and:

$$p_y = D\frac{y'}{\sqrt{1+x'^2+y'^2}} \approx y' \tag{64}$$

where the prime indicates the derivative with respect to the path length s,

$$D = \sqrt{1+\frac{2\delta}{\beta_0}+\delta^2} \tag{65}$$

and the approximations hold for $x'^2+y'^2 \ll 1$, and $\delta \ll 1$.

Finally, let us consider the evolution of the dynamical variables in a drift space (field-free region) of length L. The Hamiltonian is (56):

$$H = \frac{\delta}{\beta_0} - \sqrt{\left(\frac{1}{\beta_0}+\delta\right)^2 - p_x^2 - p_y^2 - \frac{1}{\beta_0^2\gamma_0^2}} \tag{66}$$

Since there is no dependence on the coordinates, the momenta are constant:

$$\Delta p_x = 0, \qquad \Delta p_y = 0, \qquad \Delta\delta = 0 \tag{67}$$

The transverse coordinates change as follows:

$$\frac{\Delta x}{L} = \frac{p_x}{\sqrt{\left(\frac{1}{\beta_0}+\delta\right)^2 - p_x^2 - p_y^2 - \frac{1}{\beta_0^2\gamma_0^2}}} \qquad (68)$$

$$\frac{\Delta y}{L} = \frac{p_y}{\sqrt{\left(\frac{1}{\beta_0}+\delta\right)^2 - p_x^2 - p_y^2 - \frac{1}{\beta_0^2\gamma_0^2}}} \qquad (69)$$

From (57), we have the change in the longitudinal coordinate:

$$\frac{\Delta z}{L} = \frac{1}{\beta_0} - \frac{\frac{1}{\beta_0}+\delta}{\sqrt{\left(\frac{1}{\beta_0}+\delta\right)^2 - p_x^2 - p_y^2 - \frac{1}{\beta_0^2\gamma_0^2}}} \qquad (70)$$

Equations (67), (68), (69) and (70) constitute the dynamical map for a drift space: they tell us how to calculate the values of the dynamical variables at the exit of the drift space, given the values at the entrance.

Note that the map is nonlinear: the changes in the variables have a nonlinear dependence on the initial values of the variables.

However, we can make Taylor expansions for the changes in the coordinates, (68), (69) and (70). For small values of the canonical momenta, first-order expansions provide reasonable accuracy for most applications. We can then write the *transfer map* as a matrix...

For a drift space, we can write:

$$\vec{x}\,(s = L) \approx R \cdot \vec{x}\,(s = 0) \tag{71}$$

where:

$$\vec{x} = \begin{pmatrix} x \\ p_x \\ y \\ p_y \\ z \\ \delta \end{pmatrix} \qquad R = \begin{pmatrix} 1 & L & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & L & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \frac{L}{\beta_0^2\gamma_0^2} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{72}$$

The above approximation is valid only for $\delta \ll 1$, $p_x \ll 1$, $p_y \ll 1$, and $\gamma_0 \gg 1$.

We obtained this map by making a linear approximation to the exact solutions to the equations of motion for the Hamiltonian (66). There is a danger that we lost symplecticity by this approach (in fact, in this case we are safe); there is an alternative method...

To obtain the linear map (72) for a drift space, we solved the equations of motion for the exact Hamiltonian, then made a linear approximation to the solutions.

Alternatively, we can expand the Hamiltonian to second order in the dynamical variables, then solve the new Hamiltonian exactly, to get a linear map. In other words, we approximate the Hamiltonian, rather than the equations of motion.

The advantage of this approach is that the solution is guaranteed to be symplectic. If we have to make an approximation somewhere, we would rather have an approximate map that is symplectic, than an approximate map that is not symplectic.

Expanding the Hamiltonian (66) to second order in the dynamical variables (and dropping constant terms that make no contribution to the equations of motion), we construct the Hamiltonian:

$$H_2 = \frac{1}{2}p_x^2 + \frac{1}{2}p_y^2 + \frac{1}{2}\frac{\delta^2}{\beta_0^2\gamma_0^2} \tag{73}$$

This is much simpler than Hamiltonians we have recently looked at! Solving the equations of motion is very easy, and we find once again that the transfer matrix for a drift of length L is given by:

$$R = \begin{pmatrix} 1 & L & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & L & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \frac{L}{\beta_0^2\gamma_0^2} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{74}$$

For describing particle motion in high-energy accelerators, we use a relativistic Hamiltonian, and momentum variables normalised to a *reference momentum*. The beamline is generally designed for particles with momenta close to the reference momentum.

The fields (and hence the Hamiltonian) change continually along the beamline. This means it is more convenient to work with the *path length* as the independent variable, rather than the time.

We can construct *linear maps* for accelerator components by expanding the appropriate relativistic Hamiltonian to second order in the dynamical variables. The advantage of this approach is that the map that we produce is guaranteed to be symplectic; but for the expansion to be valid, the values of the dynamical variables must be small.