## Linear Dynamics, Lecture 10

# Effects of Linear Imperfections

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## Course Outline

Part II (Lectures 6 – 10): Description of beam dynamics using optical lattice functions.

6. Linear optics in periodic, uncoupled beamlines
7. Including longitudinal dynamics
8. Bunches of many particles
9. Coupled optics
10. Effects of linear imperfections

In this course, we have learned how to track relativistic particles through electromagnetic fields, and derived symplectic transfer matrices to describe the linear dynamics. We have learned how the linear dynamics in complicated configurations of electromagnetic fields (that occur, for example, in an accelerator beamline) can be described using the "lattice functions" of beam optics.

So far, we have assumed that the lattice is perfect: in other words, we have learned how to analyse and describe and accelerator that exists as a design on a computer. When an accelerator gets built, things are never exactly the way the designer intended them to be: fields have the wrong strength, and magnets are in the wrong position.





Understanding and dealing with machine errors is a significant challenge in modern accelerators. The machines may be very large, involving hundreds or thousands of magnets, each one of which can be slightly wrong. Achieving the intended performance levels requires errors to be reduced to levels where they have no significant effect on the machine: and modern machines can be *very* sensitive to errors. For example, in the International Linear Collider, movement of some magnets in the beam delivery system by a few nanometers can have a severe impact on the machine performance.

A comprehensive survey of machine errors and how to deal with them requires a course in itself. In many ways we are still learning the subject: people are actively working on new ways to locate the sources of errors and to compensate for them.

In this lecture, I shall present a brief introduction to some of the more important errors and their effects, focusing on storage rings. Specifically, I shall look at:

- steering errors, which lead to distortion of the closed orbit;
- focusing errors, which lead to changes in the tunes and the lattice functions (Twiss parameters);
- coupling errors, which lead to beam coupling.

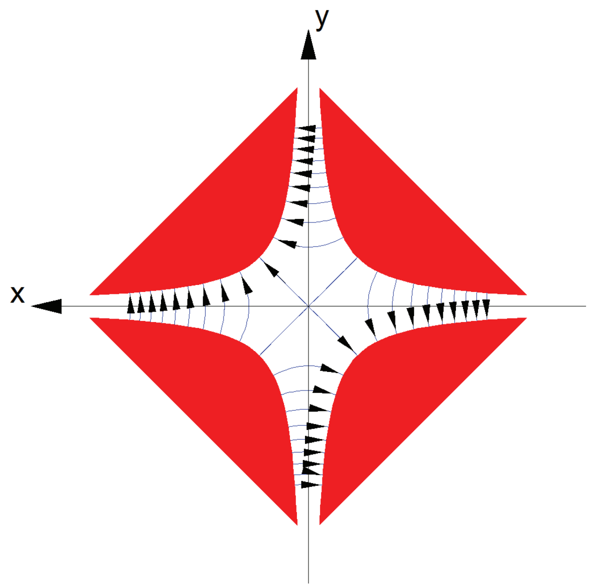


## Steering Errors

The reference trajectory is usually defined in such a way that a particle with the reference momentum initially on the reference trajectory will continue to travel on the reference trajectory. The steering effects of dipoles can be accounted for by having a curved reference trajectory.

If there is a dipole field error, then a particle following the reference trajectory will be steered away from the reference trajectory when it reaches the region of the dipole field error.

Dipole field errors can come from dipoles that don't have exactly the correct strength or have some rotation about the reference trajectory; or from quadrupoles that don't have the correct horizontal or vertical alignment.




With respect to the center of the quadrupole, the field is:

$$B_x = b_2 \frac{y}{r_0}, \qquad B_y = b_2 \frac{x}{r_0}, \qquad B_z = 0 \tag{1}$$





If the axis of the quadrupole is displaced vertically a distance $\Delta y$ from the reference trajectory, then the field *with respect to the reference trajectory* can be found by making the substitution:

$$y \to y - \Delta y \tag{2}$$

in equation (1) to obtain:

$$B_x = b_2 \frac{y}{r_0} - b_2 \frac{\Delta y}{r_0}, \qquad B_y = b_2 \frac{x}{r_0}, \qquad B_z = 0 \tag{3}$$

The second term in the expression for $B_x$ in equation (3) is independent of the coordinates: it appears as a dipole field superposed on the quadrupole field.

A storage ring is designed so that the reference trajectory closes on itself after one complete turn. If there is a dipole error in the storage ring, then a particle initially following the reference trajectory is deflected off the reference trajectory by the error. However, there may still be a *closed orbit*, which is defined as the trajectory of a particle that closes on itself after one turn.

The closed orbit must exist in an operating storage ring, otherwise a beam cannot be stored. However, mathematically, the closed orbit does not always have to exist. In fact, one of the first tasks when commissioning a newly-built storage ring is to tune the machine into a state where the closed orbit *does* exist.



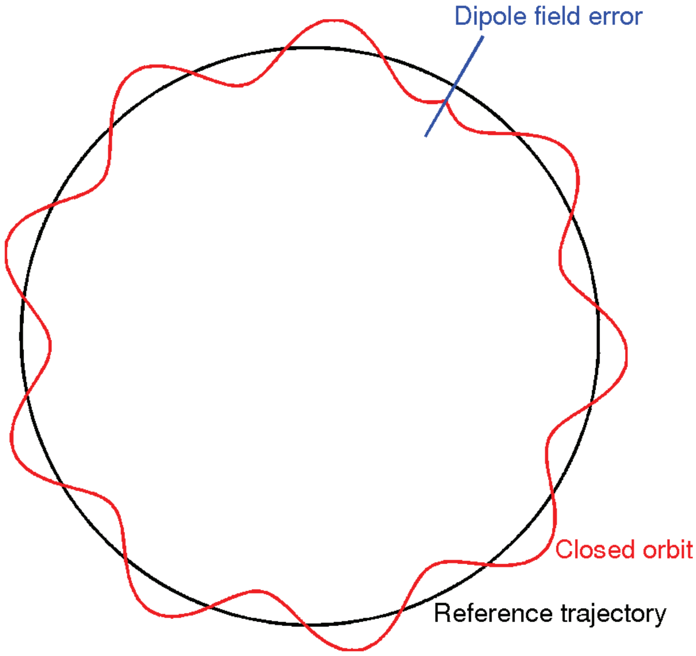


Consider a storage ring with a single dipole error at the location $s = 0$ (as usual, the independent variable $s$ measures the distance along the reference trajectory). Let us see if we can find an expression for the closed orbit: by doing this, we also find the conditions necessary for the closed orbit to exist.

Let us suppose that the dipole field error can be represented by a short horizontal field, of integrated strength $B_x \Delta s$. Then, the change in vertical momentum of a particle moving through the field is:

$$\Delta p_y = \frac{q}{P_0} B_x \Delta s = \Delta \theta_y \tag{4}$$

where $q$ is the charge of the particle, and $P_0$ is the reference momentum.




A dipole field error causes a distortion of the closed orbit. There is a "kink" in the closed orbit at the location of the dipole field error.





Let us suppose that there is no coupling in the storage ring. Then, because the field error is a horizontal dipole field, it has no effect on the horizontal trajectory of a particle: the effects are only in the vertical degree of freedom.

Now recall that the trajectory of a particle moving through an accelerator can be written in terms of the lattice functions and the action-angle variables:

$$y = \sqrt{2\beta_y J_y}\cos\phi_y \quad (5)$$

$$p_y = \sqrt{\frac{2J_y}{\beta_y}}(\sin\phi_y + \alpha_y\cos\phi_y) \quad (6)$$

Let us try to find a trajectory that closes on itself after one turn, starting from just *after* the dipole field error. We shall express this trajectory in terms of some initial (and conserved) action $J_{y0}$, and initial angle $\phi_{y0}$.

If the total phase advance over one turn of the storage ring is $\mu_y$, then the conditions for the trajectory of a particle to close on itself can be written, from (5):

$$\sqrt{2\beta_y J_{y0}}\cos(\phi_{y0} + \mu_y) = \sqrt{2\beta_y J_{y0}}\cos(\phi_{y0}) \tag{7}$$

and from (6):

$$\sqrt{\frac{2J_{y0}}{\beta_y}}(\sin(\phi_{y0} + \mu_y) + \alpha_y \cos(\phi_{y0} + \mu_y)) + \Delta\theta_y = \sqrt{\frac{2J_{y0}}{\beta_y}}(\sin\phi_{y0} + \alpha_y \cos\phi_{y0}) \tag{8}$$

where the Twiss parameters are to be evaluated at $s = 0$. Equations (7) and (8) may be solved to give:

$$J_{y0} = \frac{\beta_y \Delta\theta^2}{8\sin^2 \pi\nu_y}, \qquad \phi_{y0} = \pi\nu_y \tag{9}$$

where $\nu_y = \mu_y/2\pi$ is the vertical tune.





Consider the expression from (9) for the action of the closed orbit:

$$J_{y0} = \frac{\beta_y \Delta\theta_y^2}{8\sin^2 \pi\nu_y} \tag{10}$$

We immediately see that if $\nu_y = 0$, in other words if the tune is on the integer resonance, then the denominator vanishes, and there is no solution for the action. This is the condition for the closed orbit to exist: the tune of the machine must be off the integer resonance.

Even if the lattice is close to the integer resonance, though not exactly on the resonance, then the factor $1/\sin^2 \pi\nu_y$ makes the closed orbit very sensitive to dipole field errors: even a small error can lead to a large closed orbit distortion. This can make practical operation of the storage ring very difficult, because alignment tolerances on the quadrupoles become very demanding.

The other point to notice in the expression for the action of the closed orbit (9):

$$J_{y0} = \frac{\beta_y \Delta\theta_y^2}{8\sin^2 \pi\nu_y} \qquad (11)$$

is the dependence on the beta function. The larger the beta function at the location of the dipole field error, the larger the closed orbit distortion for a given size of field error. This is of great practical significance: the larger the beta function, the more sensitive the beam is to dipole field errors (or similar effects causing deflections). Generally, lattice designers try to keep beta functions low, a few 10's of metres. In some cases, very large beta functions (even 10's of kilometres) may be unavoidable, for example in the final focus systems of colliders. With large beta functions, sensitivity to errors is very much a concern.





Usually, in a storage ring, the closed orbit at a number of locations around the ring is measured with beam position monitors (BPMs). The corresponding momentum is not usually observable.

Using equations (9) we can write the vertical coordinate of the closed orbit resulting from a single dipole field error at $s = 0$:

$$y_{\text{co}}(s) = \frac{\sqrt{\beta_y(0)\beta_y(s)}}{2\sin \pi\nu_y} \Delta\theta_y \cos(\pi\nu_y + \mu_y(s)) \qquad (12)$$

where $\mu_y(s)$ is the phase advance to a point $s$ from $s = 0$.

Note that the closed orbit at any point depends on the beta function at that point, as well as the beta function at the location of the dipole field error.

Usually, of course, there is not just one error in a storage ring, but a large number of errors distributed around the ring. If the orbit distortion is not too large, then we may simply use linear superposition to add together the effects from all the various errors. For example, for a horizontal dipole field error $B_x(s)$ which is a function of position, the closed orbit distortion is:

$$y_{\text{co}}(s) = \int_0^{C_0} \frac{\sqrt{\beta_y(s')\beta_y(s)}}{2\sin\pi\nu_y} \frac{q}{P_0} B_x(s') \cos(\pi\nu_y + \mu_y(s', s)) ds' \quad (13)$$

where $C_0$ is the length of the reference trajectory, and $\mu_y(s', s)$ is the phase advance from point $s'$ to point $s$ along the reference trajectory.





When coupling is present, complicated effects can occur from dipole field errors. For example, consider introducing a vertical dipole field error at a location where the reference trajectory is curved: for example, we might adjust the power supply on a main bending magnet. This leads to a change in the horizontal closed orbit; but because of the curvature of the reference trajectory, this then leads to a change in the total path length around the ring. If the RF frequency is kept fixed, then the energy of the particles in the beam must change to maintain synchronisation with the RF field in the cavities. But the change in energy leads to an *additional* change in the closed orbit, because of the dispersion in the ring.

Despite the complexity of the response of the closed orbit to steering errors in a coupled storage ring, there is in fact an easy way to compute the closed orbit in the presence of a *single* dipole error. This follows from writing the closed orbit condition:

$$R \cdot \vec{x}_{\mathrm{co}} + \Delta\vec{\theta} = \vec{x}_{\mathrm{co}} \tag{14}$$

where $R$ is the single-turn matrix from the point immediately following the (short) region of the field error, and $\Delta\vec{\theta}$ describes the deflection from the field error (which can be horizontal, vertical, or a change in energy deviation).

From equation (15) we find:

$$\vec{x}_{\mathrm{co}} = (I - R)^{-1} \cdot \Delta\vec{\theta} \tag{15}$$

where $I$ is the identity matrix. The condition for the closed orbit to exist is that the matrix $(I - R)$ is invertible.





Note that equation (15):

$$\vec{x}_{\mathrm{co}} = (I - R)^{-1} \cdot \Delta\vec{\theta}$$

gives the closed orbit at a single location in the ring, which is the location of the field error. To propagate the closed orbit around the ring, one needs to use the transfer matrices from the location of the error to other locations of interest (for example, the locations of the BPMs).

Although equation (15) takes a nice simple form, applying it to find the closed orbit in a specific case does require the inversion of a $6 \times 6$ matrix. Also, looked at in this way, we don't immediately see how the tunes and the lattice functions affect the sensitivity of the closed orbit to dipole field errors.

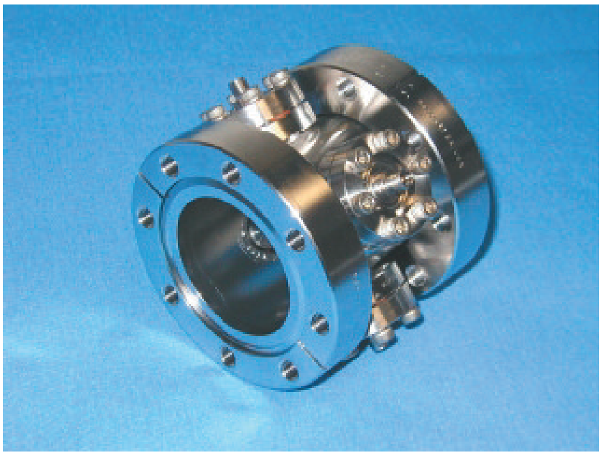
Control of the closed orbit in a storage ring is fundamental to its effective operation. It is necessary in a light source for directing the synchrotron radiation accurately down the beamlines to the synchrotron light users; and in a collider for making sure that the beams collide. Orbit control is also the first step in correcting focusing and coupling errors, which we shall discuss shortly.





*Steering magnet from SPARC*

Generally, one controls the orbit in a beamline (storage ring or linear beamline) using a set of small dipole magnets placed at intervals along the beamline. These *steering magnets* or *orbit correctors* allow the operators to apply controlled "dipole field errors" to compensate the errors that are not controlled (those arising from misalignments of quadrupoles, or faults in dipole magnet power supplies).



*Four-button BPM*

In addition to the orbit correctors, the other essential instruments in orbit control are beam position monitors (BPMs) that measure the transverse position of the closed orbit at various locations around the ring. One can construct an *orbit response matrix*, $M$; the component $M_{ij}$ of the orbit response matrix is the change in reading on the $i$th BPM, in response to a given change in the strength of the $j$th orbit corrector.





If $\Delta\vec{c}$ is a vector constructed from the change in orbit corrector strengths, and $\Delta\vec{x}$ is the corresponding change in BPM readings, then:

$$\Delta\vec{x} = M \cdot \Delta\vec{c} \qquad (16)$$

where $M$ is the orbit response matrix. $M$ may be calculated from a model of the machine, or it may be measured by adjusting the strength of each corrector magnet in turn, and measuring the resulting change in closed orbit.

Determining the changes in corrector strengths necessary to achieve a desired set of readings on the BPMs involves inverting the orbit response matrix $M$. Generally, the number of BPMs is different from the number of corrector magnets, so $M$ is not a square matrix. However, it can still be inverted, using clever algorithms such as Singular Value Decomposition.

Another common field error in a storage ring is an error in the strength of a quadrupole magnet. Assuming there are no closed orbit distortions, and the reference trajectory passes through the center of the quadrupole, then changing the strength of a quadrupole does not change the closed orbit. However, changing the strength of a quadrupole does result in a *focusing error* that changes the phase advances (tunes) and the lattice functions.





Note that focusing errors can also come from horizontal misalignments of sextupole magnets. Recall that the field in a normal sextupole with axis on the reference trajectory can be written:

$$B_x = 2b_3 \frac{xy}{r_0^2}, \qquad B_y = b_3 \frac{(x^2 - y^2)}{r_0^2}, \qquad B_z = 0 \tag{17}$$

A horizontal misalignment of the sextupole by distance $\Delta x$ can be represented by the transformation:

$$x \to x - \Delta x \tag{18}$$

Under the transformation (18):

$$x \to x - \Delta x \tag{19}$$

the field in the sextupole becomes:

$$B_x = 2b_3 \frac{xy}{r_0^2} - 2b_3 \frac{\Delta x}{r_0} \frac{y}{r_0} \tag{20}$$

$$B_y = b_3 \frac{(x^2 - y^2)}{r_0^2} - 2b_3 \frac{\Delta x}{r_0} \frac{x}{r_0} + b_3 \frac{\Delta x^2}{r_0^2} \tag{21}$$

$$B_z = 0 \tag{22}$$

The terms linear in $\Delta x$ result in an effective quadrupole field, of strength $b_2 = -2b_3 \Delta x / r_0$, superposed on the sextupole field. There is also a dipole term, which will lead to a steering error, but this is second order in the misalignment $\Delta x$, which we hope is small.





Let us consider the effects of a focusing error in an uncoupled storage ring. We write the single-turn transfer matrix for the horizontal motion at the location of the quadrupole (without the focusing error) in the usual form:

$$R = \begin{pmatrix} \cos\mu_x + \alpha_x \sin\mu_x & \beta_x \sin\mu_x \\ -\gamma \sin\mu_x & \cos\mu_x - \alpha_x \sin\mu_x \end{pmatrix} \tag{23}$$

where $\mu_x = 2\pi\nu_x$ is the total phase advance, and $\nu_x$ is the tune. We represent the focusing error as:

$$R_{\text{err}} = \begin{pmatrix} 1 & 0 \\ -\Delta K & 1 \end{pmatrix} \tag{24}$$

Note that a positive value for $\Delta K$ represents an *increase* in strength for a horizontally focusing magnet.

The single-turn transfer matrix including the focusing error, $\tilde{R}$, is obtained simply by multiplying the original transfer matrix $R$ with the matrix $R_{\text{err}}$ representing the error:

$$\tilde{R} = R \cdot R_{\text{err}} \tag{25}$$

We can also express $\tilde{R}$ in terms of the new phase advance $\tilde{\mu}_x$ and lattice functions:

$$\tilde{R} = \begin{pmatrix} \cos\tilde{\mu}_x + \tilde{\alpha}_x \sin\tilde{\mu}_x & \tilde{\beta}_x \sin\tilde{\mu}_x \\ -\tilde{\gamma}\sin\tilde{\mu}_x & \cos\tilde{\mu}_x - \tilde{\alpha}_x \sin\tilde{\mu}_x \end{pmatrix} \tag{26}$$

If we equate the right-hand sides of equations (25) and (26), and make some approximations for small $\Delta K$, we find:

$$\Delta\nu_x \approx \frac{\Delta K \beta_x}{4\pi}, \qquad \tilde{\beta}_x \approx \frac{\beta_x}{1 + \frac{1}{2}\Delta K \beta_x \cot\mu_x} \tag{27}$$

where $\Delta\nu_x$ is the change in the tune as a result of the focusing error.





The change in the tune resulting from a focusing error is (27):

$$\Delta\nu_x \approx \frac{\Delta K \beta_x}{4\pi} \tag{28}$$

Note that there is again a dependence on the beta function: the larger the beta function at the location of the error, the greater the change in tune that results from the focusing error.

Note also that for *positive* $\Delta K$, i.e. an *increase* in focusing strength, the change in the tune is also positive. An increase in focusing strength at one point in the lattice leads to an overall increase in phase advance around the storage ring.

The change in the beta function resulting from a focusing error is (27):

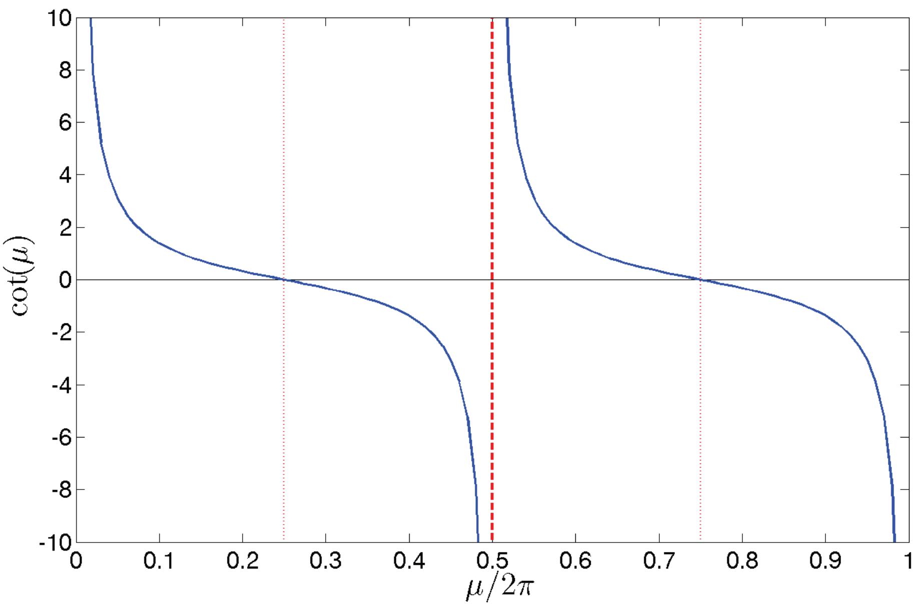
$$\tilde{\beta}_x \approx \frac{\beta_x}{1 + \frac{1}{2}\Delta K \beta_x \cot \mu_x} \tag{29}$$

Whether the beta function increases or decreases for $\Delta K > 0$ depends on the sign of $\cot \mu_x$, i.e. on the value of the tune.

As usual, the sensitivity to the error depends on the value of the beta function at the location of the error: the larger the beta function, the greater the sensitivity. However, the sensitivity also depends on the tune: if the fractional part of the tune is close to 0 or 0.5, then the beta functions become very sensitive to focusing errors. This is consistent with our observation in the previous lecture, that the dynamics become unstable on the integer and half-integer resonances.








$$\tilde{\beta}_x \approx \frac{\beta_x}{1 + \frac{1}{2}\Delta K \beta_x \cot \mu_x} \tag{30}$$

Unwanted skew quadrupole or solenoid fields in a beamline lead to coupling errors. Skew quadrupole fields can arise from the tilt of normal quadrupoles about the beam trajectory, or from vertical misalignments of sextupoles. We won't go through this case in detail, but simply note that the expression for the tune shift from the uncoupled case (28) generalises for the coupled case. In particular, if the focusing error is represented by the transfer matrix:

$$R_{\text{err}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -\Delta K_{11} & 1 & -\Delta K_{13} & 0 & -\Delta K_{15} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -\Delta K_{31} & 0 & -\Delta K_{33} & 1 & -\Delta K_{35} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -\Delta K_{51} & 0 & -\Delta K_{53} & 0 & -\Delta K_{55} & 1 \end{pmatrix} \tag{31}$$

then the tune shifts resulting from the focusing error can be written:

$$\Delta\nu_k \approx \frac{1}{4\pi} \sum_{i,j=1,3,5} \beta_{ij}^k \Delta K_{ij} \tag{32}$$





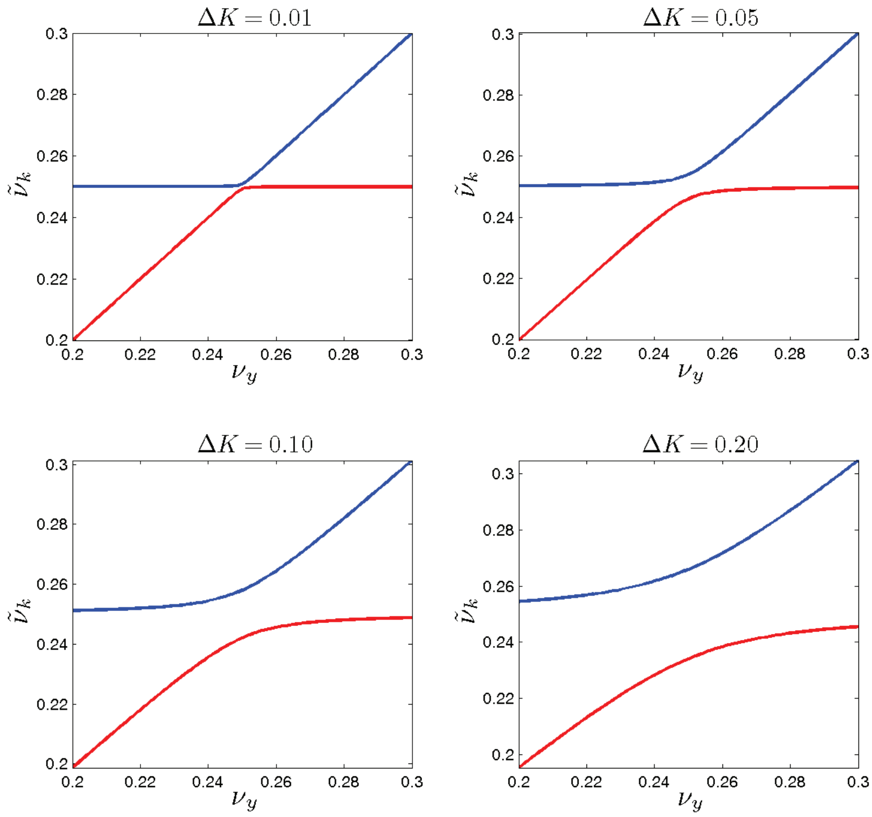
Some interesting features of coupling emerge if we compute the tunes from the eigenvalues of the transfer matrix in the presence of a coupling error. Consider the transverse motion in a storage ring. Initially, the ring is uncoupled and free of errors: let the betatron tunes in this case be $\nu_x$ and $\nu_y$. In the presence of a coupling error, we can calculate the "perturbed" tunes $\tilde{\nu}_{\text{I}}$ and $\tilde{\nu}_{\text{II}}$ from the eigenvalues of the matrix $\tilde{R}$:

$$\tilde{R} = R(\nu_x, \nu_y) \cdot R_{\text{err}} \tag{33}$$

where the matrix $R_{\text{err}}$ represents a skew quadrupole field error:

$$R_{\text{err}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\Delta K & 0 \\ 0 & 0 & 1 & 0 \\ -\Delta K & 0 & 0 & 1 \end{pmatrix} \tag{34}$$

By adjusting quadrupole strengths we can adjust the tunes. Let us suppose we keep $\nu_x$ fixed and vary $\nu_y$. Then we plot the "perturbed" tunes $\tilde{\nu}_{\text{I}}$ and $\tilde{\nu}_{\text{II}}$ as a function of $\nu_y$ for a fixed value of $\Delta K$...








It can be shown that the distance between the "perturbed" tunes $|\tilde{\nu}_{\rm I} - \tilde{\nu}_{\rm II}|$ at $\nu_y = \nu_x$ ("on the coupling resonance") is given by:

$$|\tilde{\nu}_{\rm I} - \tilde{\nu}_{\rm II}|_{\nu_y=\nu_x} \approx \frac{\sqrt{\beta_x \beta_y}}{2\pi} \Delta K \qquad (35)$$

In a storage ring, we can control the tunes by adjusting the quadrupole strengths. We can also measure the tunes $\tilde{\nu}_{\rm I}$ and $\tilde{\nu}_{\rm II}$ by resonant excitation of the beam at the betatron frequencies. Thus, we can measure the closest approach of the tunes in a real machine: this allows us to characterise the coupling errors.

Dipole field errors result in distortion of the closed orbit. Dipole field errors can be caused by variation in the field strength in dipoles, and by transverse misalignments of quadrupoles.

Focusing errors from normal quadrupoles lead to changes in the betatron tunes, and in variation of the beta functions. Focusing errors can be caused by variations of the field strength in quadrupoles, and by horizontal misalignments of sextupoles.

Skew quadrupole field errors lead to beam coupling. Skew quadrupole field errors can be caused by tilts of normal quadrupoles around the magnetic axis, and by vertical misalignments of sextupoles.

In all cases, the beta function at the location of the error is an important quantity: the larger the beta function, the greater the sensitivity to the error.