Revision of Electromagnetic Theory

Lecture 2

Waves in Free Space
Waves in Materials
Waves in Bounded Regions
Generation of Electromagnetic Waves

Andy Wolski

University of Liverpool, and the Cockcroft Institute

Maxwell's Equations

In the previous lecture, we discussed the physical interpretation of Maxwell's equations:

$$\nabla \cdot \vec{D} = \rho, \tag{1}$$

$$\nabla \cdot \vec{B} = 0, \tag{2}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \tag{3}$$

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}, \tag{4}$$

and looked at some applications in accelerators, involving static (time independent) fields.

We also saw how the fields could be expressed in terms of the potentials:

$$\vec{B} = \nabla \times \vec{A}, \tag{5}$$

$$\vec{E} = -\nabla\varphi - \frac{\partial \vec{A}}{\partial t}. \tag{6}$$

Electromagnetism in Accelerators

In this lecture, we shall:

- show that Maxwell's equations lead to wave equations for the electric and magnetic fields, and explore the solutions to these equations in free space;
- discuss the behaviour of electromagnetic waves in conducting materials;
- derive the solutions to the wave equations in regions bounded by conductors;
- consider some applications in accelerators, in particular in waveguides and rf cavities;
- discuss (briefly) the *generation* of electromagnetic waves.

Electromagnetic Waves in Free Space

In free space, Maxwell's equations (3) and (4) take the form:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \tag{7}$$

$$\nabla \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}, \tag{8}$$

where μ_0 and ε_0 are respectively the permeability and permittivity of free space (fundamental physical constants).

If we take the curl of (7), then, using a vector identity, we find:

$$\nabla \times \nabla \times \vec{E} \equiv \nabla \left(\nabla \cdot \vec{E} \right) - \nabla^2 \vec{E} = -\frac{\partial}{\partial t} \nabla \times \vec{B}. \tag{9}$$

In free space, we have $\nabla \cdot \vec{E} = 0$; then, substituting for $\nabla \times \vec{B}$ from (8) we obtain the wave equation:

$$\nabla^2 \vec{E} - \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} = 0. \tag{10}$$

A solution to the wave equation (10) can be written:

$$\vec{E}(\vec{r},t) = \vec{E}_0 e^{i\left(\vec{k}\cdot\vec{r}-\omega t\right)}, \tag{11}$$

where $\vec{E}_0$ and $\vec{k}$ are constant vectors, ω is a constant, and $\vec{r} = (x, y, z)$ is the position vector. Substituting this solution into the wave equation yields the relationship between ω and $\vec{k}$:

$$\frac{\omega}{\left|\vec{k}\right|} = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = c. \tag{12}$$

Equation (12) gives a relationship between the wavelength $\lambda = 2\pi/\left|\vec{k}\right|$ and frequency $f = \omega/2\pi$ of the wave; such an equation is known as a *dispersion relation*.

Since we obtained the wave equation by taking additional derivatives of Maxwell's equations, we should check that the solution (11) satisfies the original equations. In this case, we find that Maxwell's equation (1) (in free space):

$$\nabla \cdot \vec{E} = 0, \tag{13}$$

gives a relation between the amplitude vector $\vec{E}_0$ and the wave vector $\vec{k}$:

$$\vec{k} \cdot \vec{E}_0 = 0. \tag{14}$$

Since $\vec{E}_0$ gives the direction of the electric field at all points in space, and $\vec{k}$ gives the direction of propogation of the wave, equation (14) tells us that plane electric waves in free space are *transverse* waves.

We can also derive a wave equation for the magnetic field, starting from Maxwell's equation (8) (in free space):

$$\nabla \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}.$$

Taking the curl, as before, and using a vector identity gives:

$$\nabla \times \nabla \times \vec{B} \equiv \nabla \left(\nabla \cdot \vec{B} \right) - \nabla^2 \vec{B} = \mu_0 \varepsilon_0 \frac{\partial}{\partial t} \nabla \times \vec{E}. \qquad (15)$$

Using $\nabla \cdot \vec{B} = 0$, and substituting for $\nabla \times \vec{E}$ from Maxwell's equation (3) gives:

$$\nabla^2 \vec{B} - \mu_0 \varepsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2} = 0, \qquad (16)$$

which has the solution:

$$\vec{B}(\vec{r}, t) = \vec{B}_0 e^{i\left(\vec{k} \cdot \vec{r} - \omega t\right)}, \qquad (17)$$

As before, we have to check that our solution for the magnetic wave equation satisfies Maxwell's equations. In this case, we find from $\nabla \cdot \vec{B} = 0$ that the wave is once again a transverse wave:

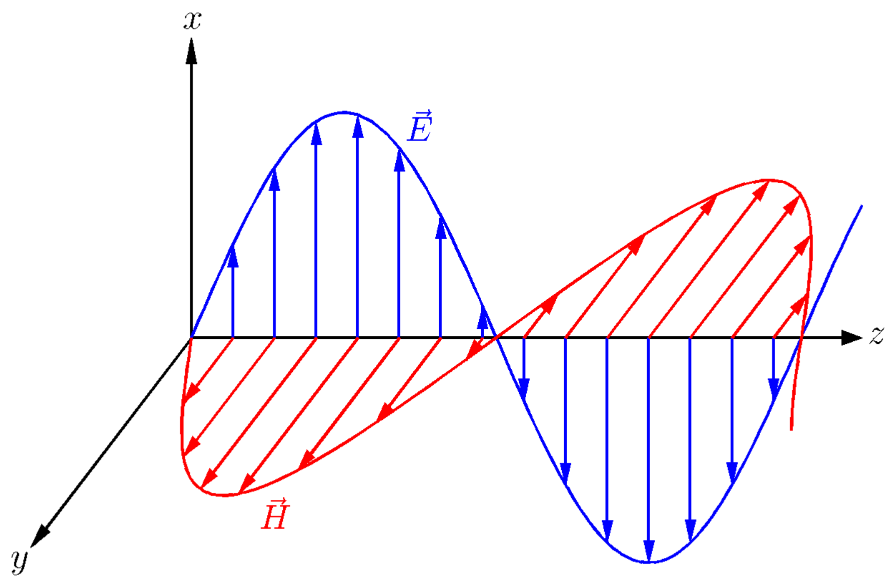
$$\vec{k} \cdot \vec{B}_0 = 0. \qquad (18)$$

We also have to verify that our solutions for the electric and magnetic fields *simultaneously* satisfy Maxwell's equations. We find:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \Rightarrow \quad \vec{k} \times \vec{E}_0 = \omega \vec{B}_0, \qquad (19)$$

$$\nabla \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \quad \Rightarrow \quad \vec{k} \times \vec{B}_0 = \frac{\omega}{c^2} \vec{E}_0. \qquad (20)$$

These results tell us that the electric and magnetic fields and the wave vector must be mutually perpendicular...



We also find the relationship between the amplitudes of the electric and magnetic fields:

$$\frac{\left|\vec{E}_0\right|}{\left|\vec{B}_0\right|} = \frac{\omega}{\left|\vec{k}\right|} = c. \tag{21}$$

In terms of the magnetic intensity $\vec{H}_0 = \vec{B}_0/\mu_0$:

$$\frac{\left|\vec{E}_0\right|}{\left|\vec{H}_0\right|} = \mu_0 c = \sqrt{\frac{\mu_0}{\varepsilon_0}}, \tag{22}$$

where we have used:

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}. \tag{23}$$

The ratio between the amplitudes of $\vec{E}$ and $\vec{H}$ in a plane wave in free space defines the *impedance of free space*, Z_0:

$$Z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} \approx 376.730\,\Omega. \tag{24}$$

Electromagnetic waves in dielectrics generally behave in a similar way to electromagnetic waves in free space, except that the phase velocity v_ϕ is modified by the relative permeability and relative permittivity:

$$v_\phi = \frac{1}{\sqrt{\mu\varepsilon}}, \tag{25}$$

where $\mu = \mu_r \mu_0$ and $\varepsilon = \varepsilon_r \varepsilon_0$.

The *refractive index* n is defined by:

$$n = \frac{c}{v_\phi} = \sqrt{\mu_r \varepsilon_r}. \tag{26}$$

Generally, $v_\phi < c$, so $n > 1$.

At certain frequencies, resonance can occur between the electric field in the wave, and the oscillation of electrons in the atoms and molecules of the dielectric. In such regimes, the refractive index can become a complex function of the frequency.

Electromagnetic waves in conductors do behave differently in some important respects to waves in free space and in dielectrics. The differences arise from the fact that there will be electric currents in the conductor driven by the electric field.

For an ohmic conductor, the current density is proportional to the electric field:

$$\vec{J} = \sigma \vec{E}. \tag{27}$$

Maxwell's equation (4) then becomes:

$$\nabla \times \vec{B} = \mu\sigma\vec{E} + \mu\varepsilon\frac{\partial \vec{E}}{\partial t}. \tag{28}$$

Proceeding as before, we take the curl, and then substitute for $\nabla \times \vec{E}$ from (3), to find:

$$\nabla^2 \vec{B} - \mu\sigma\frac{\partial \vec{B}}{\partial t} - \mu\varepsilon\frac{\partial^2 \vec{B}}{\partial t^2} = 0. \tag{29}$$

Similarly, starting from (3), we find:

$$\nabla^2 \vec{E} - \mu\sigma \frac{\partial \vec{E}}{\partial t} - \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} = 0. \tag{30}$$

Note the presence of the first derivative with respect to time. This represents a "damping" term, that will lead to decay of the wave amplitude. The solutions to the wave equations (29) and (30) can be written:

$$\vec{B} = \vec{B}_0 e^{-\vec{\beta}\cdot\vec{r}} e^{i(\vec{\alpha}\cdot\vec{r}-\omega t)}, \tag{31}$$

$$\vec{E} = \vec{E}_0 e^{-\vec{\beta}\cdot\vec{r}} e^{i(\vec{\alpha}\cdot\vec{r}-\omega t)}. \tag{32}$$

Substituting this solution into the wave equation gives the dispersion relations:

$$2\vec{\alpha}\cdot\vec{\beta} = \mu\sigma\omega, \tag{33}$$

$$\vec{\alpha}^2 + \vec{\beta}^2 = \mu\varepsilon\omega^2. \tag{34}$$

The dispersion relations are more complicated in a conductor than in free space. In a conductor, the phase velocity is a function of the frequency:

$$\frac{|\vec{\alpha}|}{\omega} = \sqrt{\mu\varepsilon}\left(\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{\sigma^2}{\omega^2\varepsilon^2}}\right)^{\frac{1}{2}}. \tag{35}$$

We can define a "good conductor" as one for which:

$$\sigma \gg \omega\varepsilon. \tag{36}$$

(Note that this condition depends on the frequency of the wave). Then, we can make the approximation for the phase velocity:

$$\frac{|\vec{\alpha}|}{\omega} \approx \sqrt{\frac{\mu\sigma}{2\omega}}. \tag{37}$$

The amplitude of the wave in a conductor falls by a factor $1/e$ in a distance $1/\left|\vec{\beta}\right|$: this defines the *skin depth*, δ. The skin depth can be found from the dispersion relations (33) and (34).

In a good conductor ($\sigma \gg \omega\varepsilon$), the skin depth is given approximately by:

$$\delta = \frac{1}{\left|\vec{\beta}\right|} \approx \sqrt{\frac{2}{\mu\sigma\omega}}. \tag{38}$$

Note that this distance is shorter than the wavelength by a factor 2π:

$$\lambda = \frac{2\pi}{|\vec{\alpha}|} \approx 2\pi\sqrt{\frac{2}{\mu\sigma\omega}}. \tag{39}$$

As well as a decay in the amplitude of the wave in a conductor, there is also a phase difference between the electric and magnetic fields. This follows from the relationship between the amplitudes, which can be found from Maxwell's equation (3):

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}.$$

With the solutions (31) and (32), we find:

$$\left(i\vec{\alpha} - \vec{\beta}\right) \times \vec{E}_0 = i\omega\vec{B}_0. \tag{40}$$

For a good conductor ($\sigma \gg \omega\varepsilon$), we have:

$$|\vec{\alpha}| \approx \left|\vec{\beta}\right| \approx \sqrt{\frac{\mu\sigma\omega}{2}}. \tag{41}$$

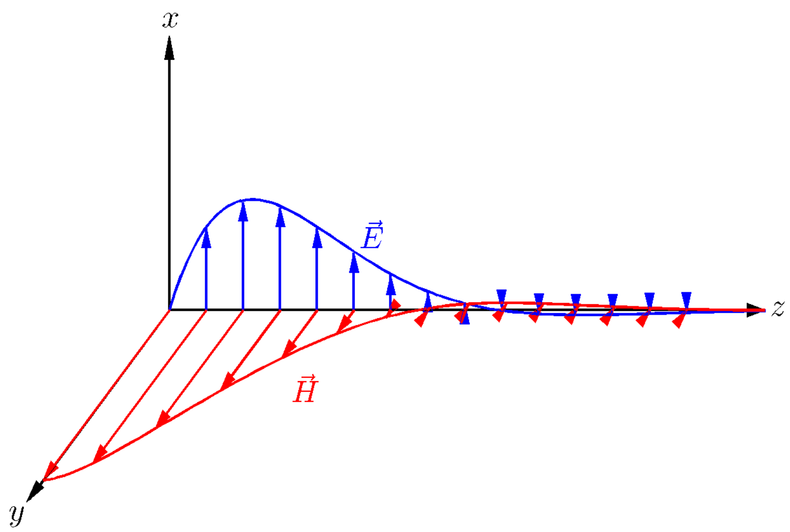
Hence, the ratio of field amplitudes is given by:

$$\frac{\left|\vec{E}_0\right|}{\left|\vec{B}_0\right|} \approx (i-1)\sqrt{\frac{\omega}{2\mu\sigma}}. \tag{42}$$

Electromagnetic Waves in Conductors

In a good conductor:

- the wave amplitude decays exponentially over a distance shorter than the wavelength;
- the electric and magnetic fields are out of phase by approximately 45°.



Boundary Conditions on Electromagnetic Fields

The electromagnetic waves we encounter in accelerators generally exist within bounded regions, for example within vacuum chambers, rf cavities, or waveguides.

The presence of boundaries imposes conditions on the waves that can exist within a given region: often, only particular frequencies and wavelengths are allowed for waves in a bounded region. This is in contrast to waves in free space, where *any* frequency of wave is allowed.

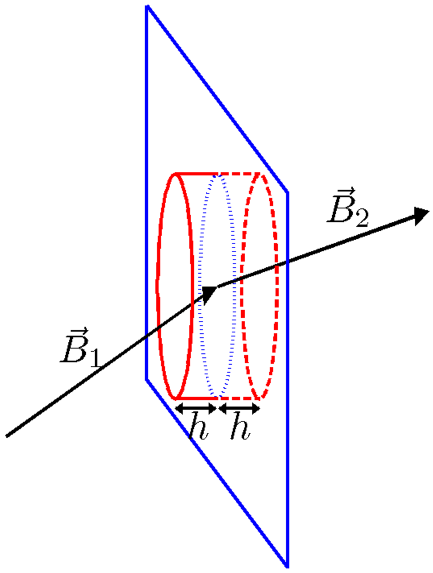
To understand the constraints on waves in bounded regions, we first have to understand the constraints on the fields themselves at boundaries. These "boundary conditions" can be derived from Maxwell's equations.

Boundary Conditions 1: Normal Component of $\vec{B}$

First, consider the magnetic field $\vec{B}$ at the boundary between two media. The field must satisfy Maxwell's equation, in integral form:

$$\int_{\partial V} \vec{B} \cdot d\vec{S} = 0. \qquad (43)$$

If we take the region V with boundary ∂V to be a thin "pill box" just crossing the boundary, then, taking the limit that the width of the pill box approaches zero, we find the boundary condition:



$$B_{2\perp} - B_{1\perp} = 0, \qquad (44)$$

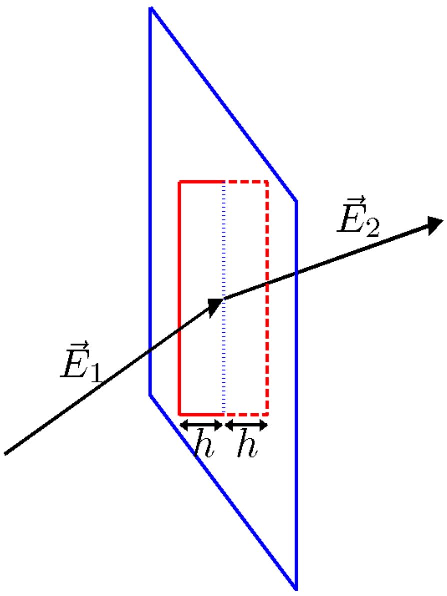
where $B_{2\perp}$ is the normal component of the magnetic field on one side of the boundary, and $B_{1\perp}$ is the normal component on the other side of the boundary.

Boundary Conditions 2: Parallel Component of $\vec{E}$

Next, consider the electric field $\vec{E}$ at a boundary. The field must satisfy Maxwell's equation, in integral form:

$$\int_{\partial S} \vec{E} \cdot d\vec{\ell} = -\frac{\partial}{\partial t} \int_{S} \vec{B} \cdot d\vec{S}. \qquad (45)$$

If we take the area S with boundary ∂S to be a thin strip just crossing the boundary, then, taking the limit that the width of the strip approaches zero, we find the boundary condition:



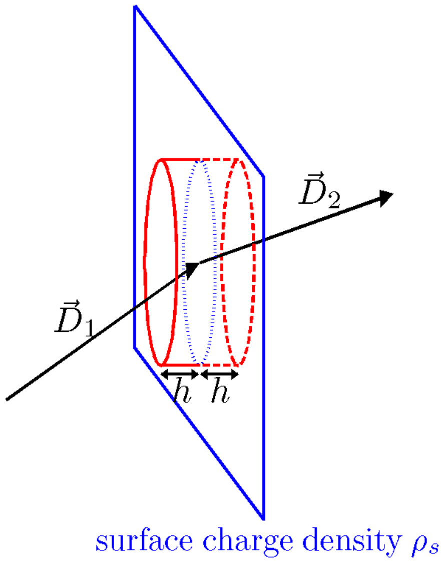
$$E_{2\|} - E_{1\|} = 0, \qquad (46)$$

where $E_{2\|}$ is the parallel component of the electric field on one side of the boundary, and $E_{1\|}$ is the parallel component on the other side of the boundary.

Now, consider the electric displacement $\vec{D}$ at a boundary. The field must satisfy Maxwell's equation, in integral form:

$$\int_{\partial V} \vec{D} \cdot d\vec{S} = q, \qquad (47)$$

where q is the electric charge enclosed by the surface ∂V. If we take the region V with boundary ∂V to be a thin "pill box" just crossing the boundary, then, taking the limit that the width of the pill box approaches zero, we find the boundary condition:



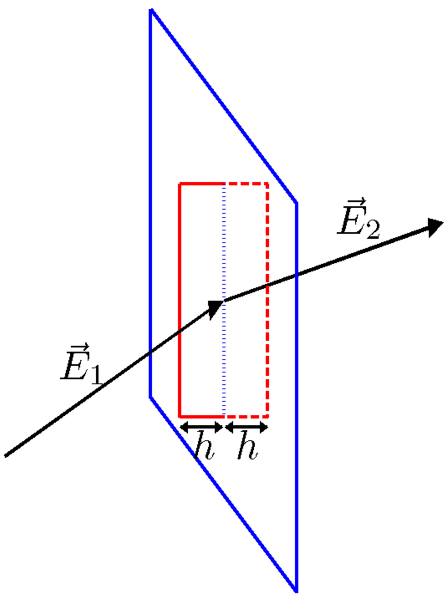
$$D_{2\perp} - D_{1\perp} = \rho_s, \qquad (48)$$

where ρ_s is the *surface charge density* (i.e. the charge per unit area) existing on the boundary between the media.

Finally, consider the magnetic intensity $\vec{H}$ at a boundary. The field must satisfy Maxwell's equation, in integral form:

$$\int_{\partial S} \vec{H} \cdot d\vec{\ell} = \int_S \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right) \cdot d\vec{S}. \qquad (49)$$

If we take the area S with boundary ∂S to be a thin strip just crossing the boundary, then, taking the limit that the width of the strip approaches zero, we find the boundary condition:



$$H_{2\parallel} - H_{1\parallel} = J_s, \qquad (50)$$

where J_s is the *surface current density* flowing on the boundary, perpendicular to the magnetic intensity. If the media have finite conductivity, we can expect this surface current density to be zero.

Laws of Reflection and Refraction

The boundary conditions applied to the fields in electromagnetic waves lead to the familiar phenomena associated with reflection and refraction, including:

- the law of reflection, and the law of refraction (Snell's law);
- the intensity of reflected and transmitted waves, as functions of the angle of incidence and polarisation of the wave (Fresnel's equations);
- total internal reflection (critical angle);
- polarisation by reflection (Brewster angle).

However, we will be mostly concerned with waves on the surface of a "good" conductor...

Boundary Conditions on the Surface of a Good Conductor

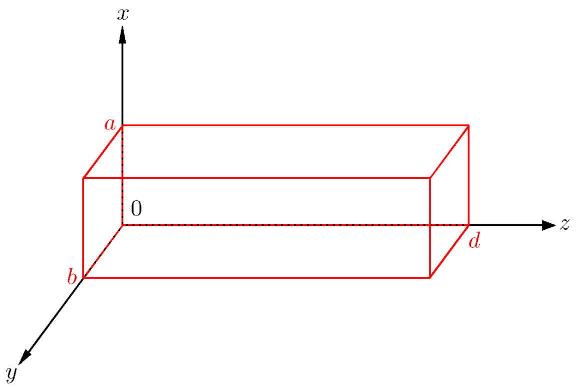
Electromagnetic waves at the surface of good conductors (including superconductors) require rather careful analysis. However, the significant phenomena (for our purposes) can be understood if we *assume* that the skin depth approaches zero as the conductivity becomes infinite. In this approximation, electromagnetic waves cannot penetrate into a good conductor, and we can assume that the fields inside the conductor are zero.

Then, applying the boundary conditions, we find (for medium 1 air or vacuum, and medium 2 a good conductor):

$$\begin{aligned} B_{1\perp} &= 0 & \qquad B_{2\perp} &= 0 \\ E_{1\parallel} &= 0 & \qquad E_{2\parallel} &= 0 \\ D_{1\perp} &= -\rho_s & \qquad D_{2\perp} &= 0 \\ H_{1\parallel} &= J_s & \qquad H_{2\parallel} &= 0 \end{aligned}$$

Physically, the charges at the boundary respond to the fields in any incident wave to suppress any fields within the conductor.

Consider a vacuum within a rectangular box with side lengths a, b and d, and perfectly conducting walls.



We know that within the cavity, the electromagnetic fields must satisfy the wave equations:

$$\nabla^2 \vec{E} - \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} = 0, \tag{51}$$

$$\nabla^2 \vec{B} - \mu_0 \varepsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2} = 0. \tag{52}$$

However, now we also have to satisfy the boundary conditions on the walls of the box.

The appropriate solutions for the electric field take the form of *standing waves*:

$$E_x = E_{x0} \cos k_x x \sin k_y y \sin k_z z \, e^{-i\omega t} \tag{53}$$

$$E_y = E_{y0} \sin k_x x \cos k_y y \sin k_z z \, e^{-i\omega t} \tag{54}$$

$$E_z = E_{z0} \sin k_x x \sin k_y y \cos k_z z \, e^{-i\omega t} \tag{55}$$

where the wave equation is satisfied if:

$$k_x^2 + k_y^2 + k_z^2 = \mu_0 \varepsilon_0 \omega^2. \tag{56}$$

Maxwell's equation $\nabla \cdot \vec{E} = 0$ is satisfied if:

$$k_x E_{x0} + k_y E_{y0} + k_z E_{z0} = 0. \tag{57}$$

And finally, the boundary condition (that the parallel component of the electric field vanishes at the walls of the box) is satisfied if:

$$k_x = \ell \frac{\pi}{a}, \quad k_y = m \frac{\pi}{b}, \quad k_z = n \frac{\pi}{d}, \tag{58}$$

for *integer* ℓ, m and n.

Equations (56), (57), and (58) have important consequences. First, equations (56) and (58) mean that the (standing) waves within the box can only take particular, *resonant* frequencies:

$$\omega_{\ell mn} = \pi c \sqrt{\frac{\ell^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{d^2}}, \tag{59}$$

where $c = 1/\sqrt{\mu_0 \varepsilon_0}$.

From equation (57) we see that at least two of ℓ, m and n must be non-zero, otherwise the wave vanishes entirely. Thus, the lowest frequency wave that can exist within the box has frequency (assuming $a < b$ and $a < d$):

$$\omega_{011} = \pi c \sqrt{\frac{1}{b^2} + \frac{1}{d^2}}. \tag{60}$$

The $(0, 1, 1)$ mode has electric field given by:

$$E_x = E_{x0} \sin k_y y \sin k_z z \, e^{-i\omega t}, \tag{61}$$

$$E_y = 0, \tag{62}$$

$$E_z = 0, \tag{63}$$

where:

$$k_y = \frac{\pi}{b}, \quad \text{and} \quad k_z = \frac{\pi}{d}. \tag{64}$$

The magnetic field is related to the electric field by Maxwell's equations. (Formally, we have to verify that our solutions satisfy all of Maxwell's equations and the boundary conditions simultaneously.) For the $(0, 1, 1)$ mode, the magnetic field is given by:

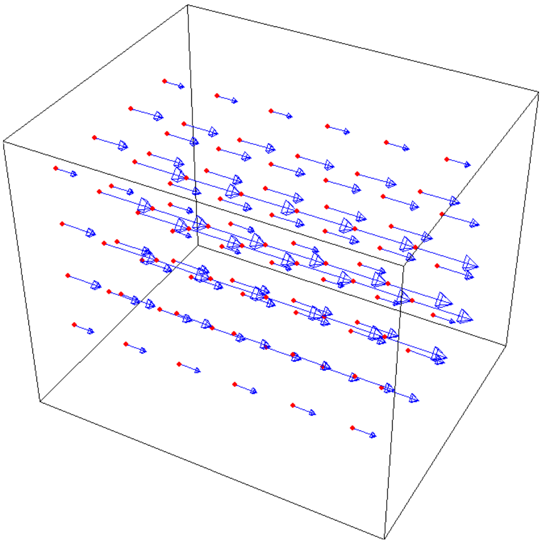
$$B_x = 0, \tag{65}$$

$$B_y = -\frac{i}{\omega\mu} k_z E_{x0} \sin k_y y \cos k_z z \, e^{-i\omega t}, \tag{66}$$

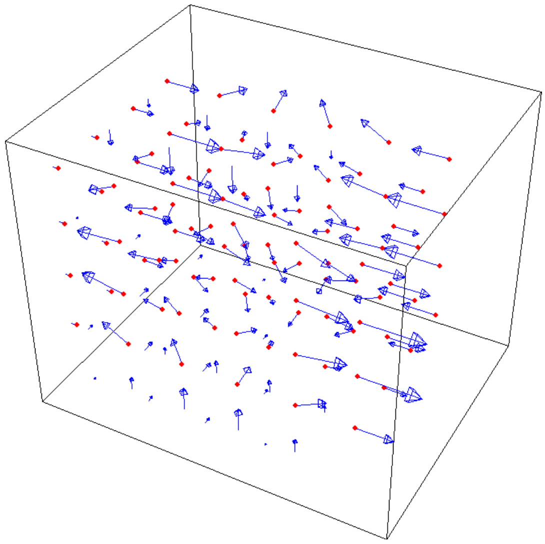
$$B_z = \frac{i}{\omega\mu} k_y E_{x0} \cos k_y y \sin k_z z \, e^{-i\omega t}. \tag{67}$$

Electromagnetic Waves in a Metal Box

Each electromagnetic field "pattern" (with its own frequency) is known as a "mode". Electromagnetic field modes within a metal cavity are analogous to standing waves or modes that can exist on a stretched wire fixed at each end.



Mode $(0, 1, 1)$

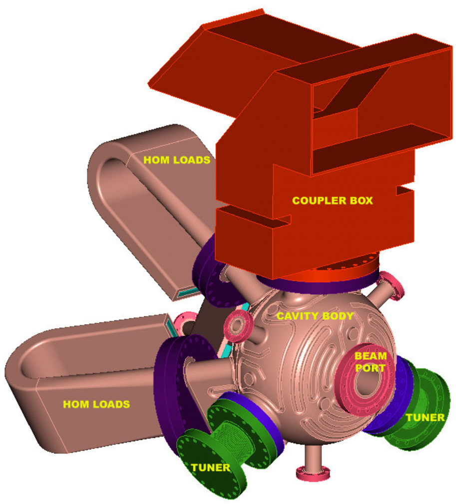


Mode $(1, 2, 1)$

RF Cavities in Accelerators

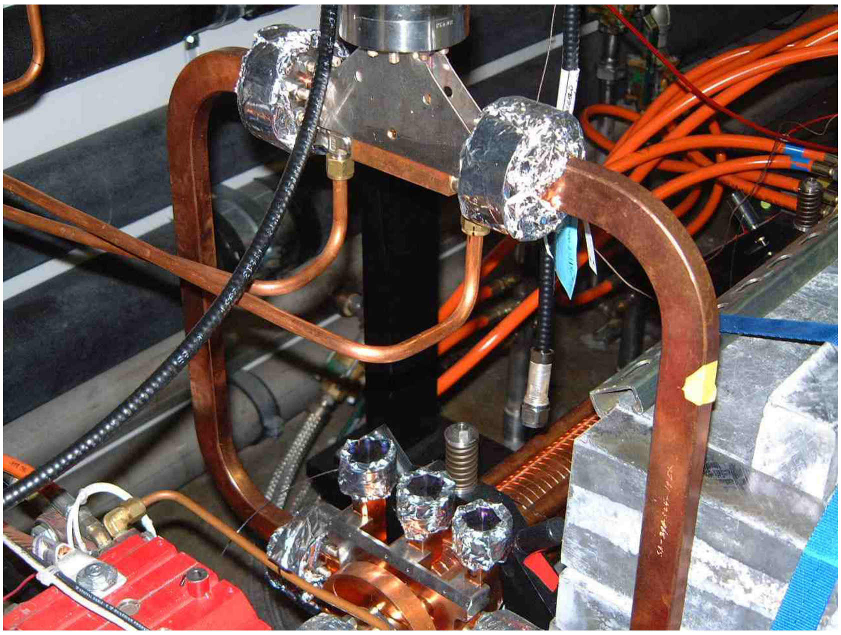
RF cavities are commonly used in accelerators to provide an accelerating (longitudinal electric) field. Issues include:

- geometry design to maximize the field on-axis, while minimizing the field near the walls (which can lead to ohmic losses);
- design of the "couplers" that carry electromagnetic energy into the cavity to excite the desired mode;
- design of "higher order mode dampers" that extract electromagnetic energy from undesired modes.



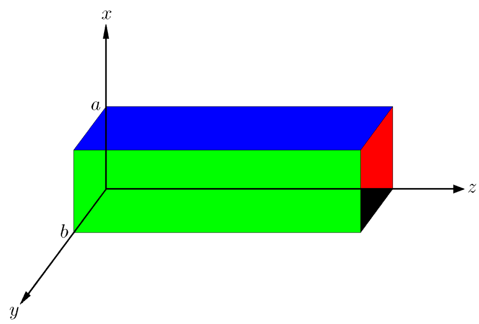
Waves in Waveguides

In its simplest form, a waveguide is simply a long metal pipe. However, in accelerators, waveguides are extremely useful for carrying electromagnetic energy in the form of waves from a source (e.g. a klystron) to an rf cavity.



Waves in Waveguides

The analysis of the fields in a rectangular waveguide is similar to the analysis of fields in a rectangular cavity... except that one dimension is extended to infinity.



In the "open" dimension, the solution to the wave equation takes the form of a travelling wave; in the bounded directions, the solution has the form of a standing wave.

$$E_x = E_{x0} \cos k_x x \sin k_y y \, e^{i(k_z z - \omega t)}, \quad (68)$$

$$E_y = E_{y0} \sin k_x x \cos k_y y \, e^{i(k_z z - \omega t)}, \quad (69)$$

$$E_z = iE_{z0} \sin k_x x \sin k_y y \, e^{i(k_z z - \omega t)}, \quad (70)$$

where:

$$k_x = \ell \frac{\pi}{a}, \quad \text{and} \quad k_y = m \frac{\pi}{b}. \quad (71)$$

Note that the boundary conditions impose constraints on k_x and k_y. In principle, there is no constraint on k_z, and for given (integer) mode numbers ℓ and m, any frequency ω is allowed.

However, the wave equation leads to the relationship:

$$k_z^2 = \frac{\omega^2}{c^2} - k_x^2 - k_y^2. \tag{72}$$

If ω is too small, then k_z^2 will be less than zero; k_z will be an imaginary number. In this case, the wave does not propagate along the waveguide, but instead the fields decay exponentially with distance along z.

The lowest frequency for a wave to propagate along the waveguide is known as the *cut-off frequency*, and is given (for $b < a$) by:

$$\omega_{\text{cut-off}} = \frac{\pi}{a} c. \tag{73}$$

The phase velocity of a wave in a waveguide is given by:

$$v_p = \frac{\omega}{k_z} = \frac{\sqrt{k_x^2 + k_y^2 + k_z^2}}{k_z} c. \tag{74}$$

It appears that the phase velocity is greater than the speed of light in vacuum. This is perfectly legitimate, since energy propagates at the *group velocity* rather than the phase velocity... and as we shall see, the group velocity is less than the speed of light.

Consider a wave with frequency ω and wave number k:

$$y = \cos(kz - \omega t). \tag{75}$$

Suppose we superpose another wave on top of this one; but the new wave has slightly different frequency, $\omega + d\omega$ and wave number $k + dk$ ($d\omega \ll \omega$, and $dk \ll k$). The total "displacement" is given by:

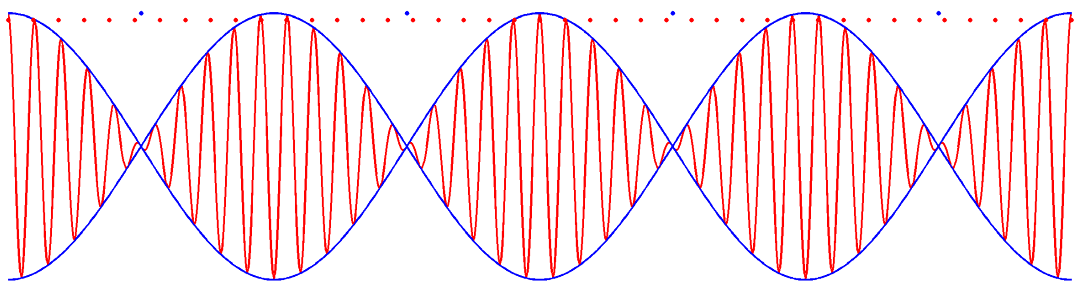
$$\begin{aligned} y &= \cos(kz - \omega t) + \cos\left((k + dk)z - (\omega + d\omega)t\right) \\ &= 2\cos\left((k + \frac{dk}{2})z - (\omega + \frac{d\omega}{2})t\right)\cos(dk\,z - d\omega\,t). \end{aligned} \tag{76}$$

We see that the wave is now modulated; the velocity of the modulation is:

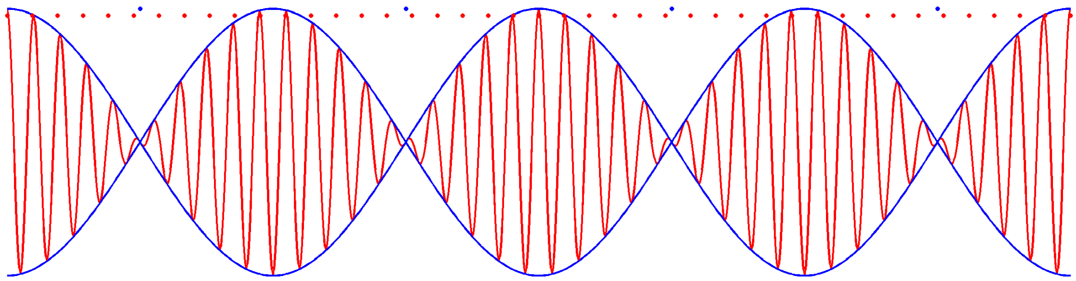
$$v_g = \frac{d\omega}{dk}. \tag{77}$$

This is the *group velocity*. Since the energy in a wave is proportional to the square of the amplitude, the energy in a wave propagates at the group velocity.

Zero dispersion: $\omega = v_p k$



With dispersion: $\omega = v_p\sqrt{k^2 + \alpha^2}$



The dispersion relation for a wave in a waveguide is, from (72):

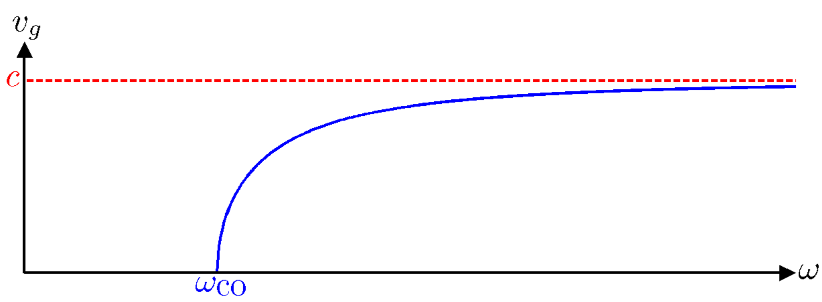
$$\omega = c\sqrt{k_x^2 + k_y^2 + k_z^2}. \tag{78}$$

Hence, the group velocity is:

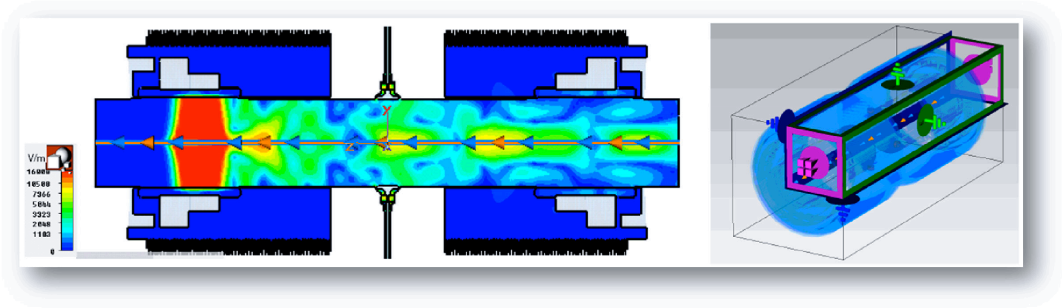
$$v_g = \frac{\partial \omega}{\partial k_z} = \frac{ck_z}{\sqrt{k_x^2 + k_y^2 + k_z^2}}. \tag{79}$$

For real values of k_x and k_y, the group velocity is less than the speed of light. In fact, we find that $v_p v_g = c$.

For given mode numbers ℓ and m, we can plot the group velocity as a function of frequency:



We have seen that it is possible to find analytical solutions to Maxwell's equations in bounded regions with simple geometries. In more realistic situations, where the geometries involved are more complicated, it is usually necessary to solve Maxwell's equations numerically, using a specialised computer code.

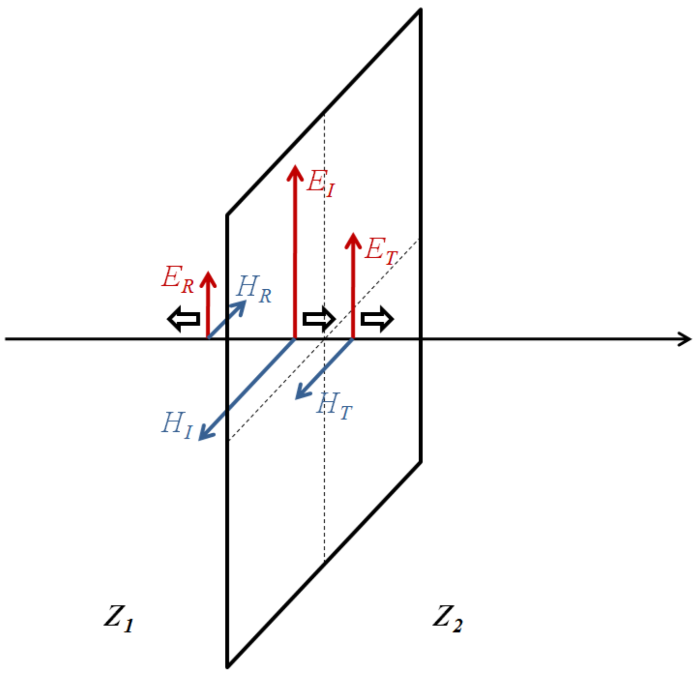


However, one further important situation that is amenable to analysis concerns waves incident on boundaries. We will not treat the general case, but only consider *normal* incidence. This introduces the important concept of impedance.

Electromagnetic Waves Normally Incident on a Boundary

For simplicity, we shall consider an infinite plane electromagnetic wave striking a boundary between two dielectrics at normal incidence.

To satisfy the boundary conditions, we need, as well as the incident wave, a transmitted and a reflected wave.



Write the electric fields and the magnetic intensities in the waves as:

$$
\begin{aligned}
E_I &= E_{I0} e^{i(kz-\omega t)} & H_I &= \frac{E_{I0}}{Z_1} e^{i(kz-\omega t)} \\
E_R &= E_{R0} e^{i(kz+\omega t)} & H_R &= \frac{E_{R0}}{Z_1} e^{i(kz+\omega t)} \\
E_T &= E_{T0} e^{i(kz-\omega t)} & H_T &= \frac{E_{T0}}{Z_2} e^{i(kz-\omega t)}
\end{aligned}
$$

Here, we have used the fact that the amplitudes of the electric and magnetic fields in plane wave are related by:

$$\frac{E_0}{H_0} = \sqrt{\frac{\mu}{\varepsilon}} = Z. \qquad (80)$$

Z is the *impedance* of the medium through which the wave is travelling.

Now we use the boundary conditions, which tell us that the tangential components of $\vec{E}$ and $\vec{H}$ must be continuous across the boundary. These conditions can be written:

$$E_T = E_I + E_R, \quad \text{and} \quad H_T = H_I - H_R, \qquad (81)$$

where the minus sign in the equation for the magnetic intensity takes into account the direction of the field in the reflected wave.

We find that the boundary conditions are satisfied if:

$$\frac{E_{T0}}{E_{I0}} = \frac{2Z_2}{Z_1 + Z_2}, \quad \text{and} \quad \frac{E_{R0}}{E_{I0}} = \frac{Z_2 - Z_1}{Z_2 + Z_1}. \qquad (82)$$

Hence, the amplitudes of the reflected and transmitted waves are determined by the impedances of the media on either side of the boundary. Note that if $Z_1 = Z_2$, then the wave propagates as if there was no boundary.

Impedance

Although we have dealt with a very special case, the concept of impedance is used very widely in electromagnetism.

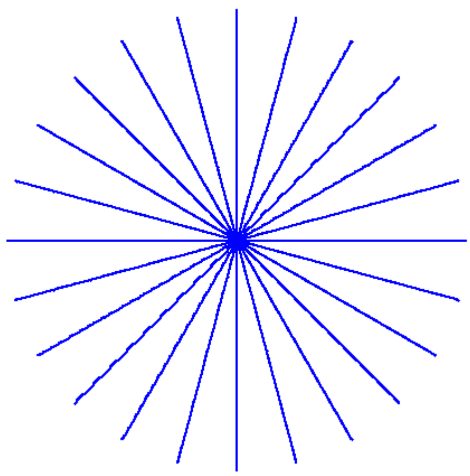
In general, impedance is used to relate:

- the ratio of the electric field to the magnetic intensity in an electromagnetic wave;
- the ratio of voltage to current in a circuit;
- the voltage acting back on a beam in an accelerator as a result of the beam current.

Any discontinuity in a medium or circuit through which energy is flowing as a wave will be associated with a change in impedance. Knowing the impedance is important for understanding the transmission and reflection of energy at the discontinuity.

Sources of Electromagnetic Radiation

If a point charge, initially at rest, is moved from one place to another, changes in the field around that charge will propagate through space at the speed of light.



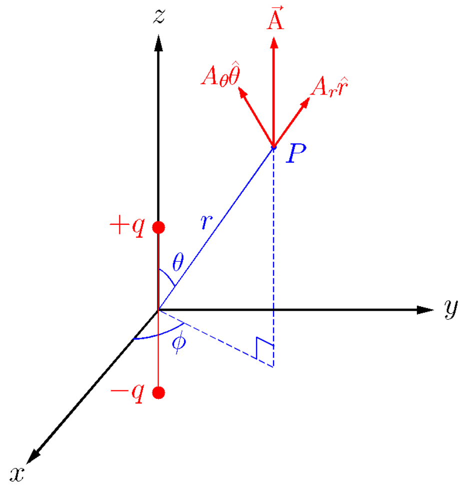
The "disturbance" in the electromagnetic field carries energy. Electromagnetic radiation is emitted whenever electric charges undergo acceleration.

The Hertzian dipole is one of the simplest "radiating systems" to analyse. This consists of an infinitesimal pair of point charges, oscillating so that the electric dipole moment is:

$$\vec{p} = q\vec{\Delta z} = \vec{p}_0 e^{-i\omega t}. \qquad (83)$$

The oscillation may be represented as an alternating current flowing between two fixed points. The vector potential can then be found from:

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}\,', t - |\vec{r}-\vec{r}\,'|/c)}{|\vec{r}-\vec{r}\,'|}\,dV'. \qquad (84)$$



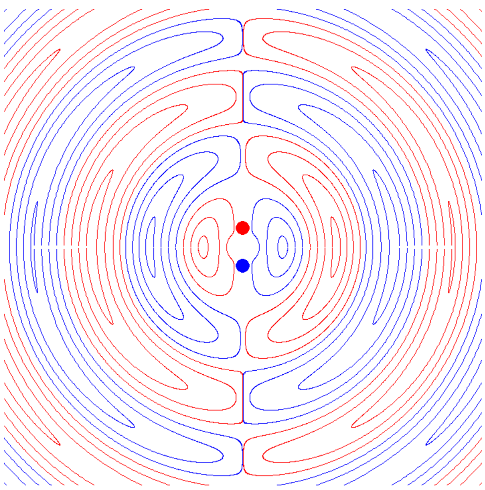
The magnetic field is found from (5):

$$\vec{B} = \nabla \times \vec{A},$$

and the electric field from Maxwell's equation (3):

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}.$$

We do not go into the detailed analysis of the Hertzian dipole. However, the result is that the dipole radiates energy, that (in the mid-plane and at large distances from the dipole) looks like plane waves. For more details, see e.g. Grant and Phillips.



Understanding sources of radiation is important in accelerators, particularly in the context of synchrotron radiation.

You should be able to:

- derive, from Maxwell's equations, the wave equations for the electric and magnetic fields in free space, dielectrics, conductors;
- derive the principal properties of the waves, including: speed of propagation, relative directions and amplitudes of the fields, and (in conductors) the skin depth;
- derive the solutions to the wave equations in regions bounded by conductors;
- explain the properties of electromagnetic waves in rf cavities and waveguides, and discuss how these components are useful in accelerators;
- explain that electromagnetic waves are generated whenever charges undergo acceleration.