Revision of Electromagnetic Theory

Lecture 1

Maxwell's Equations
Static Fields
Electromagnetic Potentials
Electromagnetism and Special Relativity

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Electromagnetism in Accelerators

Electromagnetism has two principle applications in accelerators:

- magnetic fields can be used to steer and focus beams;
- electric fields can be used to control the energy of particles.

However, understanding many diverse phenomena in accelerators, and the design and operation of many different types of components, depends on a secure knowledge of electromagnetism.

In these lectures, we will revise the fundamental principles of electromagnetism, giving specific examples in the context of accelerators. In particular, in this first lecture, we will:

- revise Maxwell's equations and the associated vector calculus;
- discuss the physical interpretation of Maxwell's equations, with some examples relevant to accelerators;
- see how the dynamics of particles in an accelerator are governed by the Lorentz force;
- discuss the electromagnetic potentials, and their use in solving electrodynamical problems;
- briefly consider energy in electromagnetic fields;
- show how the equations of electromagnetism may be written to show explicitly their consistency with special relativity.

The next lecture will be concerned with electromagnetic waves.

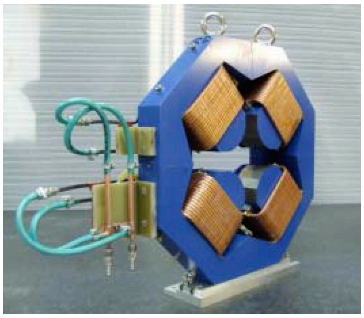
Further Reading

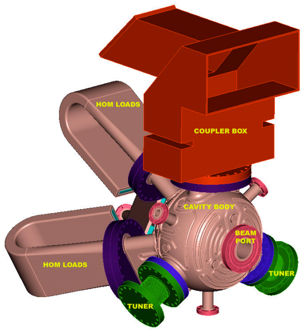
- **Recommended text:**
  I.S.Grant and W.R. Phillips, "Electromagnetism"
  Wiley, 2nd Edition, 1990

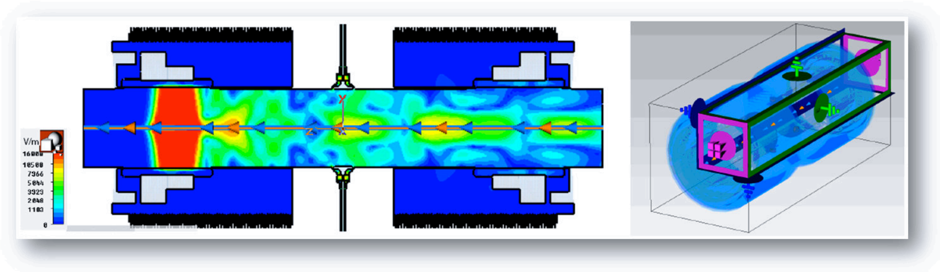
- **Free to download:**
  B. Thide, "Electromagnetic Field Theory"
  http://www.plasma.uu.se/CED/Book/index.html

- **Comprehensive reference for the very ambitious:**
  J.D.Jackson, "Classical Electrodynamics"
  Wiley, 3rd Edition, 1998

Understanding the properties of electromagnetic fields is essential for understanding the design, operation and performance of a wide variety of accelerator components.







Maxwell's equations determine the electric and magnetic fields in the presence of sources (charge and current densities), and in materials of given properties.

James Clerk Maxwell, 1831–1879

$$\nabla \cdot \vec{D} = \rho, \quad (1)$$

$$\nabla \cdot \vec{B} = 0, \quad (2)$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \quad (3)$$

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}. \quad (4)$$

ρ is the electric charge density, and $\vec{J}$ the electric current density.

The electric field $\vec{E}$ and magnetic field (flux density) $\vec{B}$ determine the force on a charged particle (moving with velocity $\vec{v}$), according to the Lorentz force equation:

$$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right). \quad (5)$$

The electric displacement $\vec{D}$ and magnetic intensity $\vec{H}$ are related the the electric and magnetic fields by:

$$\vec{D} = \varepsilon_r \varepsilon_0 \vec{E}, \tag{6}$$
$$\vec{B} = \mu_r \mu_0 \vec{H}. \tag{7}$$

The quantities ε_0 and μ_0 are fundamental physical constants; respectively, the permittivity and permeability of free space:

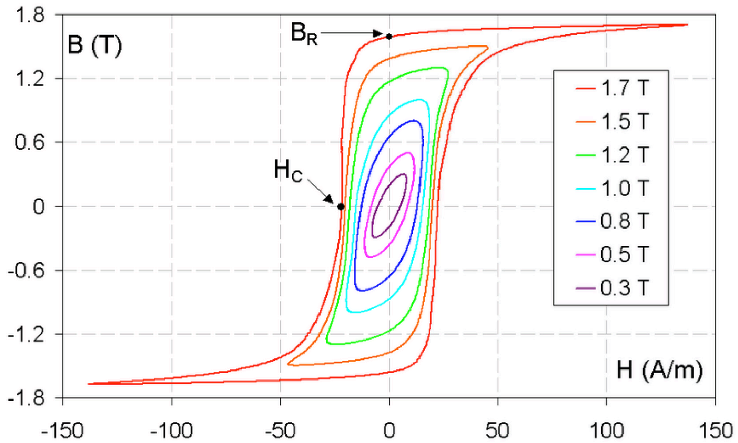
$$\varepsilon_0 \approx 8.854 \times 10^{-12}\,\mathrm{Fm}^{-1}, \tag{8}$$
$$\mu_0 = 4\pi \times 10^{-7}\,\mathrm{Hm}^{-1}. \tag{9}$$

Note that:

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} \approx 2.998 \times 10^8\,\mathrm{ms}^{-1}, \qquad Z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} \approx 376.73\,\Omega. \tag{10}$$

The relative permittivity ε_r and relative permeability μ_r are dimensionless quantities that characterise the response of a material to electric and magnetic fields.

Often, the relative permittivity and permeability of a given material are approximated by constants; in reality, they are themselves functions of the fields that are present, and functions also of the frequency of oscillations of the external fields.



Maxwell's equations are written in differential form using the operators div (divergence):

$$\nabla \cdot \vec{F} \equiv \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}, \tag{11}$$

and curl:

$$\nabla \times \vec{F} \equiv \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}, \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}, \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \equiv \begin{vmatrix} \widehat{x} & \widehat{y} & \widehat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}. \tag{12}$$

We will also encounter the grad (gradient) operator:

$$\nabla \phi \equiv \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right), \tag{13}$$

and the Laplacian:

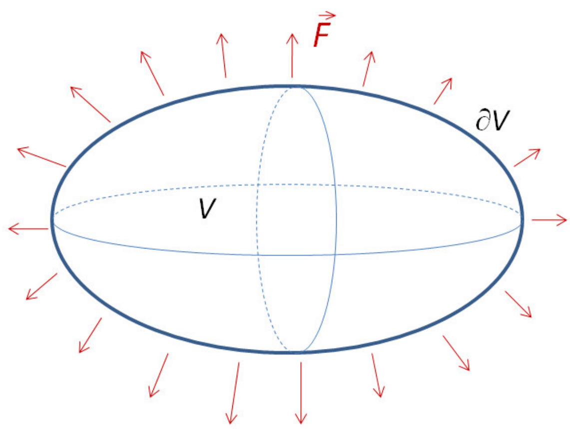
$$\nabla^2 \phi \equiv \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}. \tag{14}$$

We wrote down Maxwell's equations as a set of differential equations; however, the physical interpretation becomes clearer if the equations are converted to integral form. This may be achieved using two useful integral theorems.

First, we consider Gauss' theorem:

$$\int_V \nabla \cdot \vec{F}\, dV \equiv \oint_{\partial V} \vec{F} \cdot d\vec{S}, \tag{15}$$

where ∂V is a closed surface enclosing volume V.



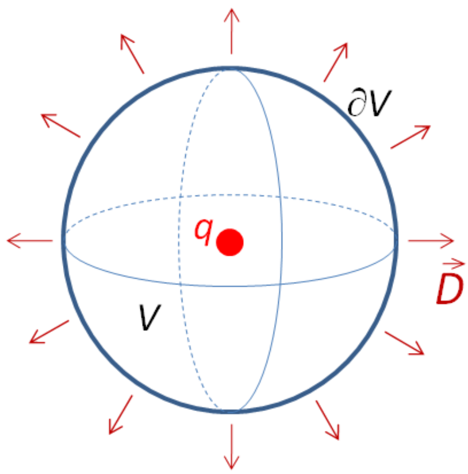
Let us apply Gauss' theorem to Maxwell's equation:

$$\nabla \cdot \vec{D} = \rho, \tag{16}$$

to give:

$$\oint_{\partial V} \vec{D} \cdot d\vec{S} = \int_V \rho \, dV. \tag{17}$$

This is the integral form of Maxwell's equation (1). From this form, we can find a physical interpretation of Maxwell's equation (1). The integral form of the equation tells us that the electric displacement integrated over any closed surface equals the electric charge contained within that surface.



In particular, if we consider a sphere of radius r bounding a spherically symmetric charge q at its centre, then performing the integrals gives:

$$4\pi r^2 \left| \vec{D} \right| = q, \tag{18}$$

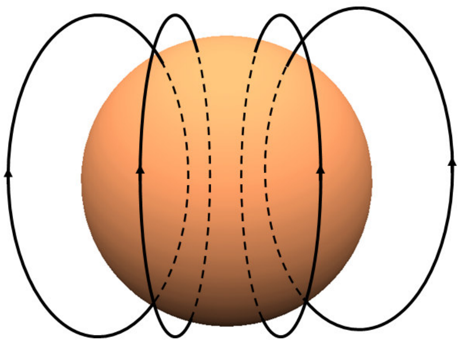
which, using $\vec{D} = \varepsilon_r \varepsilon_0 \vec{E}$ becomes Coulomb's law:

$$\vec{E} = \frac{q}{4\pi \varepsilon_r \varepsilon_0 r^2} \hat{r}. \tag{19}$$

Similarly, we can find an integral form of Maxwell's equation (2):

$$\oint_{\partial V} \vec{B} \cdot d\vec{S} = 0.$$

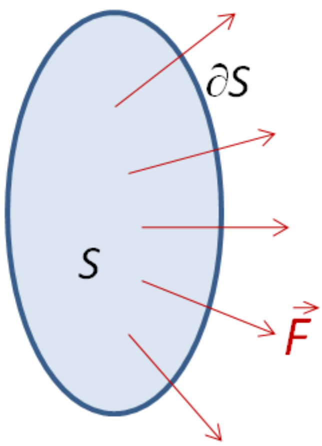
In this case, we see that the total magnetic flux crossing any closed surface is zero. In other words, lines of magnetic flux only occur in closed loops. Put yet another way, there are no magnetic monopoles.



Next, we consider Stokes' theorem:

$$\int_S \nabla \times \vec{F} \cdot d\vec{S} \equiv \oint_{\partial S} \vec{F} \cdot d\vec{\ell}, \qquad (20)$$

where ∂S is a closed loop bounding a surface S. This tells us that the integral of the curl of a vector field over a surface is equal to the integral of that vector field around a loop bounding that surface.



If we apply Stokes' theorem to Maxwell's equation (3):

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t},$$

we obtain Faraday's law of electromagnetic induction:

$$\oint_{\partial S} \vec{E} \cdot d\vec{\ell} = -\frac{\partial}{\partial t} \int_S \vec{B} \cdot d\vec{S}. \qquad (21)$$

This is the integral form of Maxwell's equation (3). It is often written as:

$$\mathcal{E} = -\frac{\partial \Phi}{\partial t}, \qquad (22)$$

where $\mathcal{E}$ is the electromotive force around a loop, and Φ the total magnetic flux passing through that loop.

Finally, we can apply Stokes' theorem to Maxwell's equation (4):

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t},$$

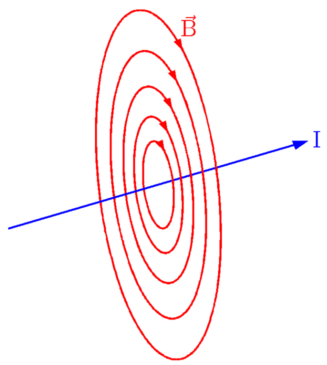
to find the integral form:

$$\oint_{\partial S} \vec{H} \cdot d\vec{\ell} = I + \frac{\partial}{\partial t} \int_S \vec{D} \cdot d\vec{S}, \qquad (23)$$

where I is the total electric current flowing through the closed loop ∂S.

Note that there are two terms on the right hand side of (23). The first term, I, is simply a flow of electric current; in this context, it is sometimes called the *conduction current*. In the case of a uniform current flow along a long straight wire, the conduction current term leads to Ampère's law:

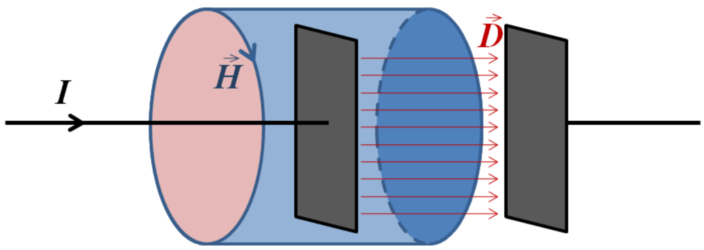
$$\vec{H} = \frac{I}{2\pi r}\hat{\theta}. \tag{24}$$

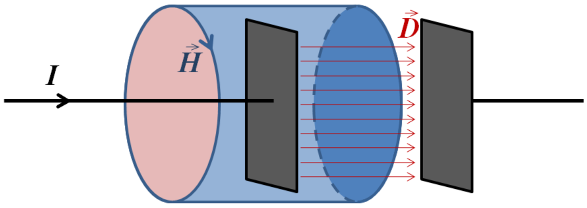


The second term on the right hand side of the integral form of Maxwell's equation (23) is sometimes called the *displacement current*, I_D:

$$I_D = \frac{\partial}{\partial t}\int_S \vec{D} \cdot d\vec{S}. \tag{25}$$

The need for this term can be understood by considering the current flow into a parallel-plate capacitor. We must find the same magnetic field around the wire using Stokes' theorem, whether we integrate over a surface cutting through the wire, or over a "deformed" surface passing between the plates...





Using Maxwell's equation (1):

$$\nabla \cdot \vec{D} = \rho, \tag{26}$$

and applying Gauss' theorem over a volume enclosing the surface of one plate of the capacitor, we find:

$$\int_V \nabla \cdot \vec{D} \, dV = \int_S \vec{D} \cdot d\vec{S} = q, \tag{27}$$

where q is the charge on one plate. The conduction current is:

$$I = \frac{dq}{dt} = \frac{\partial}{\partial t} \int_S \vec{D} \cdot d\vec{S} = I_D. \tag{28}$$

In this case, we see that the displacement current between the plates is equal to the conduction current carrying charge onto the plates. Hence, we arrive at the same magnetic field, no matter which surface we choose.

The displacement current term in Maxwell's equation (4) has an important physical consequence. If we take the divergence of this equation, we obtain:

$$\nabla \cdot \nabla \times \vec{H} = \nabla \cdot \vec{J} + \frac{\partial}{\partial t} \nabla \cdot \vec{D}. \tag{29}$$

Now, if we use the vector identity:

$$\nabla \cdot \nabla \times \vec{F} \equiv 0, \tag{30}$$

and Maxwell's equation (1), we find:

$$\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0. \tag{31}$$

This is the *continuity equation*, which expresses *local* conservation of electric charge.

The physical interpretation of the continuity equation becomes clearer if we use Gauss' theorem, to put it into integral form:

$$\int_{\partial V} \vec{J} \cdot d\vec{S} = -\frac{\partial}{\partial t} \int_V \rho \, dV. \qquad (32)$$

We see that the rate of decrease of electric charge in a region V is equal to the total current flowing out of the region through its boundary, ∂V. Since current is the rate of flow of electric charge, the continuity equation expresses the *local* conservation of electric charge.

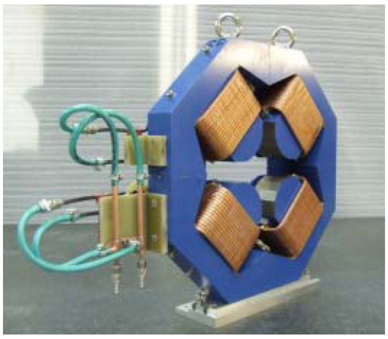
Donald Kirst with the world's first induction accelerator, at the University of Illinois, 1940.

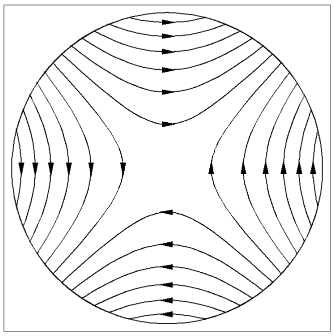
A betatron uses a time-dependent magnetic field passing through a toroidal vacuum chamber to accelerate a beam. By Maxwell's equation (3):

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t},$$

the varying magnetic field induces an electromotive force around the vacuum chamber, which accelerates the particle.

Example 2: Quadrupole Magnet





A quadrupole magnet is used to focus a beam of particles. Inside a quadrupole, the magnetic field is given by:

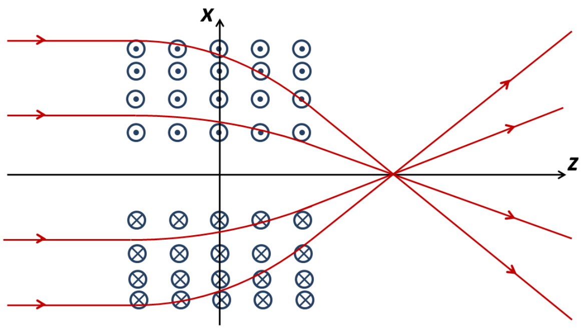
$$B_x = k_1 y, \quad B_y = k_1 x, \quad B_z = 0. \tag{33}$$

It is straightforward to show that (assuming a material of constant permeability, e.g. a vacuum, between the poles of the magnet):

$$\nabla \cdot \vec{B} = 0, \quad \text{and} \quad \nabla \times \vec{H} = 0. \tag{34}$$

The field (33) is therefore a valid solution to Maxwell's equations, in the absence of electric currents and time-dependent electric fields.

Example 2: Quadrupole Magnet

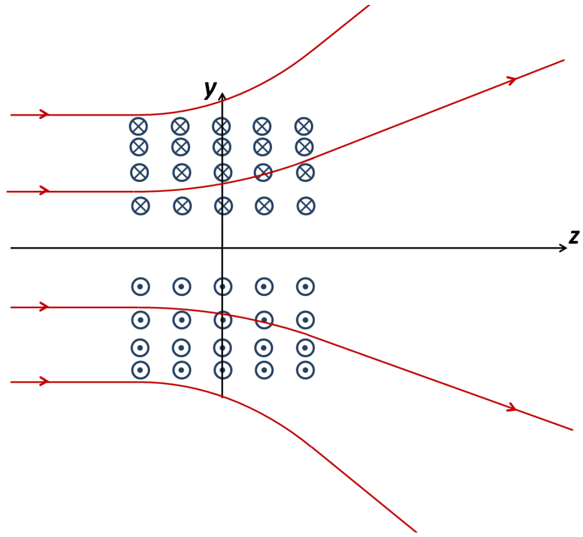


If we consider a particle with charge q travelling at speed v parallel to the z axis through the quadrupole field, we find that the change in horizontal momentum over a (short) distance Δz is given by:

$$\Delta p_x = -qvB_y\,\Delta t = -qk_1 x\,\Delta z. \tag{35}$$

In other words, in passing through the quadrupole, the particle receives a horizontal "kick" that is proportional to its horizontal position with respect to the centre of the quadrupole, and directed towards the z axis. This is just the property we require of a focusing lens.

Example 2: Quadrupole Magnet



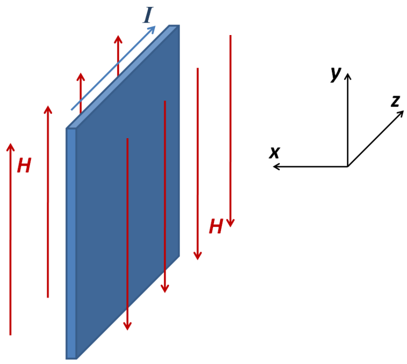
The change in vertical momentum is given by:

$$\Delta p_y = qvB_x \, \Delta t = qk_1 y \, \Delta z. \tag{36}$$

Here, the vertical kick is proportional to the vertical offset, but directed away from the z axis. The quadrupole is defocusing in the vertical direction.

Example 3: Septum Magnet

As a final example, consider a thin metal "blade" of width Δx in the $y - z$ plane, carrying current $J \, \Delta x$ in the z direction per unit length in the y direction.



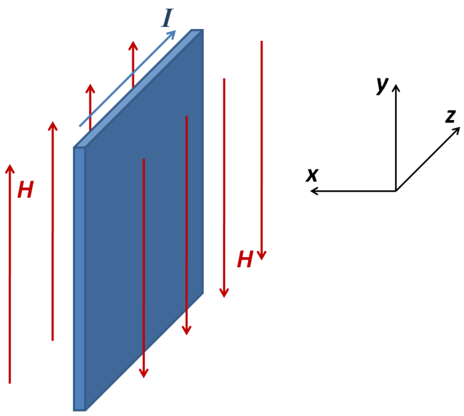
Using Maxwell's equation (4):

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t},$$

and Stokes' theorem, we find that:

$$\int_S \nabla \times \vec{H} \cdot d\vec{S} = \oint_{\partial S} \vec{H} \cdot d\vec{\ell} = 2H \, \Delta y = J \, \Delta x \, \Delta y. \tag{37}$$

Example 3: Septum Magnet



The magnetic field is in the y direction. If we superpose a field from an external source with $H = J\,\Delta x/2$ in the y direction, then we obtain a total field with uniform $H = J\,\Delta x$ in the y direction on one side of the blade, and zero on the other side.

This kind of "septum" magnet has important applications for beam injection and extraction.

Electromagnetic Potentials

Some problems in electromagnetism are solved more easily by working with the electromagnetic potentials φ and $\vec{A}$, rather than with the fields directly.

The potentials are functions of position and time. The electromagnetic fields are obtained by taking derviatives of the potentials:

$$\vec{B} = \nabla \times \vec{A}, \tag{38}$$

$$\vec{E} = -\nabla\varphi - \frac{\partial \vec{A}}{\partial t}. \tag{39}$$

To find the fields in a particular system, with specified currents, charges, and boundary conditions, we have to solve Maxwell's equations. Since the fields are derived from the potentials, there must be equivalent equations that allow us to determine the potentials in a given system.

First, note that from the vector identities:

$$\nabla \cdot \nabla \times \vec{F} \equiv 0, \quad \text{and} \quad \nabla \times \nabla f \equiv 0, \tag{40}$$

for *any* vector field $\vec{F}$ and scalar field f, if the fields $\vec{B}$ and $\vec{E}$ are given by (38) and (39), then the homogeneous Maxwell's equations (2) and (3):

$$\begin{aligned} \nabla \cdot \vec{B} &= 0, \\ \nabla \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t}, \end{aligned}$$

are automatically satisfied.

The inhomogeneous equations give:

$$\nabla \times \vec{B} = \nabla \times \nabla \times \vec{A} = \mu \vec{J} + \mu\varepsilon \frac{\partial \vec{E}}{\partial t}, \tag{41}$$

and:

$$\nabla \cdot \vec{E} = -\nabla \cdot \nabla \varphi - \frac{\partial}{\partial t} \nabla \cdot \vec{A} = \frac{\rho}{\varepsilon}. \tag{42}$$

Now we use two further identities:

$$\nabla \times \nabla \times \vec{F} \equiv \nabla(\nabla \cdot \vec{F}) - \nabla^2 \vec{F}, \tag{43}$$

$$\nabla \cdot \nabla f \equiv \nabla^2 f, \tag{44}$$

to write the equations for the potentials:

$$\nabla^2 \vec{A} - \mu\varepsilon \frac{\partial^2 \vec{A}}{\partial t^2} - \nabla \left(\nabla \cdot \vec{A} + \mu\varepsilon \frac{\partial \varphi}{\partial t} \right) = -\mu \vec{J}, \tag{45}$$

$$\nabla^2 \varphi - \frac{\partial}{\partial t} \nabla \cdot \vec{A} = -\frac{\rho}{\varepsilon}. \tag{46}$$

The equations for the potentials look very complicated. However, we note that *if* the potentials happen to satisfy the Lorenz gauge condition:

$$\nabla \cdot \vec{A} + \mu\varepsilon \frac{\partial \varphi}{\partial t} = 0, \tag{47}$$

then the equations for the potentials become:

$$\nabla^2 \vec{A} - \mu\varepsilon \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu \vec{J}, \tag{48}$$

$$\nabla^2 \varphi - \mu\varepsilon \frac{\partial^2 \varphi}{\partial t^2} = -\frac{\rho}{\varepsilon}. \tag{49}$$

In fact, for any given fields $\vec{E}$ and $\vec{B}$, it is *always* possible to find potentials φ and $\vec{A}$ that satisfy the condition (47), as we shall now show...

First, we note that if we have potentials φ and $\vec{A}$ that generate a given electromagnetic field, then the potentials:

$$\varphi' = \varphi - \frac{\partial \psi}{\partial t}, \quad \text{and} \quad \vec{A}' = \vec{A} + \nabla\psi, \tag{50}$$

for any function ψ, generate the *same* electromagnetic field. The transformation (50) is known as a gauge transformation.

Now, let us suppose we have potentials that give:

$$\nabla \cdot \vec{A} + \mu\varepsilon \frac{\partial \varphi}{\partial t} = f, \tag{51}$$

where f is some function of position and time. If we make a gauge transformation generated by ψ, where ψ satisfies:

$$\nabla^2 \psi - \mu\varepsilon \frac{\partial^2 \psi}{\partial t^2} = -f, \tag{52}$$

then the new potentials φ' and $\vec{A}'$ satisfy the Lorenz gauge condition (47):

$$\nabla \cdot \vec{A}' + \mu\varepsilon \frac{\partial \varphi'}{\partial t} = 0, \tag{53}$$

Electromagnetic Potentials in the Lorenz Gauge

In the Lorenz gauge, the electromagnetic potentials satisfy the wave equations with sources, (48) and (49):

$$\nabla^2 \vec{A} - \mu\varepsilon \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu \vec{J},$$

$$\nabla^2 \varphi - \mu\varepsilon \frac{\partial^2 \varphi}{\partial t^2} = -\frac{\rho}{\varepsilon}.$$

Note that these equations are decoupled ($\vec{A}$ and φ appear only in one equation or the other). This makes them easier to solve in general than Maxwell's equations, in which the fields tend to be coupled. In fact, these equations have standard solutions. For example, for the scalar potential:

$$\varphi(\vec{r}, t) = \frac{1}{4\pi\varepsilon} \int \frac{\rho(\vec{r}', t')}{|\vec{r} - \vec{r}'|} \, d^3\vec{r}', \tag{54}$$

where:

$$t' = t - \frac{|\vec{r} - \vec{r}'|}{v}, \quad \text{and} \quad v = \frac{1}{\sqrt{\mu\varepsilon}}. \tag{55}$$

Electromagnetic Potentials, Energy and Momentum

The motion of a charged particle in an electromagnetic field is determined by the Lorentz force (5):

$$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right). \tag{56}$$

The motion can also be obtained from conservation of the total energy $\mathscr{E}$ and total momentum $\vec{\mathscr{P}}$, defined by:

$$\mathscr{E} = \gamma m c^2 + q\varphi, \tag{57}$$

$$\vec{\mathscr{P}} = \vec{\beta}\gamma m c + q\vec{A}, \tag{58}$$

where $\vec{\beta}$ is the velocity divided by the speed of light, and γ is the relativistic factor.

If the potentials are independent of time, then the total energy $\mathscr{E}$ is conserved.

If the potentials are independent of a space coordinate (x, y or z), then the corresponding component of the total momentum $\vec{\mathscr{P}}$ is conserved.

Since a charged particle moving in an electromagnetic field can gain or lose energy, and the total energy within a closed system is conserved, we expect that energy can be stored within the electromagnetic field itself. The energy within an electromagnetic field is made explicit using Poynting's theorem.

From Maxwell's equations, we can write:

$$\vec{H} \cdot \nabla \times \vec{E} - \vec{E} \cdot \nabla \times \vec{H} = -vecH \cdot \frac{\partial \vec{B}}{\partial t} - \vec{E} \cdot \vec{J} - vecE \cdot \frac{\partial \vec{D}}{\partial t}. \quad (59)$$

Using a vector identity, we have:

$$\vec{H} \cdot \nabla \times \vec{E} - \vec{E} \cdot \nabla \times \vec{H} \equiv \nabla \cdot \left(\vec{E} \times \vec{H} \right). \quad (60)$$

Combining these results, using Gauss' theorem, and assuming we are working in a linear medium (constant permittivity and permeability) we arrive at Poynting's theorem:

$$\frac{\partial}{\partial t} \int_V \frac{1}{2} \vec{D} \cdot \vec{E} + \frac{1}{2} \vec{B} \cdot \vec{H} \, dV = - \int_V \vec{E} \cdot \vec{J} \, dV - \int_{\partial V} \vec{E} \times \vec{H} \cdot d\vec{S}. \quad (61)$$

Poynting's theorem (61) can be written:

$$\frac{\partial}{\partial t} \int_V \frac{1}{2} \vec{D} \cdot \vec{E} + \frac{1}{2} \vec{B} \cdot \vec{H} \, dV = - \int_V \vec{E} \cdot \vec{J} \, dV - \int_{\partial V} \vec{E} \times \vec{H} \cdot d\vec{S}.$$

The first term on the right is the rate at which the electric field does work on charges within the volume V. We then make the following interpretations:

- The expression on the left hand side of (61) represents the rate of change of energy in the electromagnetic field within the volume V. The *energy density* in the electric and magnetic fields is then $\frac{1}{2}\vec{D} \cdot \vec{E}$ and $\frac{1}{2}\vec{B} \cdot \vec{H}$, respectively.

- The second term on the right hand side represents the flux of energy in the electromagnetic field. That is, the *Poynting vector* $\vec{E} \times \vec{H}$ represents the flow of energy in the electromagnetic field per unit time per unit area perpendicular to $\vec{E} \times \vec{H}$.

Finally (for this lecture), we note that the theory of electromagnetism based on Maxwell's equations is fully consistent with Special Relativity.

In particular:

- Electric charge is a Lorentz scalar: all observers will measure the same charge on a given particle, irrespective of that particle's motion.
- The current and charge densities can be combined into a four-vector $(\vec{J}, c\rho)$ that transforms (under a Lorentz boost) the same way as the space-time four-vector $(\vec{x}, ct)$.
- Similarly, the vector and scalar potentials can be combined into a four-vector, $(\vec{A}, \phi/c)$.
- The components of the electromagnetic field can be combined into a second rank tensor, with well-defined transformation properties under a Lorentz boost.

The Lorentz transformation of any four-vector can be written in matrix form. For example, for a boost along the x axis:

$$\begin{pmatrix} x' \\ y' \\ z' \\ ct' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ ct \end{pmatrix}, \tag{62}$$

where:

$$\beta = \frac{v}{c}, \quad \text{and} \quad \gamma = \frac{1}{\sqrt{1-\beta^2}}. \tag{63}$$

The current density $(\vec{J}, c\rho)$ and four-potential $(\vec{A}, \phi/c)$ transform in exactly the same way as the space-time four-vector $(\vec{x}, ct)$.

The transformation of the electromagnetic fields is just a little more complicated...

Using index notation, we can write the transformation of a four-vector as:

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}. \quad (64)$$

where the indices μ and ν range from 1 to 4.

Note that we use the *summation convention*, so that a summation (from 1 to 4) is implied whenever an index appears twice in any term.

The transformation of the electromagnetic fields can be deduced from the transformations of the potentials, and the space-time coordinates. The result is:

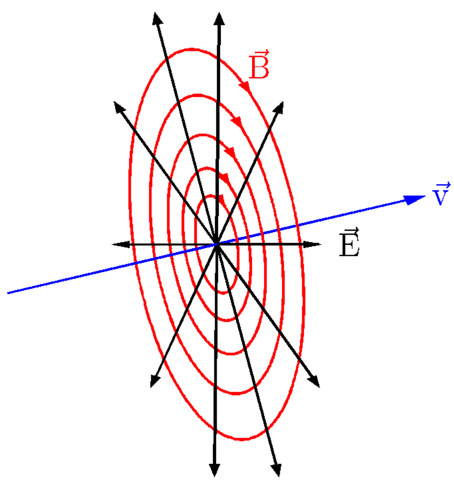
$$F'^{\mu\nu} = \Lambda^{\mu}{}_{\alpha} \Lambda^{\nu}{}_{\beta} F^{\alpha\beta}, \quad (65)$$

where $F^{\alpha\beta}$ is the matrix:

$$F = \begin{pmatrix} 0 & B_z & -B_y & -E_x/c \\ -B_z & 0 & B_x & -E_y/c \\ B_y & -B_x & 0 & -E_z/c \\ E_x/c & E_y/c & E_z/c & 0 \end{pmatrix}. \quad (66)$$

Applying the Lorentz transformation to the field around a static point charge, we find that the fields around a relativistic moving charge are "flattened" towards the plane perpendicular to the motion of the charge.

We also find a cancellation in the forces on an adjacent, co-moving charge, from the electric and magnetic fields around a moving point charge. This has important consequences for bunches of particles in accelerators, which hold together much longer at higher energies, because the repulsive Coulomb forces are mitigated by relativistic effects.



Note that any two four-vectors may be combined to form a Lorentz scalar, using the *metric tensor* g. For example:

$$x_\mu x^\mu \equiv g_{\mu\nu} x^\nu x^\mu \tag{67}$$

where:

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{68}$$

is a Lorentz scalar (invariant under a boost).

Note that the metric tensor is used to change an "up" index to a "down" index (a covariant vector to a contravariant vector). In any summation over an index, the index should appear once in the "up" position, and once in the "down" position.

To write the equations of electromagnetism in *explicitly covariant form* (i.e. in a form where the transformations under a boost are obvious), we need the four-vector differential operator:

$$\partial^\mu = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, -\frac{1}{c}\frac{\partial}{\partial t} \right). \tag{69}$$

The second-order differential operator, $\Box$, is called the d'Alembertian, and is defined by:

$$\Box \equiv \partial_\mu \partial^\mu = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}. \tag{70}$$

With the covariant quantities we have defined, Maxwell's equations may be written:

$$\partial_\mu F^{\mu\nu} = -\mu_0 J^\nu, \qquad (71)$$

$$\partial^\lambda F^{\mu\nu} + \partial^\mu F^{\nu\lambda} + \partial^\nu F^{\lambda\mu} = 0. \qquad (72)$$

The continuity equation may be written as:

$$\partial_\mu J^\mu = 0. \qquad (73)$$

The fields may be derived from the potentials using:

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu. \qquad (74)$$

The Lorenz gauge condition is:

$$\partial_\mu A^\mu = 0. \qquad (75)$$

In the Lorenz gauge, the potentials satisfy the inhomogeneous wave equation:

$$\Box A^\mu = -\mu_0 J^\mu. \qquad (76)$$

Summary

You should be able to:

- write down Maxwell's equations in differential form, and use Gauss' and Stokes' theorems to change them to integral form;
- discuss the physical interpretation of Maxwell's equations, with some examples relevant to accelerators;
- explain how the dynamics of particles in an accelerator are governed by the Lorentz force;
- state the relationship between the electromagnetic potentials and the electromagnetic field, and explain how the potentials can be used in solving electrodynamical problems;
- derive Poynting's theorem, and explain the physical significance of the different terms;
- write down the equations of electromagnetism in explicitly covariant form.