Stability and Supersymmetry

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16 May 2019

Contents

1 Introduction

This is an extended version of a talk given at the Working Seminar on Bridgeland Stability Conditions in the Department of Mathematical Sciences at the University of Liverpool in May 2019. The talk aims to explain the physical background of some of the concepts used in the context of stability conditions to an audience of mathematicians with no particular background in physics. After a brief review of how particles and fields relate to representations of the Poincaré group, we introduce Poincaré Lie superalgebras, central charges and the BPS bound. Then we investigate the stability of BPS states and use the $\mathcal{N} = 2$ $SU(2)$ gauge theory as an example of a physical theory where the space of ground states ('moduli space') separates into chambers with distinct BPS spectra, separated by 'walls of marginal stability.' Finally we augment this example by an embedding into string theory, where BPS states are realized by D-branes, and conclude with some comments on how this relates to the topics covered in the seminar.

2 Prelude: space, time and matter

2.1 Spacetime without gravity

Spacetime. Space and time are combined into *spacetime*. If effects of gravity can be neglected, spacetime is modelled by *Minkowski space* (also called Minkowski spacetime). We denote by $\mathbb{R}^{1,3}$ the vector space $\mathbb{R}^4$ equipped with a non-degenerate bilinear form η of signature $(1,3)$,

$$\eta(x,y) = -x^0y^0 + x^1y^1 + x^2y^2 + x^3y^3 = \eta_{\mu\nu}x^\mu y^\nu \ . \tag{1}$$

Index notation. We use Einstein's summation convention for the indices $\mu,\nu = 0,1,2,3$: any index which appears twice in a monomial, once as upper once as lower index, is understood to be summed over its full range. $(\eta_{\mu\nu}) = \mathrm{diag}(-1,1,1,1)$ is the matrix of the bilinear form η with respect to an orthonormal basis $\{e_\mu\}$, $\eta(e_\mu, e_\nu) = \eta_{\mu\nu}$. The non-degenerate bilinear form η allows us to identify $V = \mathbb{R}^{1,3}$ with its dual

$$V \xrightarrow{\cong} V^* \ , \quad v \mapsto \eta(v, \cdot) \ .$$

Defining the dual (or reciprocal) basis $\{e^\mu\}$ of V by

$$\eta(e_\mu, e_\nu) = \eta_{\mu\nu} \ ,$$

we can assign to each vector v contravariant components v^μ and covariant components v_μ:

$$v = v^\mu e_\mu = v_\mu e^\mu \ .$$

Using that

$$\eta(e^\mu, e^\nu) = \eta^{\mu\nu}$$

are the components of the inverse matrix of $(\eta_{\mu\nu})$ (which happens to be equal to the matrix itself), and using Einstein's summation convention, we can 'raise and lower indices'

$$V^\mu = \eta^{\mu\nu}V_\nu \ , \quad V_\mu = \eta_{\mu\nu}V^\nu \ ,$$

and have a certain flexibility in writing expressions such as scalar products:

$$\eta(v,w) = \eta_{\mu\nu}v^\mu w^\nu = \eta^{\mu\nu}v_\mu w_\nu = v_\mu w^\mu = v^\mu w_\mu \ .$$

Well formed equations in index notation satisfy the following criteria: all indices are either summation indices or free indices. Summation indices appear twice in a given monomial, once in upper once in lower position, and are understood to be summed over. Free indices must be in the same position on each monomial of a well formed equation, and it is understood that the equation holds for all values of the index. We remark that the formalism reviewed here generalises 'in the obvious way' from orthonormal to general bases of V.

Minkowski space. *Minkowski space* $\mathbb{M}$ is the affine space modelled on $\mathbb{R}^{1,3}$ (or, in general dimension on $\mathbb{R}^{1,n}$). By common abuse of notation, the same symbol $\mathbb{R}^{1,3}$ is often used for both the affine space and the underlying vector space.

The connected isometry group of Minkowski space is the connected Poincaré group $SO_0(1,3) \ltimes \mathbb{R}^4$, acting as

$$x \mapsto \Lambda x + a$$

where $\Lambda \in SO_0(1,3)$ and $a \in \mathbb{R}^4$.

Pseudo-orthogonal groups and their Lie algebras. When considered as a matrix group, $SO_0(t,s)$ is the connected component of the identity of the pseudo-orthogonal group

$$O(t,s) = \{M \in GL(t+s,\mathbb{R}) | M^T \eta M = \eta\} \;, \quad \eta = \mathrm{dia}(+1,\ldots+1,-1,\ldots,-1) \;.$$

Its Lie algebra is

$$\mathfrak{p}(t,s) = \mathfrak{so}(t,s) + \mathbb{R}^{t+s} \;.$$

Angular momentum, momentum and mass. The generators of the Lie algebra $\mathfrak{so}(1,3)$ of $SO_0(1,3)$ are denoted $M_{\mu\nu}$, $\mu < \nu$ and P_μ, where $\mu,\nu = 0,1,2,3$. $M_{\mu\nu}$ generate rotations (the rotation subgroup $SO(3) \subset SO_0(1,3)$) and 'boosts.' Boosts are 'hyperbolic rotations' in planes spanned by a timelike and a spacelike coordinate axis. The orbits in such planes are hyperbolas. In physics such a hyperbolic rotation corresponds to a transformations between frames (or 'observers'), which are moving with constant relative velocity along the spacelike axis. The generators P_μ generate translations. By the *first Noether theorem*, the invariance of a physical theory (defined by an action principle) under the action of a finite-dimensional Lie group implies the existence of a conserved quantity. The conserved quantities associated with the generators of the Poincaré group are the *relativistic angular momentum* and *relativistic momentum*. The relativistic momentum combines energy and momentum:

$$P^\mu = (E, p^1, p^2, p^3) \;,$$

where $E =$ energy, $\vec{p} = (p^1,p^2,p^3)$ is momentum and where we use units where the speed of light is $c = 1$. The components of P^μ are not independent:

$$P^\mu P_\mu = -E^2 + (p^1)^2 + (p^2)^2 + (p^3)^2 = -M^2 \;,$$

where M is the mass. For $M^2 > 0$ this relation defines a two-sheeted hyperboloid called the mass shell, and the relation between the components of P^μ is called the mass shell condition. The existence of two sheets is related to the existence of anti-particles: if a particle is distinct from its anti-particle (e.g.

electron, in fact most particles) one uses both sheets, but if a particle is identical to its anti-particle (photon) one used only one sheet. For $M^2 = 0$ the mass hyperboloid degenerates into a cone. This relates to the fact that massive and massless particles are qualitatively different. Representations with $M^2 < 0$ would correspond to particles moving faster than the speed of light ('tachyons') and are discarded on grounds of *causality*.

Representations of the Poincaré group. The irreducible unitary[1] representations of the Poincaré group as classified by two labels:

- The eigenvalue of the operator $P^\mu P_\mu$, which is related to the mass M by $M^2 = -P^\mu P_\mu$. In 'physical' representations $M^2 \geq 0$. Representations with $M^2 > 0$ are called massive representations, representations with $M = 0$ are called massless representations.

- A representation of the *little group*, which is the isotropy subgroup of the Lorentz group $SO_0(1,3)$ acting on the generators P^μ of translations.

  - For $M^2 > 0$ the little group is $SO(3)$. Its unitary irreducible representations R have dimension $0, 1, 3, 5, \ldots$, and can be labeled by the *spin* $s = 0, 1, 2, \ldots$. The dimension of the spin-s representation R_s is $\dim R_s = \frac{1}{2}s(s+1)$.
  - For $M = 0$ the little group is isomorphic to the two-dimensional Euclidean group $SO(2) \ltimes \mathbb{R}^2$. For 'physical' representations the factor $\mathbb{R}^2$ is represented trivially, so that the little group is effectively $SO(2) \cong U(1)$. Its unitary irreducible representations are one-dimensional, and are labeled by the helicity $h = 0, \pm 1, \pm 2, \ldots$.

2.2 Spacetime with gravity

If gravitational effects are relevant, spacetime is modelled by a *semi-Riemannian manifold* (M, g) with an indefinite metric of signature $(1,3)$ (in general dimension $(1,n)$). Following Einstein, gravity is not treated as a force, but encoded in the geometry of spacetime. The metric g is determined by the distribution of matter (the energy-momentum tensor) through the Einstein equations. If the particular process one is interested has negligible 'backreaction' on the spacetime geometry, one can treat g is given, and (M, g) as a non-dynamical 'background.' Minkowski spacetime is an example of a flat spacetime. It also is *maximally symmetric*, i.e. it hast the maximal number $\frac{1}{2}d(d+1)$ of isometries, where d is the dimension. A generic spacetime does not have any isometries. The other two maximally symmetric spacetimes are de Sitter space and anti-de Sitter space, which have constant positive and negative curvature, respectively. The global existence of a Lorentzian metric on a manifold is in general obstructed. For example, the only closed Riemann surface admitting a global Lorentzian metric is the torus (genus 1).

[1] We require unitary representations for consistency with quantum theory, where symmetries have to be implemented as unitary group representations.

2.3 Matter

Matter comes in two forms:

- *Particles* are discrete and localized. They are characterized by their *mass*, *spin*, and possibly *charges*. Mass and spin assign them to irreducible unitary representations of the Poincaré group. Charges are labels corresponding to further 'internal' group actions.

- *Fields* are continuous and extend over all of space. Geometrically they are sections of tensor bundles $\mathcal{T} \to M$ over spacetime M. Here tensor bundles refers to bundles which can be constructed out of the tangent bundle $TM \to M$ by taking duals and tensor products,

$$\mathcal{T} = (TM \otimes TM \otimes \cdots) \otimes (T^*M \otimes T^*M \otimes \cdots) \to M ,$$

  and applying symmetrization, anti-symmetrization and contraction. More generally fields may be sections of vector bundles of the form $\mathcal{T} \otimes V \to M$, where $V \to M$ is a vector bundle associated to a G-principal bundle over M by some G-representation. The group G contains all 'internal' group actions, where internal means any group action not acting on space-time. Examples are the group actions associated to electromagnetism and other interactions. Fields are subject to Lorentz-covariant (Lorentz-equivariant) field equations. The linear part of such a field equation selects one (or several) unitary irreducible representation of the Poincaré group. (The non-linear part encodes interactions.) Thus fields like particles carry mass and spin (and possibly charges, related to internal G-actions).

Example: The Fourier transform of the Klein-Gordon equation is the relativistic mass shell condition:

$$-\Box\phi + M^2\phi = 0 \Leftrightarrow P^\mu P_\mu + M^2 = 0 ,$$

where $\Box = \partial^\mu \partial_\mu$ is the d'Alembertian or wave operator.

Remark: Other relativistic wave equations (Maxwell equations, Dirac equation) have several components, with each component satisfying the Klein-Gordon equation as a consistency condition. The Klein-Gordon equation selects the mass (representation of the translation group), whereas the algebraic relations between the components select the spin/helicity (representation of the little group).

Remark: While the linear part of a relativistic field equation determines the mass and spin/helicity of a field, non-linear terms introduce interactions (note that linear superpositions of solutions will in general not be solutions any longer). For example

$$-\Box\phi + M^2\phi = g\phi^{N-1} ,$$

where $N \geq 3$ corresponds to a so-called ϕ^N -theory. The constant g is the so-called coupling constant which measures the strength of the interaction. In particular, $g = 0$ is the free (non-interacting) limit.

2.4 Quantum aspects

Bosons and fermions. The *states* of a quantum system are described by vectors v in a separable Hilbert space $\mathcal{H}$. Inequivalent states do not correspond to vectors $v \in \mathcal{H}$ but to *rays* $[v] = \mathbb{C}^* v \in P\mathcal{H}$. Group actions on rays must only be representations up to a factor $\omega \in \mathbb{C}^*$:

$$[R_g(R_h v)] = [R_{gh} v] \Leftrightarrow R_g R_h = \omega(g,h) R_{gh} \ , \quad \omega(g,h) \in \mathbb{C}^* \ .$$

Associativity of group multiplication implies that ω must be a two-cocycle. Representations of a group G 'up to factors $\mathbb{C}^*$' are called projective representations of G. Projective representations of G are representations of a central extension of G by $\mathbb{C}^*$. In quantum theory, group actions must be unitary. This restricts us to unitary ray representations of G, where ω takes values in $U(1)$. If the cocycle ω is trivial, a projective or ray representation is equivalent to an 'ordinary representation.' Bargmann's theory of ray representations provides a criterion for testing whether ray representations are equivalent to ordinary representations which only involves 'factors' ω close to the unit element. If this local criterion is met all ray representations of a connected, simply connected Lie group are ordinary representations, and all projective representations of its non-simply connected factor groups lift to ordinary representations of their universal cover. This criterion is met by all semi-simple groups, and, by inspection, also for the connected Poincaré group.

It is convenient to work with ordinary representations of the universal covering group $\tilde{G}$ instead of projective representations of a non-simply connected group G. In this sense the 'quantum mechanical rotation group' is not $SO(3)$, but is universal cover $\mathrm{Spin}(3) \cong SU(2)$, while the 'quantum mechanical Lorentz group' is not $SO_0(1,3)$ but $\mathrm{Spin}_0(1,3) \cong SL(2,\mathbb{C})$. In terms of spin and helicity this means that we need to allow half-integer values, $s = 0, \frac{1}{2}, 1, \ldots$ and $h = 0, \pm\frac{1}{2}, \pm 1, \ldots$. Particles with integer spin are called *bosons*, particles with half integer spin are called *fermions*. Fields with half integer spin/helicity correspond to sections of spinor bundles $S \to M$ over spacetime M. Loosely speaking, spinor bundles arise as 'square roots' of the tangent bundle (quite literally for the tangent bundle of a Riemann surface without boundary). More precisely, the tangent bundle is a $GL(d,\mathbb{R})$ vector bundle associated to the frame bundle. If we require the existence of a Lorentzian metric (which is in general obstructed), the structure group can be reduced to $SO_0(1,3)$. One can then try to lift the transition functions of the tangent bundle to $\mathrm{Spin}_0(1,3)$ and thus obtain a $\mathrm{Spin}_0(1,3)$ vector bundle S, In general, it is not possible to perform this lifting consistently globally on M. The obstruction is measured by certain Stiefel-Whitney classes. Manifolds which admit spin bundles are called *spin manifolds*.

Particles and fields. By combining quantum theory with special relativity (that is, by formulating quantum theory as a theory on Minkowski space with the Poincaré group as invariance group) we obtain (relativistic) *quantum field theory*. In quantum field theory there is no fundamental distinction between particles and fields, which just correspond to different types of states. The states of a single particle of given mass, spin (and possible charges) is described by its one-particle Hilbert space $\mathcal{H} \cong L^2(\mathbb{R}^3, \mathbb{C}^N, d\mu)$. The elements of $\mathcal{H}$ can be interpreted as 'momentum-space' wave functions, that is they encode the probability distribution for measuring a particular value of the momentum $\vec{p} \in \mathbb{R}^3$. The multiple components $\mathbb{C}^N$ encode a unitary irreducible representation of the Poincaré group determined by the mass and spin/helicity of the particle, plus possibly further internal G-representations corresponding to its charges. $d\mu$ is the Lorentz invariant measure on the mass shell $P^\mu P_\mu = -M^2$

Multi-particle states for a given particle species (same mass, spin and charges) are obtained by taking tensor products of one-particle Hilbert spaces $\mathcal{H}$. Since quantum mechanical particles are *indistinguishable*, one does not take the full tensor product, but either its symmetric or its antisymmetric part. To avoid causality-violating effects which propagate faster than the speed of light, one must take the symmetric tensor product for bosons (integer spin) and the anti-symmetric tensor product for fermions (half-integer spin).

$$\mathcal{F}_{\text{boson}} = \mathbb{C} \oplus \mathcal{H} \oplus (\mathcal{H} \otimes_s \mathcal{H}) \oplus \cdots , \tag{2}$$

$$\mathcal{F}_{\text{fermion}} = \mathbb{C} \oplus \mathcal{H} \oplus (\mathcal{H} \wedge \mathcal{H}) \oplus \cdots . \tag{3}$$

This incorporates the *Pauli principle* which states that two fermsions can never occupy the same state (have the same one-particle wave function). This principle explains the stability of atoms and of atomic matter. Without it molecules and by extension cells and living organisms could not exist, because electrons would preferably[2] occupy their lowest energy state. However the existence of molecules and and other compounds of atoms relies on electrons being forced to occupy higher states.

The multiparticle Hilbert spaces $\mathcal{F}_{\text{boson/fermion}}$ are called Fock spaces. The above decomposition corresponds to eigenstates of the particle number: $\mathbb{C}$ contains the zero-particle state ('vacuum'), $\mathcal{H}$ one-particle states, and $\mathcal{H}^{\otimes_s N}$ and $\mathcal{H}^{\wedge N}$ N-particle states. General states do not have a fixed particle number and therefore have a non-trivial projection onto more than one 'particle number sector.' [3] What is called 'fields' in classical physics, that is smooth extended distributions of energy and momentum, corresponds to so-called coherent quantum states which have contributions from all particle numbers N. Fock spaces arise naturally when 'quantizing' a classical field theory.

[2] At zero temperature all electrons would sit in the ground state. At finite temperature one would obtain a thermal distribution favouring low energy states.

[3] Fineprint: this only holds if there is no 'central observable' defining 'superselection sectors'. In this case sums of states with different particle number are 'mixed states' rather than 'pure states.' Electrical charge is an example of a central observable defining superselection sectors.

3 Supersymmetry, BPS states and central extensions

3.1 Poincaré Lie superalgebras

There are two types of symmetries in a quantum (field) theory: space-time symmetries, that is the Poincaré group, and internal symmetries. One natural question is whether both types of symmetries could be part of a single simple group, thus 'unifying' all symmetries. A no-go theorem of Coleman and Mandula states that if in a quantum field theory the Poincaré group and internal groups do not form a direct product, the theory must be non-interacting.[4] A later theorem by Haag, Sohnius and Lopuszanzki showed that the no-go theorem can be circumvented if one allows a more general concept of symmetry, dubbed supersymmetry, which uses a 'graded generalisation' of the concepts of Lie group and Lie algebra. We take the Lie algebra point of view in the following.

We denote by $V = \mathbb{R}^{t,s}$ the standard real vector space of dimension $t+s$, equipped with a bilinear form of signature (t,s). We may use same symbol for the affine space over V. The Lie algebra of isometries of the affine space V is the Poincaré Lie algebra:

$$\mathfrak{p}(V) = so(V) \oplus V \; .$$

Its extension to a Lie superalgebra, is called a Poincaré Lie superalgebra. As $\mathbb{Z}_2$-graded vector spaces Poincaré Lie superalgebras take the form

$$\mathfrak{sp}(V,S) := (\mathfrak{so}(V) \oplus V) \oplus S = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \; .$$

where S is a spinor module which may be reducible. The real spinor module $S_{\mathbb{R}}$ is the Spin(V)-module obtained by restricting an irreducible module of the real Clifford algebra. The complex spinor module $\mathbb{S}$ is the Spin(V)-module obtained by restricting an irreducible module of the complex Clifford algebra $\mathbb{C}l(V) = Cl(V) \otimes \mathbb{C}$. For $V = \mathbb{R}^{1,3}$

$$Cl_{1,3} \cong \mathbb{R}(4) \Rightarrow S_{\mathbb{R}} = \mathbb{R}^4$$

which is irreducible as a Spin representation ('Majorana spinors'), and

$$\mathbb{C}l_4 \cong \mathbb{C}(4) \Rightarrow \mathbb{S} = \mathbb{C}^4 \; ,$$

which is reducible as a Spin representation ('Dirac spinors'), $\mathbb{S} \cong S_{\mathbb{R}} \oplus S_{\mathbb{R}}$. For $V = \mathbb{R}^{t,s}$ the most general choice for S is

$$S = \bigoplus_{i=1}^{\mathcal{N}} S_{\mathbb{R}} \; .$$

The case $\mathcal{N} = 1$ is called *minimal supersymmetry*, while the case $\mathcal{N} > 1$, where several copies of the irreducible Spin module are used, is called *extended supersymmetry.*

[4]More precisely, the asymptotic time evolution operator ('S-matrix') is the identity.

Following physicist's conventions, we denote antisymmetric brackets (such as commutators in the case of an underlying associative algebra) by $[\cdot,\cdot]$, and symmetric brackets (such as anti-commutators in the case of an underlying associative algebra) by $\{\cdot,\cdot\}$. A Lie superalgebra structure requires:

$$[\mathfrak{g}_0,\mathfrak{g}_0]\subset\mathfrak{g}_0\;,\quad[\mathfrak{g}_0,\mathfrak{g}_1]\subset\mathfrak{g}_1\;,\quad\{\mathfrak{g}_1,\mathfrak{g}_1\}\subset\mathfrak{g}_0\;.$$

The bracket on $\mathfrak{g}_0$ is the one of the Poincaré Lie algebra. The bracket $[\mathfrak{so}(V),S]$) is fixed by S being an $\mathfrak{so}(\mathfrak{V})\cong\mathfrak{spin}(V)$ module. Further constraints arise from the $\mathbb{Z}_2$-graded version of the Jacobi identity. The bracket between V and S must be trivial, and it remains to specify

$$\{S,S\}\subset V$$

This is equivalent to specifying a real, symmetric, $\mathfrak{spin}(V)$ equivariant bilinear form on the spinor module S. All such forms have been classified for arbitrary signature (t,s) and spinor module S.

The Lie superbracket extends the Lie bracket on $\mathfrak{g}_0$ to bracket with is $\mathbb{Z}_2$-graded symmetry: antisymmetric on $\mathfrak{g}_0$ and between $\mathfrak{g}_0$ and $\mathfrak{g}_1$, but symmetric on $\mathfrak{g}_1$. The bracket between $so(V)$ and S is fixed by S being an $so(V)$ module. It turns out that bracket has to vanish between V and S, so that the only choice one has to make is a the bracket on S. It turns out that this has to be valued in V ('supertransformations are square roots of translations'), and that the extension of $\mathfrak{p}(V)$ to a Lie superalgebra $\mathfrak{sp}(V)$ is equivalent to the choice of a real, symmetric, vector-valued, $\mathfrak{spin}(V)$-equivariant bilinear form on the spinor module S:

$$\{\cdot,\cdot\}\;:S\times S\to V\;.$$

N-extended supersymmetry algebra based on $V=\mathbb{R}^{1,3}$. In physicist's notation the symmetric bracket $S\times S\to V$ is specified as

$$\{Q^i_\alpha,Q^j_\beta\}=(C\gamma^\mu)_{\alpha\beta}P_\mu\delta^{ij}\;.$$

and this is referred to as the 'supersymmetry algebra' (or 'supertranslation algebra'). The *supercharges* $Q^i_\alpha, i=1,\ldots,N,\alpha=1,\ldots,4$ are the generators of $\mathfrak{sp}(V)$ belonging to S, while P_μ, $\mu=0,1,2,3$ are the components of the relativistic momentum, aka generators of translations. γ^μ are the Dirac-γ matrices, which represent the generators of $Cl(V)$ on $S_\mathbb{R}$. Finally C is the so-called charge conjugation matrix, which relates particles and anti-particles. The matrix C is either symmetry or antisymmetric, and has the property that the matrix $C\gamma^\mu$ is symmetric for all μ. This is necessary in order that the rhs is symmetric in the multi-indices (α,i), (β,j), as required to define a symmetric bracket.

We can rephrase this as follows. It is sufficient to consider the minimal case $\mathcal{N}=1$, which is equivalent to taking $i=j$ fixed. We need a real, vector-valued, symmetric, $\mathfrak{spin}(V)$-equivariant bilinear form $\Pi\in(\mathrm{Sym}(S^*\times S^*)\times V)^{\mathfrak{spin}(V)}$, so that we can define

$$\{s,t\}=\Pi(s,t)\;,\quad\forall s,t\in S\;.$$

It can be shown that vector space $(\mathrm{Sym}(S^* \times S^*) \times V)^{\mathfrak{spin}(V)}$ is spanned by vector-valued bilinear forms Π_β which are defined in terms of so-called super-admissible bilinear forms S by

$$\langle \Pi(s,t), v\rangle = \beta(vs,t)\ , \quad \forall s,t \in S, \forall v \in V$$

where $\langle\cdot,\cdot\rangle$ is the scalar product on V, and where vs denote Clifford multiplication, that is the action of $V \subset Cl(V)$ on its module S. Now introduce bases Q_α, P_μ for S, V and expand $s = s^\alpha Q_\alpha$, $t = t^\alpha Q_\alpha$, $v = v^\mu P_\mu$. Then

$$\Pi(s,t) = s^\alpha t^\beta \{Q_\alpha, Q_\beta\} = s^\alpha t^\beta M^\mu_{\alpha\beta} P_\mu$$

where $M^\mu_{\alpha\beta}$ are the coefficients of Π with respect to the basis. Taking the scalar product with v gives

$$\langle \Pi(s,t), v\rangle = s^\alpha t^\beta v_\mu M^\mu_{\alpha\beta}\ .$$

Since Π is obtained by taking a scalar bilinear form β and performing Clifford multiplication in the first argument, the coefficients $M^\mu_{\alpha\beta}$ take the special form[5]

$$M^\mu_{\alpha\beta} = (C\gamma^\mu)_{\alpha\beta}$$

where C is the matrix of the bilinear form β. Therefore

$$\{Q_\alpha, Q_\beta\} = M^\mu_{\alpha\beta} P_\mu = (C\gamma^\mu)_{\alpha\beta} P_\mu$$

The standard choice for β is the bilinear form defined by the charge conjugation matrix C. This matrix defines an isomorphism $\mathbb{S} \xrightarrow{\cong} \mathbb{S}^*$ from the complex spinor module to its dual. Therefore it also defines a complex, bilinear form

$$\beta_C : \mathbb{S} \times \mathbb{S} \to \mathbb{C}\ ,$$

which can be shown to be either symmetric or antisymmetric, and to be Spin invariant. Moreover this bilinear form is super-admissible, meaning that it defines a complex, symmetric, Spin equivariant bilinear form

$$\Pi_C\ : \mathbb{S} \times \mathbb{S} \to V \otimes \mathbb{C}\ .$$

Using that $V \cong V^*$ we can write this as

$$\Pi_C(\gamma.,\cdot,\cdot)\ : V \otimes \mathbb{C} \times \mathbb{S} \times \mathbb{S} \to \mathbb{C}\ , \quad (v,s,t) \mapsto \beta_C(vs,t) = \beta_C(\gamma_v s, t)$$

where $vs = \gamma_v s = v_\mu \gamma^\mu s$ denotes Clifford multiplication.

The complex spinor module $\mathbb{S}$ always carries at least either a Spin-invariant quaternionic structure or a Spin-invariant real structure. For $V = \mathbb{R}^{1,3}$ there exists a Spin-invariant real structure ρ, whose fixed points can be identified with

[5] Depending on conventions for index notation, this term might take different forms. In my favourite convention it's actually not $(C\gamma^\mu)_{\alpha\beta}$ but $(\gamma^\mu C^{-1})_{\alpha\beta}$ ...

the real spinor module, $\mathbb{S}^\rho \cong S_\mathbb{R}$. This allow to restrict Π_C to real, symmetric, vector-valued, Spin-equivariant bilinear form on $S_\mathbb{R}$:

$$(\Pi_C)_{|S_\mathbb{R} \times S_\mathbb{R}} \;: S_\mathbb{R} \times S_\mathbb{R} \to V \;.$$

This then defines the minimal four-dimensional supersymmetry algebra. For the '$\mathcal{N}$-extended supersymmetry algebra' with $\mathcal{N} > 1$ one essentially takes $\mathcal{N}$ copies.

Remark: Restriction to $\mathcal{N} \leq 8$ Mathematically any positive integer $\mathcal{N}$ makes sense, but for theories with $\mathcal{N} > 8$ the smallest massless representation on fields contains fields of spin larger than 2. For such theories Poincaré covariant, unitary, interacting field equations can only be formulating by using infinitely many fields ('higher spin theories,' possibly a zero tension limit of string theory). Similar remarks apply to theories in dimension $D > 11$. In general dimension D, the realisation of supersymmetry in field theories requires to limit the number of real supercharges to 16 without coupling to gravity and to 32 with coupling to gravity. In four dimension this implies $\mathcal{N} \leq 8$, in $D = 11$ there is a unique theory, eleven-dimensional supergravity.

3.2 Central charges and the BPS bound

Supersymmetry and Stability. Simple supersymmetry implies the positivity of energy, $E \geq 0$. If the supersymmetry algebra admits a central extension (which happens for $\mathcal{N} > 1$, then supersymmetry implies a stronger bound: the mass M of any state is bounded from below by the modulus of its *central charge* Z, that is the eigenvalue of the central operator, likewise denoted Z (since on irreducible representation Z is proportional to the identity):

$$M \geq |Z| \;.$$

States which saturate this 'BPS bound'

$$M = |Z|$$

are called BPS states. They can only decay into other BPS states (semi-stability in the terminology of the seminar), and transform in short or BPS representations of the supersymmetry algebra, where part of the generators at trivially. Many supersymmetric theories do not have a unique ground state, but a 'moduli space of vacua' $\mathcal{M}$. Then the central charge in general depends on the choice of the ground states, and varies if we deform along $\mathcal{M}$. A subset of the BPS states is absolutely stable, that is they cannot even decay into other BPS states (stability in the terminology of the seminar). Upon variation over $\mathcal{M}$ we may encounter special loci (hypersurface) where such stable states become unstable. Such hypersurfaces are called *walls of marginal stability*, and they decompose $\mathcal{M}$ into *chambers*. Upon deformation through such a wall the BPS spectrum changes, that is *wall crossing* removes or adds states to the BPS spectrum. In the following we provide examples.

Toy example: Supersymmetric quantum mechanics. Consider the algebra:

$$2Q^2 = \{Q, Q\} = H \ .$$

Here H is the Hamilton operator (generator of time-translations, an essentially self-adjoint operator on a separable Hilbert space $\mathcal{H}$). The supercharge $Q = Q^\dagger$, is a Hermitian operator, which is the 'square root' of the Hamiltonian. Suppose the representation of this algebra on $\mathcal{H}$ is unitary, and let v be an energy eigenstate, $Hv = E_v v$. Then:

$$0 \leq 2(Qv, Qv) = 2(v, Q^\dagger Q v) = (v, 2Q^2 v) = (v, Hv) = E_v(v, v) \Rightarrow E_v \geq 0 \ ,$$

since we assumed unitarity. The spectrum of the Hamiltonian is non-negative.

Real life example: 'Simple' ($\mathcal{N} = 1$) supersymmetry in $d = 4$ and the positivity of energy.

$$\{Q_\alpha, Q_\beta\} = (C\gamma^\mu)_{\alpha\beta} P_\mu \ .$$

Consider a massive representation of $\mathfrak{p}(1,3)$. Then $P^\mu P_\mu = -M^2$, where $M^2 > 0$. By an $SO_0(1,3)$ transformation P_μ-eigenstates can be brought to the from $(M, 0, 0, 0)$ ('rest frame') and one obtains the bound

$$M \geq 0 \ .$$

Thus masses are non-negative in supersymmetric theories. More generally, energy is non-negative, because if we consider multi-particle systems, M is the total energy measured in the centre of mass frame. A famous application is Witten's proof of the Schoen-Yau positivity theorem for the ADM mass in general relativity. Note that this proof does not depend on nature being actually supersymmetric, it just uses that Einstein gravity can be embedded into a supersymmetric theory. A key concept used in the proof is the one of a Killing spinor, which is analogues to the one of a Killing vector. This concept is widely used in spin geometry.

Toy example: A stripped down version of four-dimensional $N = 2$ supersymmetry. Consider the algebra[6]

$$\{Q_i, Q_j^\dagger\} = \delta_{ij} H \ , \quad i, j = 1, 2 \ ,$$

$$\{Q_1, Q_2\} = Z = \{Q_1^\dagger, Q_2^\dagger\} \ .$$

where $H = H^\dagger$ is the Hamiltonian, and where $Z = Z^\dagger$ is central, $[Z, Q_i] = [Z, H] = 0$. Z is also called a *central charge*. By Schur's Lemma central operators act as multiplets of the identity on irreducible representations. Taking the linear combinations

$$a = \frac{1}{\sqrt{2}}(Q_1 + Q_2^\dagger) \ , \quad b = \frac{1}{\sqrt{2}}(Q_1 - Q_2^\dagger) \ ,$$

[6] While so far supercharges were real Lie superalgebra generators, here I work with complex generators which (under the hood) are complex linear combinations of real generators. This is natural for the $N = 2$ superalgebra since $S_\mathbb{R} \oplus S_\mathbb{R} \cong \mathbb{S}$. In this example Z is Hermitan (real eigenvalues), for (my) convenience. In examples with interesting stability properties Z has a complex spectrum.

the algebra can be rewritten:[7]

$$\{a, a^\dagger\} = H + Z \ , \quad \{b, b^\dagger\} = H - Z \ ,$$

where $\{a, a\} = 0$, etc. Then on unitary, irreducible representations, for an energy eigenstate v:

$$0 \leq ((a^\dagger \pm a)v, (a^\dagger \pm a)v) = (v, \{a, a^\dagger\}v) = (v, (H + Z)v) = (E_v + Z)(v, v)$$

Here we used that Z has the same eigenvalue on all states of an irreducible representation. For $v \neq 0$ it follows that $E_v + Z \geq 0$. Therefore $H \geq -Z$. A similar computation for $b, b^\dagger$ gives $H \geq Z$. This is the BPS bound:

$$H \geq |Z| > 0 \ .$$

The spectrum of the Hamiltonian is bounded from below by the modulus of the central charge.

Thus we have examples of the following central concepts.

- The *BPS bound*: the values of 'central charges' on irreducible representations provide a lower bound for the eigenvalues of the energy:

  $$H \geq |Z| \ .$$

- *BPS states* are states which saturate the BPS bound, $Hv = Zv$. They are invariant under part of the supersymmetry algebra. Assume without loss of generality that $Z > 0$. Then

  $$0 = \ (v, (H - Z)v) = (v, \{b, b^\dagger\}v) = ((b^\dagger \pm b)v, (b^\dagger \pm b)v) \ .$$

  By unitarity this implies $(b \pm b^\dagger)v = 0$ for $v \neq 0$, and therefore $bv = b^\dagger v = 0$. Thus $b, b^\dagger$ act trivially (as multiplication by zero) on BPS states.

Real life case: Extended $\mathcal{N} = 2$ supersymmetry in $d = 4$, central charges and the BPS bound. $\mathcal{N} = 2$ supersymmetry algebra in $d = 4$ is:

$$\{Q^i_\alpha, Q^j_\beta\} = (C\gamma^\mu)_{\alpha\beta} P_\mu \delta^{ij} \ , \quad i, j = 1, 2 \ .$$

This algebra admits a complex central extension. The $\mathrm{Spin}_0(1,3)$ decomposition on the left hand side:

$$(4 \times 4)^{\mathrm{sym}} = 10 = V + \Lambda^0 V + \Lambda^{\max} V + \cdots$$

where apart from V we have only shown irreducible representations that transform trivially under $\mathrm{Spin}_0(1,3)$. Of course, on needs to make sure that the superbracket can be extended consistently. The centrally extended $N = 2$ supersymmetry algebra is:

$$\{Q^i_\alpha, Q^j_\beta\} = (C\gamma^\mu)_{\alpha\beta} P_\mu \delta^{ij} + QC_{\alpha\beta}\varepsilon^{ij} + P(C\gamma_*)_{\alpha\beta}\varepsilon^{ij}$$

[7] We remark that this is a Clifford algebra written in terms of 'fermionic ladder operators.'

where P, Q are Hermitian and commute with all other generators, where

$$(\varepsilon^{ij}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and where $-\gamma_* \propto \gamma_0\gamma_1\gamma_2\gamma_3$ represents the generator $\Lambda^{\mathrm{top}} V \subset Cl_{1,3}$. To check that the algebra is consistent, we remark that in the representation we have chosen $C_{\alpha\beta}$ and $(C\gamma_*)_{\alpha\beta}$ are antisymmetric. Thus all terms on r.h.s. are symmetric in the multi-indices (α, i), (β, j), as required.

Working out the representation theory leads to a structure analogous to the previous toy example. In particular, the BPS bound for massive representations is:

$$M \geq |Z| = \sqrt{Q^2 + P^2} \;,$$

where $Z = Q + iP$ is the complex central charge of the representation. One example of a BPS representation is $\mathcal{N} = 2$ (BPS or short) vector multiplet, which for $M = |Z| = 0$ turns into the massless vector multiplet. Vector multiplets contain a vector field, two fermions of Spin 1/2, called gaugini, and a scalar field. In more detail they have the following field content:

- Massless vector multiplet.

Field	Helicity content	degrees of freedom
Vector field A	$h = \pm 1$	2
2 fermions λ^1, λ^2	$h = \pm\frac{1}{2}, \pm\frac{1}{2}$	4
1 complex scalar a	$h = 0, 0$	2

- Short massive (BPS) vector multiplet.

Field	Spin content	degrees of freedom
Vector field W	$s = 1$	3
2 fermions λ^1, λ^2	$s = \frac{1}{2}, \frac{1}{2}$	4
1 real scalar r	$h = 0$	1

Massive and massless vector multiplets are related by a *Higgs mechanism.* Note that massive and massless vector fields have a different number of degrees of freedom, which is compensated for by the scalar degrees of freedom. A massive vector field transform in the three-dimensional spin one representation of the little group $SO(3)$, a massless vector corresponds to two irreducible representations of the little group $SO(2)$. Physically the difference is that massless vector particles propagate with the speed of light and only have two degrees of freedom transverse to the direction of motion, while a massive vector particle propagates slower than the speed of light and has a third, longitudinal degree of freedom. To make a massless vector massive we need a scalar to provide the additional degree of freedom. This is the (one version of) the Higgs effect.

3.3 Stability of BPS states

We now turn to the question under which conditions BPS particles can be unstable and decay into other particles. As an example we start with a theory that is a supersymmetric extension of Maxwell theory. It contains a massless vector multiplet with vector field A, which we will call the Maxwell vector multiplet. The couplings between the components of the multiplet are encoded in a holomorphic function $F(a)$, where a is the complex scalar field. This theory does not have a unique ground state. Ground stables are labeled by the constant part of the scalar field, called its vacuum expectation value, which we also denote a. The vacuum expectation value a is a local coordinate on $\mathcal{M}$, but not necessarily the most convenient. We denote the coordinate we choose on $\mathcal{M}$ by u.

We now consider add states in this theory which carry electric charge q or magnetic charge p with respect to the vector field A. 'Carrying charge' means that these states interact in a certain with the vector field A. Otherwise we only assume that these combine into BPS representations of the $\mathcal{N} = 2$ supersymmetry algebra, and that the interaction terms which couple them to the massless vector multiplet are supersymmetry invariant. While we do not specify any further details, we remark that there are many $\mathcal{N} = 2$ supersymmetric theories of this type. Examples that we will come to later are the $\mathcal{N} = 2$ supersymmetric $SU(2)$ gauge theory studied famously by Seiberg and Witten, and type II string theory on space-times of the form $\mathbb{R}^{1,3} \times X$, where X is a Calabi-Yau threefold ('compactification of type II string theory on X, i.e. we take the view point that X is 'small' so the effectively spacetime is four-dimensional.)

Under the stated assumptions states carry a central charge Z which depends on the magnetic and electric charge p, q, and, through $F(a)$ on the choice of a ground state:

$$Z = pa_D(u) + qa(u) \ ,$$

where

$$a_D = F'(a) \ ,$$

and where $a(u)$ is the vacuum expectation value of the complex scalar in the Maxwell vector multiplet.[8]

Consider now three BPS states with charges (p, q), (p_1, q_1) and (p_2, q_2). Can the particle with charges (p, q) decay into the others? Charge conservation implies

$$q = q_1 + q_2 \ , \quad p = p_1 + p_2 \quad \Rightarrow Z = Z_1 + Z_2 \ .$$

For the masses $M = |Z|$, $M_i = |Z_i|$, $i = 1, 2$ this implies

$$M = |Z| = |Z_1 + Z_2| \leq |Z_1| + |Z_2| = M_1 + M_2 \ .$$

[8] We will see later that generically all states except those in the Maxwell vector multiplet are massless, and thus are 'excited states'. They do not enter into the definition of the ground state, except through interactions that have already been accounted for in the prepotential F. We will see what this means concretely later.

However, energy conservation allows the decay only if

$$M \geq M_1 = M_2 \ .$$

Therefore decay is marginally possible if

$$M = M_1 + M_2 \ .$$

Since we must have $Z = Z_1 + Z_2$ and $M = M_1 + M_2$ a decay is only possible if the Z, Z_1, Z_2 are collinear (as two-component real vectors). A further constraint comes from *charge quantization* (Dirac quantization): the allowed values of electric charges lie on a lattice $\Gamma \cong \mathbb{Z} \oplus \mathbb{Z}$. We choose a normalization where $p, q \in \mathbb{Z}$. There are two cases:

1. $a_D/a \notin \mathbb{R}$. This is the generic situation (meaning: for a generic choice of the prepotential this is the situation at a generic point on $\mathcal{M}$). Then a decay is only possible if p, q are co-prime. If we draw the allowed central charges as a lattice embedded into $\mathbb{C}$, then decays are possible if the BPS states are on the same line through the origin. The lattice points closest to the origin have co-prime entries and minimal mass among the points on their line, hence they cannot decay.

   Example: $a_D = 1$, $a = i$. Then the state $(p,q) = (mp', mq')$ where $m \in \mathbb{Z}^{>0}$ can decay into m particles of charge (p', q'). States with co-prime (p,q) like $(1,n)$, $n \in \mathbb{Z}$ cannot decay.

   Along the axes the stable states are $(\pm 1, 0)$ and $(0, \pm 1)$. One decay which we will interested in later is $(1,0) \to (1,-1) + (0,1)$. This is consistent with charge conservation, but not possible for $a_D/a \notin \mathbb{R}$.

2. $a_D/a \in \mathbb{R}$. This situation is special (occuring for generic prepotentials along 'curves of marginal stability'). The conditions for decay become less restrictive, because the lattice of central charges collapses to a line. States that otherwise would be stable may now be able to decay. As an example, consider $a_D = 1$, $a = \sqrt{2}$. Then the decay $(1,0) \to (1,-1) + (0,1)$ is possible.

This resulting picture is that the ground state manifolds decomposes into chambers which are separated by 'walls of marginal stability.' Within each chamber the spectrum of BPS states does not change when we move among vacua. When reaching a wall of marginal stability, some BPS states become unstable and decay, while new BPS states may occur, and therefore the chamber on the other side has a different spectrum of BPS states. Our example was one-dimensional, but it extends to higher-dimensional ground state manifolds, where BPS states carry charges (p_i, q^i), $i = 1, \ldots, N_V$ under several Maxwell vector multiplets.

Remark: Multiple BPS bounds. In theories with higher supersymmetry $2 < N \leq 8$, we can have several charges and resulting BPS bounds:

$$M \geq |Z_1| \geq |Z_2| \geq \cdots \geq 0 \ .$$

Depending on how many bounds are saturated, there are several types of BPS representations. The smallest BPS representations are always as large as massless representations.

Remark: Supersymmetry algebras with $N > 8$ are well defined as algebras, but any realisation as a filed theory requires to admit fields with spin $s > 2$. Interacting unitary field theories seem to require infinity many fields. Such 'higher-spin' field theories fall outside the standard framework of field theory, though they may be related to the zero-tension limit of string theory.

3.4 Main example: the Seiberg Witten solution of $N = 2$ $SU(2)$ gauge theory

3.4.1 Introduction of the model

As an example we take the $\mathcal{N} = 2$ $SU(2)$ gauge theory, which contains 3 vector multiplets which transform in the adjoint representation of the 'gauge group' $SU(2)$. The three vector multiplets are labeled by their charges under the subgroup (maximal torus) $U(1) \subset SU(2)$. In the following we represent vector muliplets by their highest (spin) components, the vector fields A, W^+, W^-:

Vector multiplet	Electric charge q	Magnetic charge q
A	0	0
W^+	1	0
W^-	-1	0

The moduli space $\mathcal{M}$ of this theory is locally parametrized by the vacuum expectation value a of the complex scalar in the Maxwell vector multiplet A. Since vacua related by $a \to -a$ we can use $u := \frac{1}{2}a^2$ to parametrize $\mathcal{M} = \mathbb{C}$. It turns out that for $a \neq 0$ the $W^{\pm}$-multiplets are massive. We can then use an effective description where all relevant information is encoded in the prepotential $F(a)$ of the Maxwell vector multiplet.

3.4.2 The free theory

In the free (non-interacting limit) the prepotential is

$$F(a) = \frac{1}{2}\tau_0 a^2 ,$$

where $\tau_0 \in \mathbb{C}$ is a complex constant. Parametrizing τ_0 in the form

$$\tau_0 = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2} ,$$

we obtain two real constants: g is the *coupling constant* which (in the full, interacting theory) measures the strength of interactions, θ is the 'θ-angle' which appears as coefficient of the 'topological term' $\int_{\mathbb{R}^{1,3}} \theta F \wedge F$, where $F \in \Omega^2(\mathbb{R}^{1,3})$

is closed; therefore $\int_{\mathbb{R}^{1,3}} \theta F \wedge F$ is a boundary term which does not contribute to the equations of motion (as long as θ is constant).

The central charge of a BPS state in this theory is

$$Z_{(p,q)} = pa_D + qa$$

where

$$a_D = F'(a) = \frac{dF}{da} .$$

In the free theory

$$Z_{(p,q)} = p\tau_0 a + qa \Rightarrow M_{(p,q)} = |p\tau_0 + qa|$$

The vector multiplet A has charges $(p,q) = (0,0)$ and is massless for all $a \in \mathbb{C}$. The vector multiplets $W^{\pm}$ have charges $(p,q) = (0,\pm 1)$ and mass $M_{(0,\pm 1)} = |a|$. They are massless for $a = 0$ but massive for $a \neq 0$. This is an example of the Higgs mechanism.

The full $SU(2)$ theory also contains 'solitons' that is extended field configurations which solve the field equations and are particle-like, in the sense that they are localized and have finite mass, spin and charges. These include monopoles with charges $(p,q) = (\pm 1, 0)$ and dyons with charges $(p,q) = (\pm 1, \pm 1)$.[9] They are part of complete BPS supersymmetry representations, namely hypermultiplets which consist of 4 real scalars and two fermions. If we set $\theta = 0$, and use that in an expansion around the free limit $0 < g \ll 1$, their masses and are

$$M_{(\pm 1,0)} \approx M_{(\pm 1,\pm 1)} \approx \frac{4\pi}{g}|a| .$$

As long as $a \neq 0$, monopoles and dyons are much heavier as as the $W^{\pm}$ vector multiplets, since $g^{-1} \gg 1$. Below will see how this picture is modified by interactions, and ask whether whether BPS states are stable for all $a \in \mathbb{C}$.

3.4.3 The 'perturbative' theory, and a problem

We now turn to the interacting theory. Perturbation theory allows to compute corrections around the free limit as power series in the coupling constant g. It is widely believed that perturbation theory, and supported by empirical evidence, that perturbation theory is a valid asymptotic expansion, but it is also expected that it has zero radius of convergence. For the $N = 2$ $SU(2)$ gauge theory, the perturbative corrections can be computed exactly. The prepotential and its derivatives are

$$\begin{aligned} F(a) &= \frac{i}{2\pi} a^2 \ln \frac{a^2}{\Lambda^2} \\ a_D = F'(a) &= \frac{i}{\pi} a \left(\ln \frac{a^2}{\Lambda^2} + 1 \right) \\ \tau(a) &:= \frac{i}{\pi} \left(\ln \frac{a^2}{\Lambda^2} + 3 \right) =: \frac{\theta(a)}{2\pi} + \frac{4\pi i}{g^2(a)} \end{aligned}$$

[9] There is an infinite tower of dyons, but here we only need the lightest such states.

In the 'perturbative theory', the couplings τ, g, θ have become functions of the vacuum expectation value a. In general, through the procedure of *perturbative renormalization*, couplings become 'running couplings' which depend on the energy scale at which they are defined or measured. In our description this scale dependence is expressed through the vacuum expectation a and the scale Λ, at which the coupling g diverges,

$$g(a) \xrightarrow{a \to \Lambda} \infty \; .$$

Perturbation theory is believed to be an asymptotic expansion with zero radius of convergence. In the case at hand it is valid for $|a| \gg 1$. While we cannot draw conclusions of what exactly happens in the strong 'coupling region' of small $|a|$, we can probe the effect of the singularity (branch point) at $a = 0$ by computing the monodromy of $(a_D(u), a(u))$ along a loop around $a = 0$ inside the region $|a| \gg 1$. Using $u = \frac{1}{2}a^2$ as coordinate on $\mathcal{M} = \mathbb{C}\backslash\{0\}$ we find:

$$\begin{pmatrix} a_D \\ a \end{pmatrix} \xrightarrow{u \to e^{2\pi i} u} \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix} = M_\infty \begin{pmatrix} a_D \\ a \end{pmatrix} \; .$$

This monodromy is consistent with the singularity at $a = 0$ being due to the $W^\pm$ vector multiplets becoming massless at this point.[10] Under the transformation M_∞ the vector (a_D, a) changes, but this is a symmetry transformation in the sense that the theory is mapped back to itself, up to re-labeling states.

One can show that the picture that perturbation theory suggests for $\mathcal{M}$ is qualitatively wrong. The reason is that $\mathcal{N} = 2$ supersymmetry requires that

$$ds^2 = \mathrm{Im}\tau(a) da d\bar{a}$$

is a positive definite ('special' Kähler) metric on $\mathcal{M}$. The positivity of $\mathrm{Im}\tau$ also follows from the natural requirement that the coupling $g(a)$ should be real. One can check that $\mathrm{Im}\tau(a)$, as computed in perturbation theory is indeed positive for $|a| \gg 1$. But as the imaginary part of a holomorphic function it is harmonic and cannot have a minimum on the complex plane. Therefore $\mathrm{Im}\tau$ must become negative for sufficiently small $|a|$.

3.4.4 The Seiberg-Witten solution

Since no consistent picture of the dynamics of the theory arises from the perturbative analysis, 'non-perturbative' aspects seem to be relevant. The class of non-perturbative effects that one expects to be present are *instantons*, which in the functional integral formulation correspond to non-trivial saddle points.

[10] In perturbation theory logarithmic expressions for running couplings encode the charges of intermediate states. We are working in the set-up of an effective field theory, whose definition relies on a separation of states into light and heavy states. If a state which generically is heavy becomes light in some region of the moduli space, this inconsistency manifest themselves as a characteristic singularity.

Instanton corrections to observables come with a weight factor e^{-1/g^2}, and are therefore non-analytic in the coupling g at the expansion point $g = 0$.

At the time of the seminal paper by Seiberg and Witten, it was known that the instanton corrections to $F(a)$ take the form

$$F_{NP}(a) = \sum_{k=1}^{\infty} \mathcal{F}_k \left(\frac{\Lambda}{a} \right)^{4-k} a^2 \ ,$$

and $\mathcal{F}_1$ had been computed and found to be non-zero $\mathcal{F}_1 \neq 0$. Later F_{NP} was computed exactly by Nekrasov using localization techniques, and found to agree with the solution given by Seiberg and Witten based on consistency arguments.

Seiberg and Witten started from the observation that there was no Kähler metric on $\mathcal{M}$ with the right asymptotic behaviour at $|a| \gg 1$. Therefore there has to be more than one singularity (branchpoint). They assumed the minimal case of two singularities (which was later shown to be the only possible solution satisfying all physical requirements). Then they made an educated guess of the nature (monodromy) of the singularity. Physically acceptable (that is, interpretable) singularities on $\mathcal{M}$ arise at points where states which are otherwise massive become massless. According to the perturbative theory (and the free theory), the $W^{\pm}$-vector multiplets become massless at $a = 0$. In this region of $\mathcal{M}$ the theory is strongly coupled, that is g is large. One general idea how to deal with the large coupling behaviour of a theory is to re-write the theory in different variables, so that the theory is weakly coupled in the new variables. For gauge theories such a transformation is provided by *electric-magnetic duality*, which exchange the electric and magnetic degrees of freedom of the photon-like field A, and acts as $g \to \frac{1}{g}$ on the coupling.

The $\mathcal{N} = 2$ $SU(2)$ theory has magnetic monopoles with charges $(\pm 1, 0)$, according to the semiclassical analysis valid at $|a| \gg 1$. Seiberg and Witten postulated that one of the two singularities corresponds to monopoles becoming massless. The corresponding strong coupling regime can be reparametrized as a theory of light and weakly coupled mononoples. Monopoles with charges $(\pm 1, 0)$ become massless at $a_D = 0$, and they choose a coordinate u on $\mathcal{M}$ such that this happens at $u = 1$. The corresponding monodromy matrix M_1 is determined by the charges of the particles which become massless. What happens at the second singularity, which with their choice of u happens at $u = -1$, is determined by the monodromy M_∞ of the perturbative prepotential, $M_\infty = M_1 M_{-1}$ where M_{-1} is the monodromy around the second singularity. M_{-1} corresponds to dyons with charges $(\pm 1, \pm 1)$, which exist for $|a| \gg 1$ and become massless for $a + a_D = 0$.

For this scenario to work out the massive vector multiplets $W^{\pm}$, which exist for $|a| \gg 1$ and would become massless at $a = 0$ must become unstable and disappear from the BPS spectrum in some region around $a = 0$. In other words, there must be curve of marginal stability where $a_D/a \in \mathbb{R}$, so that the decay

$$(0, \pm 1) \to (\pm 1, \pm 1) + (\mp 1, 0)$$

is possible. The condition

$$|\pm a| = |\pm(a + a_D)| + |\mp a_D|$$

is clearly met at $u \pm 1$ where $a_D = 0$ and $a + a_D = 0$, respectively. So if the curve exists, it will run through the special points $u = \pm 1$. Seiberg and Witten conjectured that the curve roughly looks like $|u| = 1$. This turned out to be right. The curve is not the unit circle, and the analytic expression is complicated.

For completeness we sketch how Seiberg and Witten then obtain their solution. They observe

- that the monodromy group generated by M_∞, M_1, M_{-1} is $\Gamma(2) \subset SL(2,\mathbb{Z})$, the congruence subgroup mod 2 of the modular group,
- that $\mathcal{M} = \mathbb{C}\backslash\{1,-1\} \cong \mathbb{H}/\Gamma(2) \xrightarrow{6:1} \mathbb{H}/SL(2,\mathbb{Z})$ is precisely a fundamental domain for the family

  $$E_u \, y^2 = (x-1)(x+1)(x-u)$$

  of elliptic curves.

The quantities $(a_D(u), a(u))$ can be regarded as sections of a vector bundle V over $\mathcal{M}$. Seiberg and Witten argue that the fibre over u can be identified with the space of meromorphic one-forms with vanishing residue on the elliptic curve E_u, modulo differentials of meromorphic functions on E_u. Concretely, a_D and a arise as period integrals of a familiy of merormorphic differentials λ,

$$a_D = \int_{\gamma_1} \lambda \, , \quad a = \int_{\gamma_2} \lambda$$

where γ_1, γ_2 is a basis of the first homology group of E_u, normalized such that the intersection product is $\gamma_1 \gamma_2 = 1$. They then show that the unique choice (up to exact forms), which satisfies all physical requirements is

$$\lambda = \frac{\sqrt{2}(\lambda_2 - u\lambda_1)}{2\pi}$$

where

$$\lambda_1 = \frac{dx}{y} \, , \quad \lambda_2 \frac{xdx}{dy}$$

is a basis for the meromorphic differentials with zero residue module exact meromorphic differentials.

In this construction a positive definite metric arises as follows. The modular parameter τ_u of E_u is

$$\tau_u = \frac{\int_{\gamma_1} \lambda_1}{\int_{\gamma_2} \lambda_1} \, .$$

Now

$$\tau(a) = F''(a) = \frac{da_D}{da} = \frac{da_D/du}{da/du} = \frac{\int_{\gamma_1} \frac{d\lambda}{du}}{\int_{\gamma_2} \frac{d\lambda}{du}} = \frac{\int_{\gamma_1} \lambda_1}{\int_{\gamma_2} \lambda_1} = \tau_u \ .$$

Therefore the complex coupling $\tau(a)$ is identified with the modular parameter τ_u of a family of elliptic curves, and this guarantees that its imaginary part is positive.

3.5 Embedding into string theory and D-branes

The family E_u of elliptic curves in the Seiberg-Witten solution geometrizes the problem of describing the full dynamics of the gauge theory, but it is only an auxiliary geometric construction, and not a part of space-time itself. It remains a bit mysterious why periods on a family of elliptic curves control the masses of the $W^{\pm}$ vector multiplets and of monopoles and dyons. This changes once we embed the $N = 2$ $SU(2)$ gauge theory into string theory. More precisely, the gauge theory arises as very specific 'fine-tuned' limit, which decouples the infinitely many extra degrees of freedom that string theory has.

As a crude working definition, we introduce strings ($p = 1$) together with p-dimensional membranes or p-branes, ($p > 1$) as immersed submanifolds of a semi-Riemannian space-time (M, g) of signature $(1, n)$, where we allow $n > 3$.

$$\phi \ : \Sigma_{1,p} \to M \ .$$

One part of the action functional for such a p-brane is given by the volume of its worldsheet $\Sigma_{1,p}$, measure with the metric induced by the metric of M:

$$S = -T_p \int \mathrm{vol}_{\phi * g} \ .$$

The minus sign is conventional and T_p is a characteristic constant for the p-brane, its so called tension. We consider space-times of the form $\mathbb{R}^{1,3} \times X$, where X is compact and where the immersions have the form

$$\Sigma_{1,p} \to \mathbb{R}_t \times S_p \ ,$$

where $\mathbb{R}_t \subset \mathbb{R}^{1,3}$ parametrizes time, and where $S_p \subset X$ is a compact submanifold of X of dimension p. We also assume that we are interested in length scales on M which are much larger than the size of X, so that we are left with an effective theory on $\mathbb{R}^{1,3}$. From this point of view p-branes embedded as above appear as point particles. For a static particle the action is minimized by minimizing the volume of S_p so that the mass of such particles is bounded from below by the volume of S_p:

$$M \geq T\mathrm{vol}(S_p) \ .$$

Stable static configurations correspond to $M = T\mathrm{Vol}(S_p)$, where S_p is a submanifold of minimal volume.

In a supersymmetric setup, where the background $\mathbb{R}^{1,3} \times X$ arises as a solution to the field equation of supersymmetric theory, and where the dynamics of the p-brane is described by a supersymmetric action functional, the states of an embedded p-brane can be a BPS state. This imposes conditions on the submanifold S_p and as well on other data that describe how the p-branes couples to the background. If the p-brane carries central charge under a supersymmetry algebra, masses of its particle-like excitations are subject to a BPS bound $M \geq |Z(S_p)|$, which depends on S_p (and data on it). This shows that the central charge of a BPS p-brane is related to the volume of the submanifold S_p, $|Z(S_p)| = T\text{Vol}(S_p)$.

Often the effective theory on $\mathbb{R}^{1,3}$ has a moduli space of vacua, which includes moduli corresponding to deformations of the internal manifold X. Then the volumes of submanifolds S_p change. In particular, the volume might become zero, giving rise to additional massless states. Moreover, in a supersymmetric theory, upon variation of moduli we might reach a wall of marginal stability where S_p ceases to satisfy the conditions for a BPS states (remember that S_p carries additional data).

Let us turn to string theory specifically, where he fundamental objects are 1-branes, that is strings, which can be open (segments) or closed (loops). For open strings one may choose Neumann or Dirichlet boundary conditions. With Dirichlet boundary conditions there is no momentum conservation at then ends, therefore the submanifolds on which open strings can end are dynamical p-branes, called Dp–branes.

There are several type of string theories, which differ by the specification of addtionial data in the action functional. We will consider the type-IIA and type-IIB theories, which are supersymmetric. The spacetimes strings can 'live' in are subject to various consistency conditions. For type-II strings spacetime M must be ten-dimensional. Moreover for spacetimes of the form $\mathbb{R}^{1,3} \times X$ the compact factor X must be Ricci-flat (up to corrections computable in perturbation theory). If the effective theory on $\mathbb{R}^{1,3}$ is to be $\mathcal{N} = 2$ supersymmetric, as required to make contact with an $\mathcal{N} = 2$ $SU(2)$ gauge theory, then X must have Riemannian holonomy H, such that $SU(2) \subsetneq H \subset SU(3) \subset SO(6)$, that is X must be a Calabi-Yau threefold. The massless spectrum of a type-II theory on such a background contains four-dimensional $N = 2$ supergravity, several four-dimensional vector multiplets, and further neutral (uncharged) matter multiplets. By taking a limit where only one vector multiplet remains, one can account for the Maxwell vector multiplet A of the $SU(2)$ gauge theory. To describe the $W^{\pm}$ vector multiplets and the monopoles and dyons, one needs to include D-branes.

The spectrum of D-branes of a supersymmetric string theory can be found by analysing poly-vector extensions of its supersymmetry algebra. Central charges correspond to pointlike BPS states, as we have seen. By admitting poly-vector charges (that is additional terms in the supersymmetry algebra which transform as antisymmetric tensors under the Lorentz group) we obtain extended BPS p-branes. Such branes minimize a bound for the tension T_p as a function of the central charges, and have an excitation spectrum which organises into supersymmetry representations. Type-II string theories have D-branes which are BPS in this sense. The II-A theory has D2, D4 and D6 branes while the

II-B theory has D1, D3 and D5 branes.

If such a D-brane is immersed as $\Sigma_{1,p} \to \mathbb{R}_t \times S_p \subset \mathbb{R}^{1,3} \times X$, the resulting configuration is only BPS if S_p satifies certain conditions. In particular, for the D2 and D4 branes of IIA the submanifold S_p, $p = 2, 4$, must be a complex submanifold of X (that is a holomorpic curve or surfaces in the Calabi-Yau three-fold X, respectively), while for the D3 branes of IIB the submanifold S_3 must be a special Lagrangian submanifold, that is a Lagrangian manifold calibrated by the holomorphic top form.

Let us indicate how the physical properties of these BPS states are related to geometric properties of $S_p \subset X$. For concreteness we take the D3-branes of the IIB theory. We introduce a basis A^I, B_I for the third homology group $H_3(X, \mathbb{Z})$, with standard intersection products $A^I B_J = \delta^I_J = -B_J A^I$. We expand the special Lagrangian submanifold S_3 in this basis

$$[S_3] = p^I B_I - q_I A^I \ .$$

Then the expansion coefficients, $p^I, q_I \in \mathbb{Z}$, $I = 0, 1, \ldots h_{2,1}$ (where $h_{p,q}$ denotes the Hodge numbers of X) are the magnetic and electric charges of the BPS states with respect to $h_{2,1} + 1$ Maxwell-like vector fields. The central charge is

$$Z = p^I F_I - q_I X^I \ ,$$

where (F_I, X^I) are analogous to (a_D, a) in the gauge theory. The moduli space $\mathcal{M}$ relevant for this problem is the space of complex structures of X. Thus (F_I, X^I) are locally functions of $h_{2,1}$ complex parameters z^A, which parametrize the complex structures of the Calabi-Yau threefold X. Like (a_D, a) the quantities (F_I, X^I) can be expressed as period integrals

$$X^I = \int_{A^I} \Omega \ , \quad F_I = \int_{B_I} \Omega$$

where Ω is the holomorphic top-form of X.

Note that in contrast to family E_u of elliptic curves, the family X_{z^A} of Calabi-Yau threefolds is a part of space-time, and the mass-dependence of BPS states has a direct geometrical interpretation. Since S_3 is special Lagrangian, it is a calibrated submanifold, that is its volume is minimal in its homology class, and can be computed by a calibrating form, which in our case is Ω. In other words $|Z|$ is the volume of S_p, and the masses of BPS states are given by this volume multiplied with the tension T_p. Limits where BPS states become massless corresponds to special points in the moduli space of complex structures of X where some periods of Ω vanish, which in turn implies that the volume of S_p goes to zero.

For completeness, we give a specific example of a string theory background where the analogue of the Seiberg-Witten solution has been found, and where the Seiberg-Witten solution of the $N = 2$ $SU(2)$ gauge theory can be obtained by taking a limit which decouples all the extra states. To be precise we give a pair, related by *mirror symmetry*. On realization is IIA string theory on

$X^4_{1,1,2,2,6}[12]$, the degree-12 hypersurface in the weighted projective space

$$\mathbb{P}^4_{1,1,2,2,6} := \left(\mathbb{C}^5 \backslash \Sigma\right) / \left((z_1, z_2, z_3, z_4, z_5) \cong (\lambda z_1, \lambda z_2,\ \lambda^2 z_3, \lambda^2 z_4, \lambda^6 z_5)\right) \ ,$$

which has two vector multiplets, $h_{1,1} = 2$ (and $h_{2,1} + 1 = 129$ hypermultiplets). One of these vector multiplets becomes in the limit the Maxwell vector multiplet of the gauge theory, while the charged vector multiplets $W^{\pm}$ and monopoles and dyons correspond to states of D2 branes on holomorphic curves and D4 branes on holomorphic surfaces.

The second realization is by IIB theory on the mirror $\tilde{X}$ which has $\tilde{h}_{2,1} = 2$ and $\tilde{h}_{1,1} = 128$. In this description the vector multiplet moduli are complex structure moduli, which allows to compute their prepotential exactly. The mirror can be obtained by the Greene-Plesser orbifold construction, that is the mirror is realized by the same family of hypersurface as X, but with only 2 (instead of 128) complex. Explicitly, the defining polynomial is:

$$p = z_1^{12} + z_2^{12} + z_3^6 + z_4^6 + z_5^2 - 12\psi z_1 z_2 z_3 z_4 z_5 - 2\phi z_1^6 z_2^6 \ .$$

3.6 Outlook: stability conditions

We now begin to see how the stability of BPS states in string theory is related to geometrical concepts of stability. If BPS states disappear from the spectrum of a physical theory, then the corresponding geometrical objects should become unstable, too. To really make contact several further steps need to be made.

In particular, D-branes carry more data than we have specified so far. The IIA D-branes (which for historical reasons are called B-branes) do not only require a complex submanifold, but also a holomorphic vector bundle E with a Hermitian connection ∇_E. The reason is that we can have multi-D-brane states. If two D-branes are close to each other, extra light states are present, corresponding to open strings with one end of each brane. The corresponding data define a holomorphic version Yang-Mills theory, which geometrically is encoded in (E, ∇_E). The Hermiticity condition on the connection, which is part of the Yang-Mills equations, requires that the vector bundle E is μ-stable. At this point we make contact with stability conditions in the 'classical' setting.

From the string theory point of view stability conditions must be consistent with mirror symmetry. For physicists mirror symmetry means that IIA string theory on $\mathbb{R}^{1,3} \times X$, where X is a Calabi-Yau threefold is physically equivalent (same physical observables) to IIB string theory on $\mathbb{R}^{1,3} \times \tilde{X}$, where $\tilde{X}$ is another Calabi-Yau manifold, called the mirror. If true this implies that the stability conditions for D2 and D4 branes on complex submanifolds of X must somehow map to stability conditions for D3 branes on special Lagrangian submanifolds on $\tilde{X}$, which is a highly non-trivial statement.

Following Kontevich's proposal, mirror symmetry should be rephrased as homological mirror symmetry between the derived category of coherent sheaves on X and the Fukaya catetogory of Lagrangians on the mirror $\tilde{X}$. Work along this line lead to the definition of Π-stability for D-branes by Douglas, which in turn motived Bridgeland's definition.

4 Literature

[1] is a heroic effort by a mathematician to explain quantum field theory in a mathematically solid way and thus to build 'A Bridge Between Mathematicians and Physicists.' [2] is an introduction into mirror symmetry within the framework of Kontsevich proposal of homological mirror symmetry. It has been written jointly by mathematicians and physicists and covers stability in much detail. My exposition of supersymmetry is partly based on the standard text book [3], which is a hard read for mathematicians, and on [4], which is written by mathematicians and contains the classification of Poincaré Lie superalgebras in arbitrary dimension and signature. For poly-vector extensions I used [5], a good reference for the mathematical aspects is mathematicians [6] The discussion of BPS states for the $\mathcal{N} = 2$ $SU(2)$ gauge theory follows [7], who also found the exact solution for the effective low energy theory that we sketch. The generalization to string theory is due to [8], and was elaborated on in many subsequent papers. The detailed relation with the $N = 2$ $SU(2)$ gauge theory was worked out in [9]. Some elementary aspects of quantum field theory and string theory will be covered in the textbook on string theory (aiming at MSc and beginning PhD students) that I am currently writing. Access to the current 'in progress' document is possible through dropbox, with the understanding that the text is used for personal use only. If you are interested, please just write me an e-mail. Feedback on the text is welcome.

A BPS bounds and gravity

Application: Witten's proof of the Schoen-Yau positive mass theorem in general relativity. The positivity of mass in $\mathcal{N} = 1$ supersymmetry has a famous application, namely Witten's proof of the positive mass theorem (Schoen/Yau) using the Israel-Nester construction. The theorem states that in Einstein gravity coupled to matter subject to the field equations and the 'strong energy condition' the 'ADM mass' of an asymptotically flat space-time is non-negative. How does this relate to supersymmetry? We are familiar with 'bosonic symmetries of space-time,' namely isometries. These are diffeomorphisms which preserve the metric. They are generated by Killing vectors V which satisfy the Killing equation $L_V g = 0$, where L_V is the Lie derivative. This concept of a symmetry extends to other fields, in particular to tensor fields T, that is to sections of tensor bundles over space time: $L_V T = 0$. We say that a configuration $\Phi = g, \ldots$ involving serveral fields is invariant under the transformation generated by a vector field V if $L_V \Phi = 0$. Similarly, supersymmetry transformations $\varepsilon \cdot Q := \varepsilon_i^\alpha Q_\alpha^i$ can act on field configurations. A spinor ε is called a Killing spinor for Φ if $(\varepsilon Q) \cdot \Phi = 0$. Minkowksi space-time is invariant under all supersymmetry transformations, where the Killing spinors are constant, but otherwise arbitrary. On a curved background Killing spinor fields are in general not constant, but parallel (covariantly constant) in a suitable sense. In supergravity the metric g shares a supersymmetry representation with one or several gravitini ψ (spin 3/2 fields), plus possibly fields of lower spin. The set of supergravity Killing spinor equations contains the equation

$$(\varepsilon Q)\psi = D\psi + \cdots = 0 ,$$

where D is the connection induced by the Levi-Civita connection on the spin bundle. Thus Killing spinors are parallel spinors. Note that this equation may contain further terms, which can be combined with D to define a connection.

Einstein gravity can be embedded into supergravity, and this was used by Witten to prove the positivity of the ADM mass. Note that all is used for the argument is that Einstein gravity *can* be embedded into a larger supersymmetric theory. Witten's argument does *not* require that supersymmetry is actually realized in nature. NB: spinors are used in an approach Riemannian geometry, G-structures and special holonomy, which has been dubbed spin geometry.

Example: The Gibbons-Hull bound in Einstein-Maxwell theory. For Einstein-Maxwell theory the Gibbons-Hull bound is a generalization of the positivity theorem for the mass. Einstein-Maxwell theory can be embedded into $\mathcal{N} = 2$ supergravity by adding two spin 3/2 fields, called gravitini. In this theory one has asymptotically flat solutions of finite mass, electric charge Q and magnetic charge P. In the supergravity theory this corresponds to a central charge $Z = Q + iP$. The Gibbons-Hull bound applying for such space-times is the corresponding BPS bound

$$M^2 \geq Q^2 + P^2$$

The family of Reissner-Nordstrom solutions of Einstein-Maxwell theory describes static (non-rotating) black holes with electric and magnetic charges. This family has an extremal limit, which is the lowest mass solution (for given charges) where a horizon exists. (Solution beyond this limit have a naked singularity and are deemed unphysical by the cosmic censorship principle.) The extremality bound coincides with the BPS bound, and thus supersymmetry forbids naked singularities (within this family at least). The line element of the extremal Reissner-Nordstrom solution is

$$ds^2 = -H(r)^{-2}dt^2 + H(r)^2(r)(dr^2 + r^2 d\Omega_2^2)$$

whrere (t, r, θ, ϕ) are spherical coordinates and

$$H(r) = 1 + \frac{|Z|}{r} = 1 + \frac{\sqrt{Q^2 + P^2}}{r} .$$

B Poly-vector extensions and p-branes

Example: The M-algebra, poly-vector extensions and p-branes. The centrally extended $\mathcal{N} = 2$ algebra has the following Spin representation content:

$$(\mathbb{S} \times \mathbb{S})^{\text{sym}} \to V + \Lambda^0 V + \Lambda^{\max} V$$

On the rhs we have admitted, apart from the vector module V corresponding to translations to further factors which a 'central', and i.p. singlets under Spin.

Beyond central extensions we can admit 'polyvector extensions,' that is terms on the rhs which are not central, but transform in representations of the Lorentz group and which are allowed by the representations appearing on the lhs. While central charges indicate the existence of pointlike BPS states saturating a mass bound, non-central polyvector charges are carried by extended objects, membranes of dimension $p \geq 1$, or p-branes for short. BPS p-branes saturate a lower bound on the tension (energy per volume) in terms of the polyvector charge. The simplest BPS branes are infinitely extended planar objects, which have infinite volume, and therefore infinite mass and charge. This is dealt with by going to densities, or by imposing periodic boundary conditions and compactifying the extended directions into a torus. More generally, one can consider p-branes on curved space-times. Finite mass objects arise whenever the p-brane is mapped to a compact p-dimension submanifold. To have a non-zero mass, the submanifold must have a volume which is both finite and minimal within the class of submanifolds which can be reached by deforming the embedding 'with finite energy.'

As an example for a poly-vector extension consider the unique eleven-dimension supersymmetry algebra:

$$\{Q_\alpha, Q_\beta\} = (C\gamma^M)_{\alpha\beta} P_M$$

The lhs contains the following Spin representations:

$$(S_{\mathbb{R}} \times S_{\mathbb{R}})^{\text{sym}} = V \oplus \Lambda^2 V \oplus \Lambda^5 V$$

As a check compare dimensions:

$$\frac{1}{2} 32 \times 33 = 528 = 11 + 55 + 462$$

The so-called M-algebra is the maximal poly-vector extension, where all possible Spin representations appear on the rhs:

$$\{Q_\alpha, Q_\beta\} = (C\gamma^M)_{\alpha\beta} P_M + \frac{1}{2}(C\gamma^{MN})_{\alpha\beta} Z_{MN} + \frac{1}{5!}(C\gamma^{M_1M_2M_3M_4M_5})_{\alpha\beta} Z_{M_1M_2M_3M_4M_5}$$

A field configuration of eleven-dimensional supergravity carrying charge Z_{12} is an planar supermembrane solution, which we take to be extended in the $(1,2)$ plane for concreteness:

$$ds^2 = H^{-2/3} ds^2_{1,2} + H^{1/3} ds^2_{0,8} \; , \quad H = 1 + \frac{|Z_{12}|}{|\vec{x} - \vec{x}_0|^6} \; ,$$

where $\vec{x}$ is a coordinate on the transverse $\mathbb{R}^8$. This solution saturates the bound

$$T_2 \geq |Q_2|$$

where T_2 is the tension (energy per volume) and where Q_2 is the Z_{12} charge per volume.

The supermembrane solution, also called M2-brane solution, is a higher dimensional cousin of the electrically charged, extremal Reisner-Nordstrom solutions. It is charged under a four-form gauge field, which in eleven dimensions requires membrane-like rather than point-like sources.

The action principle for a membrane, and more generally for a p-brane, can be obtained by generalizing the action of a relativistic particle, or by deriving the effective action for the collective modes of a solitonic p-brane solution. (This reflects that we can view a p-brane as either fundamental or solitonic.)

The action for a relativistic particle with mass m and charge q, in a spacetime (M, g) with electromagnetic field A is

$$S = -m \int dt \sqrt{|\det \Phi^* g|} - q \int A$$

where

$$\Phi \; : \Sigma \to M$$

is a parametrized timelike curve in M, the worldline of the particle. The electromagnetic potential is a one-form, the associated electromagnetic field strength is $F = dA$.

The generalization to a p-brane is

$$S = -T \int d^p \sigma \sqrt{|\det \Phi^* g|} - Q \int A$$

where Φ is now a parametrized $(1 + p)$-dimensional surface in M, with one timelike tangent vector at each point, where T is the p-brane tension, A a $(p + 1)$-form potential with associated field strength $(p + 2)$-form $F = dA$, and Q the p-brane charge. Supermembrane actions contain additional terms involving spinorial variables on Σ. They also obey a relation between T and Q which in suitable units is $T = |Q|$. This relation corresponds to the saturation of a BPS bound.

Closed supermembranes where the ground state energy is finite and non=zero require that that the membrane is embedded into spacetime as $\Phi : \Sigma \to \mathbb{R}_t \times S \subset M$ with $S \subset M$ defining a nontrival class in $H_2(M, \mathbb{Z})$. Supermembrane charges are then the integeger expansion coefficients of S in a basis of $H_2(M, \mathbb{Z})$.

Field configurations with finite $Z_{M_1 M_2 M_3 M_4 M_5}$ are called five-branes, or M5-branes. An infinitely extended planar M5-brane has metric

$$ds^2 = H^{-1/3} ds^2_{1,5} + H^{2/3} ds^2_{0,5}$$

where

$$H = 1 + \frac{|Q_5|}{|\vec{x} - \vec{x}_0|}$$

where $\vec{x}$ are coordinates on the transverse $\mathbb{R}^5$. The bound saturated is

$$T_5 \geq |Q_5|$$

where T_5 is the brane tension and Q_5 the charge density. M5 branes are the magnetic partners of M2 branes.

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