EXERCISES ON CHOW GROUPS

(1) Fix a point $p \in \mathbb{P}^2$. We know that $\mathrm{Bl}_p \mathbb{P}^2 \cong \mathbb{F}_1 = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$. Compute $\mathrm{CH}_*(\mathrm{Bl}_p \mathbb{P}^2)$ in two ways and compare the results (i.e., using the blow-up formula and the projective bundle formula).

(2) Consider a smooth curve $\iota\colon C \hookrightarrow \mathbb{P}^2$ of degree $d \geq 1$.

   (a) Let $d = 3$. Determine the kernel of the map
   $$\iota_*\colon \mathrm{CH}_1(C) \to \mathrm{CH}_1(\mathbb{P}^2).$$
   (b) Prove that the localisation sequence
   $$\mathrm{CH}_1(C) \to \mathrm{CH}_1(\mathbb{P}^2) \to \mathrm{CH}_1(\mathbb{P}^2 \setminus C) \to 0$$
   is left exact if and only if $d \leq 2$.

(3) Let $C, C' \subset \mathbb{P}^2$ be integral plane curves that intersect properly.

   (a) Suppose C is smooth. Compute $C \cdot C'$ via Fulton's definition.
   (b) Suppose both C and C' are singular. Compute
   $$(C \times C') \cdot \Delta \in \mathrm{CH}_0(C \cap C').$$

(4) Let S be a smooth quasi-projective surface, $p \in S$ a closed point. Let $C \subset \mathrm{Bl}_p S$ be the exceptional curve of the blow-up of S at a point. Compute the self-intersection $C \cdot C \in \mathrm{CH}_0(C)$.

(5) Let X, Y be smooth projective varieties. A class $\Gamma \in \mathrm{CH}_k(X \times Y)$ is called a *correspondence*. We define the homomorphism Γ_* as follows:
$$\Gamma_* \coloneqq \mathrm{pr}_{Y,*}(\Gamma \cdot \mathrm{pr}_X^*(-))\colon \mathrm{CH}_l(X) \to \mathrm{CH}_{l+k-\dim(X)}(Y).$$

   (a) Let $\Delta \in \mathrm{CH}_{\dim(X)}(X \times X)$ be the class of the diagonal. Prove that
   $$\Delta_* = \mathrm{id}\colon \mathrm{CH}_l(X) \to \mathrm{CH}_l(X).$$
   (b) Let $f\colon X \to Y$ be any morphism and $\Gamma_f \in \mathrm{CH}_{\dim(X)}(X \times Y)$ be the class of the graph of f. Prove that
   $$(\Gamma_f)_* = f_*\colon \mathrm{CH}_l(X) \to \mathrm{CH}_l(Y).$$

(6) Let C be an elliptic curve.

   (a) Compute the self-intersection $\Delta^2 = \Delta \cdot \Delta$ of the diagonal.
   (b) Compute the following map:
   $$\mathcal{F} \coloneqq [C \times C]_* + \Delta_* + \frac{1}{2}\Delta^2_*\colon \mathrm{CH}_*(C) \to \mathrm{CH}_*(C).$$
   (This is a variant of the Fourier (–Mukai) transform induced by the Poincaré bundle.)
   (c) Compute the matrix representing the endomorphism
   $$\mathcal{F}^H\colon H^0(C, \mathbb{Q}) \oplus H^2(C, \mathbb{Q})$$
   induced by $\mathcal{F}$, using the standard basis. Does this give you an element of $\mathrm{SL}_2(\mathbb{Z})$?
   (d) Do the same for curves of other genera.

Date: 29/10/2025.

Around Chow's moving lemma. Let's first recall the statement (see Voisin II, Lemma 9.22):

Lemma 0.1. *Let X be a smooth algebraic variety, $Y \subset X$ a smooth closed subvariety, $Z \in \mathrm{CH}_k(X)$.*

Then there exists a cycle $T = \sum n_i T_i$ of X such that:

- *T is rationally equivalent to Z,*
- *for each i, T_i intersects Y properly.*
- *Moreover, we may choose T so that T_i is **generically smooth** along $Y \cap T_i$ and $T_i \cap Y$ is **generically transverse**.*

Remark 0.2. In his book, Fulton says T_i and Y intersects transversally when the intersection is **generically** transvese (see Fulton, Example 11.4.2 in page 206).

Remark 0.3. With Fulton's definitions, it seems that a transverse intersection is NOT necessarily a proper intersection. Namely, he doesn't require a transverse intersection to have a correct codimension.

For example, consider two lines $L_1 = \{x = 0\}$ and $L_2 = \{x = 1\}$ inside $\mathbb{A}^2$. Then $L_1 \cap L_2 = \emptyset$. I guess it is a transverse intersection, but not proper since

$$\dim(L_1 \cap L_2) \neq \dim L_1 + \dim L_2 - \dim \mathbb{A}^2 = 0.$$

Let's work on one exercise:

Exercise. *Let C be the exceptional curve of the blow up $f\colon \mathbb{F}_1 \to \mathbb{P}^2$ at a point. The goal is to find a cycle $T = \sum n_i T_i$ which is rationally equivalent to C and intersects properly with C.*

1. *Prove that at least one n_i is negative, i.e., such a T cannot be effective.*
2. *Let H be a hyperplane class on $\mathbb{P}^2$. Prove that $f^*\mathcal{O}(3H) \otimes \mathcal{O}(-C)$ and $f^*\mathcal{O}(3H) \otimes \mathcal{O}(-2C)$ are very ample on $\mathbb{F}_1$.*
3. *Prove that there exist $T_1 \in |3f^*H - 2C|$ and $T_2 \in |3f^*H - C|$ which intersect with C properly (indeed, transversally).*
4. *Prove that $T = T_2 - T_1$ is rationally equivalent to C.*