DERIVED CATEGORIES OF COHERENT SHEAVES

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Abstract. We give an overview of derived categories of coherent sheaves. Our main reference is [Huy06].

1. For the participants without background in algebraic geometry

1.1. **Subvarieties of $\mathbb{C}^n$.** In (complex) differential geometry one typically studies pairs $(X, \mathcal{O})$ where X is a (complex) manifold and $\mathcal{O}$ is a sheaf of functions taking values in $\mathbb{R}$ or $\mathbb{C}$ satisfying some conditions (differentiable, analytic, holomorphic). In algebraic geometry the story is similar although there the topological spaces have rather coarse topologies and one typically imposes stronger conditions on the functions. A key example to keep in mind is the *(classical) affine* n-space usually denoted $\mathbb{A}^n$ whose underlying set is the set $\mathbb{C}^n$ and where for any polynomial in $f \in \mathbb{C}[T_1, \ldots, T_n]$ we set $Z(f) := \{x \in \mathbb{C}^n \,|f(x) = 0\}$ and the closed subsets of $\mathbb{A}^n$ are exactly the subsets which are (arbitrary) intersections of subsets of the form $Z(f)$ for some polynomial f in n-variables. The space $\mathbb{A}^n$ comes equipped with a *structure sheaf* $\mathcal{O}_{\mathbb{A}^n}$ where for an open subset $U \subset \mathbb{A}^n$ we have

$$\mathcal{O}_{\mathbb{A}^n}(U) := \{f : U \to \mathbb{C} \,|f \text{ is locally a quotient of two polynomials }\}.$$

For a closed subset $X = \cap_{i \in I} Z(f_i)$ such that the ideal generated by the f_i is a prime ideal on can give X a structure sheaf $\mathcal{O}_X$ in exactly the same way as above and this is what we call a *(classical) affine variety*. Just as with manifolds (classical) affine varieties can be glued together to get a locally ringed space $(X, \mathcal{O}_X)$ which is what is called a *(classical)* variety. Given a variety X with structure sheaf $\mathcal{O}_X$ we can talk about *sheaves of $\mathcal{O}_X$-modules* $\mathcal{M}$ which is a sheaf on the topological space X such that for any open set $U \subset X$ the set $\mathcal{M}(U)$ has the structure of an $\mathcal{O}_X(U)$-module which is compatible with the restriction maps.

In this talk we will mostly be considering something called schemes which essentially are enhanced versions of classical varieties, the participants not familiar with this are encouraged to keep the above related notions in mind.

1.2. **Vector bundles.** Like in many other areas of geometry the notion of vector bundles can also be found in algebraic geometry.

Definition 1.1. Let Y be a scheme. A vector bundle $f : X \to Y$ is a morphism of schemes together with the following additional data: An open covering $\{U_i\}$ of Y, together with U_i-isomorphisms $\psi_i : f^{-1}(U_i) \to U_i \times \mathbb{A}^n_{\mathbb{Z}}$, such that for any i, j, and for any open affine subset $V = \operatorname{Spec}(A) \subset U_i \cap U_j$ and for $k = i, j$ letting $\phi_{k,V} : \mathbb{A}^n_V \to f^{-1}(V)$ denote the isomorphism induced by the projection to V and the map

$$\mathbb{A}^n_V \to \mathbb{A}^n_{U_k} \overset{\psi_k^{-1}}{\to} f^{-1}(U_k) \to X,$$

the automorphism $\Theta_{i,j,V} : \mathbb{A}^n_V \to \mathbb{A}^n_V$ given by

$$\Theta_{i,j,V} := \phi^{-1}_{j,V} \circ \phi_{i,V}$$

is induced by a linear automorphism of $A[T_1, \ldots, T_n]$, that is a map θ satisfying $\theta(a) = a$ and $\theta(T_i) = \sum a_{i,j} T_j$ for suitable $a_{i,j} \in A$. [1]

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[1] This definition is not consistent with the definition given in [GD61, Definition 1.7.8] which is more general. We give the one above because it is more similar to the analogue in differential geometry.

The definition above can be a bit of a mouthfull the first time one encounters it. Luckily for us however the category of vector bundles over a scheme X is equivalent to the category of locally free sheaves on X which we now recall.

Definition 1.2. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. We say that $\mathcal{F}$ is a *free sheaf of rank* n if there is an isomorphism $\mathcal{F} \cong \mathcal{O}_X^{\oplus n}$ for some $n \geq 0$. We shall say that $\mathcal{F}$ is *locally free* if there is an open cover of X, $\{U_i\}$ such that $\mathcal{F}|_{U_i}$ is free of some rank for each U_i. If X is connected then the rank will be the same on all the open sets U_i.

Example 1.3. The category of locally free sheaves on a scheme X does in general not have a terminal object and hence it is not abelian. To see this consider the following example with $X = \mathbb{A}^1_k$ and suppose for the sake of contradiction that the category of locally free sheaves on X does have a terminal object which we denote by $\mathcal{T}$. It is a fact that any locally free sheaf on an affine scheme must necessarily have non-trivial global sections (to painlessly see this apply the next lemma), but each of these global sections correspond to a map of $\mathcal{O}_X$-modules $\mathcal{O}_X \to \mathcal{T}$ hence $\mathcal{T}$ can not be a terminal object.

To remedy the situation we choose to not only consider vector-bundles/locally free sheaves, but also other reasonably nice $\mathcal{O}_X$-modules which are called quasi-coherent sheaves. This notion can be defined in many equivalent ways, we give two of these in the lemma below

Lemma 1.4. *Let X be a scheme and $\mathcal{F}$ be a $\mathcal{O}_X$-module. The following are equivalent:*

(1) There exists a covering $\{U_i\}$ of X such that on each U_i, $\mathcal{F}|_{U_i}$ is isomorphic to the cokernel (in the category of all $\mathcal{O}_X$-modules) of a map of two free sheaves. In other words the sequence:

$$\mathcal{O}_{U_i}^{\oplus I} \to \mathcal{O}_{U_i}^{\oplus J} \to \mathcal{F}|_{U_i} \to 0$$

is exact.

(2) For any affine open subscheme $\operatorname{Spec}(A)$ *of X and any $f \in A$ the map* $\Gamma(\operatorname{Spec}(A), \mathcal{F})_f \to \Gamma(\operatorname{Spec}(A_f), \mathcal{F})$ *induced by universal property of localization is an isomorphism.*

Definition 1.5. Let X be a scheme and $\mathcal{F}$ an $\mathcal{O}_X$-module. If $\mathcal{F}$ satisfies the equivalent conditions of Lemma 1.4 we say that $\mathcal{F}$ is *quasi-coherent.* Furhtermore if for any affine open $\operatorname{Spec}(A)$ the A-module $M = \Gamma(\operatorname{Spec}(A), \mathcal{F})$ is finitely generated and for any map $A^{\oplus n} \to M$ (not necessarily surjective) the kernel is finitely generated, then we say that $\mathcal{F}$ is a coherent sheaf.

Theorem 1.6. *Both the category of quasi-coherent sheaves* $\operatorname{QCoh}(X)$ *and the category of coherent sheaves* $\operatorname{Coh}(X)$ *on a scheme X are abelian.*

2. The derived category of coherent sheaves

2.1. **Problems with lack of injectives and how to somewhat fix this.**

Definition 2.1. Let X be a scheme. Its derived category $D^b(X)$ is by definition the bounded derived category of the abelian category $\operatorname{Coh}(X)$, i.e.,

$$D^b(X) := D^b(\operatorname{Coh}(X)).$$

Recall the following definition

Definition 2.2. A *k-linear category* is an additive category $\mathcal{A}$ such that the groups $\operatorname{Hom}(A, B)$ are k-vector spaces and such that all compositions are k-bilinear.

An additive functor F between two k-linear additive categories over a common base field k are *k-linear* if for any objects $A, B \in \mathcal{A}$ the canonical map

$$\operatorname{Hom}(A, B) \to \operatorname{Hom}(F(A), F(B))$$

is k-linear.

Definition 2.3. Two schemes X and Y are defined over a field k are called *derived equivalent* if there exists a k-linear exact equivalence

$$D^b(X) \cong D^b(Y).$$

Recall the definition of the right derived functor of a left exact functor

Definition 2.4. Suppose $\mathcal{A}$ and $\mathcal{B}$ are abelian categories and

$$F : K^+(\mathcal{A}) \to K^+(\mathcal{B})$$

is an exact functor. Then the right derived functor $RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ for F is the functor (unique up to unique isomorphism) satisfying the following properties:

(1) Letting $Q_A : K^+(\mathcal{A}) \to D^+(\mathcal{A}), Q_B : K^+(\mathcal{B}) \to D^+(\mathcal{B})$ be the canonical functors we have a natural transformation

$$Q_{\mathcal{B}} \circ F = RF \circ Q_{\mathcal{A}}$$

(2) $RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ is an exact functor of triangulated categories.

(3) Given $G : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ and any natural transformation $Q_{\mathcal{B}} \circ F \to G \circ Q_{\mathcal{A}}$ there exists a unique natural transformation $RF \to G$ making the following diagram commute

$$\begin{array}{ccc} Q_{\mathcal{B}} \circ F & \longrightarrow & G \circ Q_{\mathcal{A}} \\ & \searrow \quad \nearrow & \\ & RF \circ Q_{\mathcal{A}} & \end{array}$$

Recall from the previous talk (and the spoiler occuring in all previous talks) that for an abelian category $\mathcal{A}$ with enough injectives we have that $i : K^+(\mathcal{I}) \to D^+(\mathcal{A})$ is an equivalence of categories. For a left exact functor $F : \mathcal{A} \to \mathcal{B}$ between abelian categories we then constructed the right derived functor of F as $RF = Q_{\mathcal{B}} \circ K(F) \circ i^{-1}$.

Unfortunately, $Coh(X)$ usually contains no non-trivial injective objects. Here is a simple example to keep in mind

Example 2.5. Take $X = \mathrm{Spec}(\mathbb{Z})$ then the category of coherent sheaves on X is equivalent to the category of finitely generated abelian groups. Now note that if I is an injective object in this category then since any element $i \in I$ gives rise to a map $\mathbb{Z} \to I$ and we can consider

$$\begin{array}{ccccc} 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\cdot n} & N \\ & & \downarrow & \swarrow_{\exists} & \\ & & I & & \end{array}$$

this implies that for any $n \in \mathbb{Z}$ we must have $nI = I$. However by the structure theorem of finitely generated abelian groups it follows that no finitely generated abelian group can satisfy this.

Thus in order to compute derived functors introduced in the previous talk we have to pass to bigger abelian categories. Most often we will work with the abelian category of quasi-coherent sheaves $\mathrm{QCoh}(X)$, with its derived categories $D^*(\mathrm{QCoh}(X))$ $(* = b, +, -)$ and sometimes with the abelian category of $\mathcal{O}_X$-modules $\mathrm{Sh}_{\mathcal{O}_X}(X)$. Whenever the scheme is defined over a field k, the derived categories will tacitly be considered as k-linear categories.

Notation 2.6. In order to avoid any possible confusion between sheaf cohomology $H^i(X, \mathcal{F})$ and the cohomology $H^i(\mathcal{F}^\bullet)$ of a complex of sheaves, we will from now on write $\mathcal{H}^i(\mathcal{F}^\bullet)$ for the latter.

The following proposition shows that the category of quasi-coherent sheaves on a Noetherian scheme X has enough injectives.

Proposition 2.7. *[Huy06, Prop. 3.3] On a Noetherian scheme X any quasi-coherent sheaf $\mathcal{F}$ admits a resolution*

$$0 \to \mathcal{F} \to \mathcal{I}^0 \to \mathcal{I}^1 \to \dots$$

by quasi-coherent sheaves $\mathcal{I}^i$ which are injective as $\mathcal{O}_X$-modules.

The following proposition will (to some extent) allow us to overcome the problem with the lack of injectives in $\mathrm{Coh}(X)$.

Proposition 2.8. *[Huy06, Prop. 3.5] Let X be a Noetherian scheme. Then the natural functor*

$$D^b(X) \to D^b(\mathrm{QCoh}(X))$$

defines an equivalence betweeen the derived category $D^b(X)$ of X and the full triangulated subcategory $D^b_{coh}(\mathrm{QCoh}(X))$ of bounded complexes of quasi-coherent sheaves with coherent cohomology.

Remark 2.9. Recall that for an abelian category $\mathcal{A}$ with enough injectives we have

$$\mathrm{Ext}^i_{\mathcal{A}}(A,B) \cong \mathrm{Hom}_{D^b(\mathcal{A})}(A, B[i])$$

Thus we have for any two coherent sheaves $\mathcal{F}$ and $\mathcal{G}$ we have by the above proposition that

$$\mathrm{Ext}^i_{\mathrm{QCoh}(X)}(\mathcal{F},\mathcal{G}) \cong \mathrm{Hom}_{D^b(X)}(\mathcal{F},\mathcal{G}[i]).$$

and taken together with [Huy06, Remark. 2.5.7] we can extend this to complexes $\mathcal{E}^\bullet, \mathcal{F}^\bullet$ of coherent sheaves. If X is a proper variety over a field k one can apply Grothendiecks Coherence theorem to show that $\mathrm{Ext}^i_{\mathrm{QCoh}(X)}(\mathcal{F},\mathcal{G})$ is a finite dimensional k-vector space. It is actually possible to show the same in the case of complexes of coherent sheaves as well.

2.2. Serre functors and Serre duality.

Definition 2.10. Let $\mathcal{A}$ be a k-linear category. A *Serre functor* is a k-linear equivalence $S : \mathcal{A} \to \mathcal{A}$ such that for any two objects $A, B \in \mathcal{A}$ there exists an isomorphism

$$\eta_{A,B} : \mathrm{Hom}(A,B) \xrightarrow{\cong} \mathrm{Hom}(B, S(A))^\vee$$

of (k-vector spaces) which is functorial in A and B.

We write the induced pairing as

$$\mathrm{Hom}(B,S(A)) \times \mathrm{Hom}(A,B) \to k,\ (f,g) \mapsto \langle f|g\rangle.$$

Let now X be a smooth projective variety over a field k, and ω_X its canonical bundle.

First note that for any locally free sheaf $\mathcal{M}$ the functor $\mathrm{Coh}(X) \to \mathrm{Coh}(X)$ given by $\mathcal{F} \mapsto \mathcal{F} \otimes \mathcal{M}$ is exact. In particular, it immediately descends to an exact functor on the derived categories $D^*(X) \to D^*(X)$, which will be denoted by $\mathcal{M} \otimes ()$. Other exact functors to consider are the shift functors $[i] : D^*(X) \to D^*(X)$ with $i \in \mathbb{Z}$.

Definition 2.11. Let X be a smooth projective variety of dimension n. Then one defines the exact functor S_X as the composition

$$D^*(X) \xrightarrow{\omega_X \otimes ()} D^*(X) \xrightarrow{[n]} D^*(X)$$

Theorem 2.12. *(Serre duality) Let X be a smooth projective variety over a field k. Then*

$$S_X : D^b(X) \to D^b(X)$$

is a Serre functor*. Or in otherwords for any two complexes $\mathcal{E}^\bullet, \mathcal{F}^\bullet \in D^b(X)$ we have an isomorphism*

$$\mathrm{Hom}_{D(\mathcal{A})}(\mathcal{E}^\bullet, \mathcal{F}^\bullet) \cong \mathrm{Hom}_{D(\mathcal{A})}(\mathcal{F}^\bullet, \mathcal{E}^\bullet \otimes \omega_X[n])^\vee$$

Corollary 2.13. *Let X be a smooth projective variety over a field k. Then for any two complexes $\mathcal{E}^\bullet, \mathcal{F}^\bullet \in D^b(X)$ we have an isomorphism*

$$\mathrm{Ext}^i(\mathcal{E}^\bullet, \mathcal{F}^\bullet) \cong \mathrm{Ext}^{n-i}(\mathcal{F}^\bullet, \mathcal{E}^\bullet \otimes \omega_X)^\vee$$

which is functorial in $\mathcal{E}^\bullet$ and $\mathcal{F}^\bullet$.

Proof. We have

$$\mathrm{Ext}^i(\mathcal{E}^\bullet, \mathcal{F}^\bullet) \cong \mathrm{Hom}_{D^b(X)}(\mathcal{E}^\bullet, \mathcal{F}^\bullet[i]) \cong \mathrm{Hom}_{D(X)}(\mathcal{F}^\bullet[i], \mathcal{E}^\bullet \otimes \omega_X[n])^\vee \cong$$
$$\cong \mathrm{Hom}_{D(X)}(\mathcal{F}^\bullet, \mathcal{E}^\bullet \otimes \omega_X[n-i])^\vee = \mathrm{Ext}^{n-i}(\mathcal{F}^\bullet, \mathcal{E}^\bullet \otimes \omega_X)^\vee.$$

□

Corollary 2.14. *Suppose that $\mathcal{F}, \mathcal{G}$ are coherent sheaves on a smooth projective variety X of dimension n. Then*

$$\operatorname{Ext}^i(\mathcal{F}, \mathcal{G}) = 0 \text{ for } i > n.$$

Proof. Simply note for $n - i < 0$ we have

$$\operatorname{Ext}^i(\mathcal{E}, \mathcal{F}) \cong \operatorname{Ext}^{n-i}(\mathcal{F}, \mathcal{E} \otimes \omega_X)^\vee \cong 0.$$

□

Corollary 2.15. *Let C be a smooth projective curve. Then any object in $D^b(C)$ is isomorphic to a direct sum $\oplus \mathcal{E}_i[i]$, where $\mathcal{E}_i$ are coherent sheaves on C.*

Proof. We proceed by induction over the length of the complex. Suppose $\mathcal{E}^\bullet$ is a complex of length k with $\mathcal{H}^i(\mathcal{E}^\bullet) = 0$ for $i < i_0$. Then [Huy06, Ex.2.32] gives us a distinguished triangle of the form

$$\mathcal{H}^{i_0}(\mathcal{E}^\bullet)[-i_0] \to \mathcal{E}^\bullet \to \mathcal{E}_1^\bullet \to \mathcal{H}^{i_0}(\mathcal{E}^\bullet)[1 - i_0]$$

If this distinguished triangle splits, that is if we have that

$$\mathcal{E}^\bullet \cong \mathcal{H}^{i_0}(\mathcal{E}^\bullet)[-i_0] \oplus \mathcal{E}_1^\bullet$$

then the induction hypothesis for $\mathcal{E}_1^\bullet$ allows us to conclude. To prove that we indeed do get such a splitting it is by [Huy06, Ex. 1.38] enough to prove that

$$\operatorname{Hom}_{\operatorname{Coh}(X)}(\mathcal{E}_1^\bullet, \mathcal{H}^{i_0}(\mathcal{E}^\bullet[1 - i_0])) = 0.$$

Now by the induction hypothesis on $\mathcal{E}_1^\bullet$ together with the fact that $\mathcal{H}^i(\mathcal{E}_1^\bullet) = 0$ for $i \leq i_0$ it follows that

$$\mathcal{E}_1^\bullet \cong \oplus_{i > i_0} \mathcal{H}^i(\mathcal{E}_1^\bullet)[-i].$$

Thus

$$\operatorname{Hom}_{\operatorname{Coh}(X)}(\mathcal{E}_1^\bullet, \mathcal{H}^{i_0}(\mathcal{E}^\bullet[1 - i_0]) = \oplus_{i > i_0} \operatorname{Ext}^{1+i-i_0}(\mathcal{H}^i(\mathcal{E}_1^\bullet), \mathcal{H}^{i_0}(\mathcal{E}^\bullet)) = 0,$$

as the homological dimension of a curve is one. □

Remark 2.16. The proof applies more generally to any abelian category of homological dimension ≤ 1.

3. Derived functors in algebraic geometry

The following result tells us that under certain conditions (right) derived functors induce functors on the derived categories where one has extra conditions on the objects.

Corollary 3.1. *[Huy06, Cor. 2.68] Suppose that $F : K^+(\mathcal{A}) \to K^+(\mathcal{B})$ is an exact functor which admits a right derived functor $RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ and assume that $\mathcal{A}$ has enough injectives.*

(1) Suppose $\mathcal{C} \subset \mathcal{B}$ is a thick subcategory with $R^iF(A) \in \mathcal{C}$ for all $A \in \mathcal{A}$. Then RF takes values in $D^+_{\mathcal{C}}(\mathcal{B})$, i.e.,

$$RF : D^+(\mathcal{A}) \to D^+_{\mathcal{C}}(\mathcal{B})$$

(2) If $RF(A) \in D^b(\mathcal{B})$ for any object $A \in \mathcal{A}$, then $RF(A^\bullet) \in D^b(\mathcal{B})$ for any complex $A^\bullet \in D^b(\mathcal{A})$, i.e., RF induces an exact functor

$$RF : D^b(\mathcal{A}) \to D^b(\mathcal{B}).$$

Let X be a Noetherian scheme over a field k. The global section functor

$$\Gamma : \operatorname{QCoh}(X) \to \operatorname{Vect}_k, \mathcal{F} \mapsto \Gamma(X, \mathcal{F})$$

is a left exact functor. Since $\operatorname{QCoh}(X)$ has enough injectives, its derived functor

$$R\Gamma : D^+(\operatorname{QCoh}(X)) \to D^+(\operatorname{Vect}_k)$$

exists. The higher derived functors are denoted

$$H^i(X, \mathcal{F}^\bullet) := R^i\Gamma(\mathcal{F}^\bullet).$$

For a sheaf $\mathcal{F}$ these are just the cohomology groups $H^i(X, \mathcal{F})$ and for an honest complex $\mathcal{F}^\bullet$ they are sometimes called the *hypercohomology groups* $\mathbb{H}^i(X, \mathcal{F}^\bullet)$. Since every complex of vector spaces splits by Corollary 2.15 and Remark 2.16, one has in fact

$$R\Gamma(\mathcal{F}^\bullet) \cong \oplus H^i(X, \mathcal{F}^\bullet)[-i].$$

Recall the following standard result from algebraic geometry

Theorem 3.2. *For any quasi-coherent sheav $\mathcal{F}$ on a Noetherian topological space X one has $H^i(X, \mathcal{F}) = 0$ for $i > \dim(X)$.*

From this it follows that $R\Gamma$ induces an exact functor on the bounded derived categories

$$R\Gamma : D^b(\mathrm{QCoh}(X)) \to D^b(\mathrm{Vect}_k)$$

The following theorem is also well known

Theorem 3.3. *If $\mathcal{F}$ is a coherent sheaf on a projective scheme X over a field k, then all cohomology groups $H^i(X, \mathcal{F})$ are of finite dimension.*

Taking $i = 0$ we thus get a left exact functor

$$\Gamma : \mathrm{Coh}(X) \to \mathrm{Vect}_k^{fin}$$

However we cannot directly construct its derived functor as $\mathrm{Coh}(X)$ does not in general contain enough injectives. Instead we first consider $\Gamma' : \mathrm{Coh}(X) \to \mathrm{Vect}_k$ then using [Huy06, Prop 3.5] one obtains the right derived functor

$$R\Gamma' : D^b(X) \to D^b(\mathrm{QCoh}(X)) \to D^b(\mathrm{Vect}_k)$$

and applying Theorem 3.3 we see $R^i\Gamma(\mathcal{F}) \in \mathrm{Vect}_k^{fin}$ for all $\mathcal{F} \in \mathrm{Coh}(X)$ thus [Huy06, Cor.2.68][2] shows that $R\Gamma'$ factorises through $D^b_{\mathrm{Vect}_k^{fin}}(\mathrm{Vect}_k)$ and since finite dimensional vector spaces are thick in vector spaces it follows from [Huy06, Prop. 2.42] that we have an equivalence $D^b(\mathrm{Vect}_k^{fin})$ with $D_{\mathrm{Vect}_k^{fin}}(\mathrm{Vect}_k)$ thus we obtain a right derived functor of Γ

$$R\Gamma : D^b(X) \to D^b(\mathrm{Vect}_k^{fin}).$$

We summarise this discussion in a diagram

$$\begin{array}{ccc}
D^+(\mathrm{QCoh}(X)) & \xrightarrow{R\Gamma} & D^+(\mathrm{Vect}_k) \\
\uparrow & & \uparrow \\
D^b+(\mathrm{QCoh}(X)) & \longrightarrow & D^b(\mathrm{Vect}_k) \\
\uparrow & & \uparrow \\
D^b(X) & \longrightarrow & D^b(\mathrm{Vect}_k^{fin}).
\end{array}$$

3.1. **Direct image.** Let $f : X \to Y$ be a morphism of Noetherian schemes. The direct image is a left exact functor

$$f_* : \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y).$$

Again, we use that $\mathrm{QCoh}(X)$ has enough injectives in order to define the right derived functor

$$Rf_* : D^+(\mathrm{QCoh}(X)) \to D^+(\mathrm{QCoh}(X)).$$

The *higher direct images* $R^if_*(\mathcal{F}^\bullet)$ of a complex of sheaves $\mathcal{F}^\bullet$ are, by definition, the cohomology sheaves $\mathcal{H}^i(Rf_*(\mathcal{F}^\bullet))$ of $Rf_*(\mathcal{F}^\bullet)$. In particular to any quasi-coherent sheaf $\mathcal{F}$ on X one associates the quasi-coherent sheaves $R^if_*\mathcal{F}$ on Y.

[2] Technically speaking this corollary is only stated for an abelian category with enough injectives, however the existence of the spectral sequence needed in its proof can be obtained by viewing any coherent sheaf as a quasi-coherent sheaf.

Lemma 3.4. *The higher direct image functor induces an exact functor*

$$Rf_* : D^b(\mathrm{QCoh}(X)) \to D^b(\mathrm{QCoh}(Y))$$

Proof. Follows from the well known fact that for a morphism $f : X \to Y$ between Noetherian schemes the higher direct images $R^i F_* \mathcal{F}$ are trivial for $i > \dim(X)$ and [Huy06, Cor.2.68]. □

Lemma 3.5. *If $f : X \to Y$ is a projective (or proper) morphism of Noetherian schemes then if $\mathcal{F}$ is coherent then $f_*(\mathcal{F})$ is also coherent on Y and we have a right derived functor*

$$Rf_* : D^b(X) \to D^b(Y)$$

Proof. Using Grothendieck's coherence theorem and arguments similar to those given in the case of Γ. □

Recall that for an abelian category $\mathcal{A}$ and left exact functor $F : \mathcal{A} \to \mathcal{B}$ we say that a class of objects $I_F \subset \mathcal{A}$ stable under direct sums is F-adapted if the following conditions hold true

(1) $A^\bullet \in K^+(\mathcal{A})$ is acyclic with $A^i \in I_F$ for all i, then $F(A^\bullet)$ is acyclic.
(2) Any object in $\mathcal{A}$ can be embedded into an object of I_F.

Whenever one has such an F-adapted class the derived functor $RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ exists.

Recall that a sheaf $\mathcal{F}$ is *flabby* if for any open subset $U \subset X$ the restriction map $\mathcal{F}(X) \to \mathcal{F}(U)$ is surjective.

Lemma 3.6. *Any injective $\mathcal{O}_X$-sheaf is flabby. Any flabby sheaf $\mathcal{F}$ on X is f_*-acyclic for any morphism $f : X \to Y$, and moreover $f_*\mathcal{F}$ is again flabby.*

For a composition

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

of two morphisms one knows that

$$(g \circ f)_* = g_* \circ f_*$$

from which we want to conclude that

$$R(g \circ f)_* \cong Rg_* \circ Rf_* : D^b(\mathrm{QCoh}(X)) \to D^b(\mathrm{QCoh}(Y))$$

we recall from last talk that this will be the case if we can ensure the existence of an f_*-adapted class $I \subset \mathrm{QCoh}(X)$ such that $f_*(I)$ is contained in a g_*-adapted class . Let I be the class of injective sheaves. Then because $\mathrm{QCoh}(X)$ has enough injectives (at least for Noetherian X) it follows from the lemma above that I is an f_*-adapted class. As the direct image of a flabby sheaf is again flabby, $f_*(I)$ is contained in the g_*-adapted class of all flabby sheaves.

3.2. **Local Homs.** Let $\mathcal{F} \in \mathrm{QCoh}(X)$. Then

$$\mathcal{H}om(\mathcal{F}, -) : \mathrm{QCoh}(X) \to \mathrm{QCoh}(X)$$

is a left exact functor. Recall that for $\mathcal{F}, \mathcal{G}$ (quasi) coherent $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ is (quasi) coherent. If X is Noetherian then since $\mathrm{QCoh}(X)$ has enough injectives we may derive $\mathcal{H}om(\mathcal{F}, -)$ to obtain

$$R\mathcal{H}om(\mathcal{F}, -) : D^+(\mathrm{QCoh}(X)) \to (\mathrm{QCoh}(X)).$$

By definition

$$\mathcal{E}xt^i(\mathcal{F}, \mathcal{G}) = R^i \mathcal{H}om(\mathcal{F}, \mathcal{G}).$$

Restricting to coherent sheaves we get

$$R\mathcal{H}om(\mathcal{F}, -) : D^+(X) \to D^+(X)$$

and if we in addition assume that X is regular (smooth) then since higher ext's vanish we get

$$R\mathcal{H}om(\mathcal{F}, -) : D^b(X) \to D^b(Y)$$

3.3. **Tensor product.** Let $\mathcal{F} \in \mathrm{Coh}(X)$. The tensor product defines the right exact functor

$$\mathcal{F} \otimes (-) : \mathrm{Coh}(X) \to \mathrm{Coh}(X)$$

Any coherent sheaf $\mathcal{E}$ admits a resolution by locally free sheaves. Moreover if $\mathcal{E}^\bullet$ is an acyclic complex bounded above with all $\mathcal{E}^i$ locally free, then $\mathcal{F} \otimes \mathcal{E}^\bullet$ is still acyclic. These two facts show that the class of locally free sheaves in $\mathrm{Coh}(X)$ is adapted for the right exact functor $\mathcal{F} \otimes (-)$ thus, the left derived functor

$$\mathcal{F} \otimes^L (-) : D^-(X) \to D^-(X)$$

exists. By definition,

$$\mathcal{T}or_i(\mathcal{F}, \mathcal{E}) := \mathcal{H}^{-i}(\mathcal{F} \otimes^L \mathcal{E}).$$

For a complex $\mathcal{F}^\bullet \in K^-(\mathrm{Coh}(X))$ that is bounded above one can define an exact functor

$$\mathcal{F}^\bullet \otimes (-) : K^-(\mathrm{Coh}(X)) \to K^-(\mathrm{Coh}(X))$$

and it can be derived

$$\mathcal{F}^\bullet \otimes^L (-) : D^-(X) \to D^-(X)$$

See [Huy06] for the details.

3.4. **Inverse image.** Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Then

$$f^* : \mathrm{Sh}_{\mathcal{O}_Y}(Y) \to \mathrm{Sh}_{\mathcal{O}_X}(X)$$

is by definition the composition of the exact functor

$$f^{-1} : \mathrm{Sh}_{\mathcal{O}_Y}(Y) \to \mathrm{Sh}_{\mathcal{O}_X}(X)$$

and the right exact functor

$$\mathcal{O}_X \otimes_{f^{-1}(\mathcal{O}_Y)} (-) : \mathrm{Sh}_{f^{-1}(\mathcal{O}_Y)}(X) \to \mathrm{Sh}_{\mathcal{O}_X}(X),$$

thus f^* is right exact and if $\mathcal{O}_X \otimes^L_{f^{-1}(\mathcal{O}_Y)} (-)$ is the left derived functor of $\mathcal{O}_X \otimes_{f^{-1}(\mathcal{O}_Y)} (-)$, then

$$Lf^* := (\mathcal{O}_X \otimes^L_{f^{-1}(\mathcal{O}_Y)} (-)) \circ f^{-1} : D^-(Y) \to D^-(X).$$

Technically speaking we have only explained how to derive the tensor product over $\mathcal{O}_X$ but the more general situation is handled similarly, moverover as in many applications f is flat we then have that f^* is exact and we do not need to derive it.

3.5. **compatibilities.** Let $f : X \to Y$ be a proper morphism of schemes over a field k. We have the following natural isomorphisms

(1) (projection formula) For $\mathcal{F}^\bullet, \mathcal{E}^\bullet \in D^b(X)$

$$Rf_*(\mathcal{F}^\bullet) \otimes^L \mathcal{E}^\bullet \xrightarrow{\cong} Rf_*(\mathcal{F}^\bullet \otimes^L Lf^*(\mathcal{E}^\bullet))$$

(2) For $\mathcal{F}^\bullet, \mathcal{E}^\bullet \in D^b(Y)$ we have

$$Lf^*(\mathcal{F}^\bullet) \otimes^L Lf^*(\mathcal{E}^\bullet) \xrightarrow{\cong} Lf^*(\mathcal{F}^\bullet \otimes^L \mathcal{E}^\bullet)$$

(3) The functor Lf^* is left adjoint to Rf_*.

References

[GD61] Alexander Grothendieck and Jean Dieudonné. *Éléments de géométrie algébrique II.* Vol. 8. Publications Mathématiques. Institute des Hautes Études Scientifiques., 1961.

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