TRACES IN COMPLEX HYPERBOLIC TRIANGLE GROUPS

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ABSTRACT. We present several formulas for the traces of elements in complex hyperbolic triangle groups generated by complex reflections.

The space of such groups of fixed signature is of real dimension one. We parameterise this space by a real invariant α of triangles in the complex hyperbolic plane. The main result of the paper is a formula, which expresses the trace of an element of the group as a Laurent polynomial in $e^{i\alpha}$ with coefficients independent of α and computable using a certain combinatorial winding number. We also give a recursion formula for these Laurent polynomials and generalise the trace formulas for the groups generated by complex μ-reflections.

We apply these formulas to prove some discreteness and some non-discreteness results for complex hyperbolic triangle groups.

1. INTRODUCTION

We study representations of real hyperbolic triangle groups, i.e. groups generated by reflections in the sides of triangles in $H^2_{\mathbb{R}}$, in the holomorphic isometry group $\mathrm{PU}(2,1)$ of the complex hyperbolic plane $H^2_{\mathbb{C}}$.

We use the following terminology: A *complex hyperbolic triangle* is a triple (C_1, C_2, C_3) of complex geodesics in $H^2_{\mathbb{C}}$. If the complex geodesics C_{k-1} and C_{k+1} meet at the angle π/p_k we call the triangle (C_1, C_2, C_3) a (p_1, p_2, p_3)-*triangle*. If the complex geodesics C_{k-1} and C_{k+1} are ultra-parallel (i.e. their closures in $H^2_{\mathbb{C}} \cup \partial H^2_{\mathbb{C}}$ do not intersect) with distance ℓ_k we call the triangle (C_1, C_2, C_3) a $[\ell_1, \ell_2, \ell_3]$-*triangle*. A *complex geodesic* (or *complex slice*) in $H^2_{\mathbb{C}}$ is the fixed point set of a complex reflection. A *complex reflection* is an element of $\mathrm{PU}(2,1)$ conjugate to the map

$$[z_1 : z_2 : z_3] \mapsto [z_1 : -z_2 : -z_3].$$

For more details on complex reflections and complex and real slices see section 2.

We call a subgroup of $\mathrm{PU}(2,1)$ generated by complex reflections ι_k in the sides C_k of a complex hyperbolic (p_1, p_2, p_3)-triangle (C_1, C_2, C_3) a (p_1, p_2, p_3)-*triangle group*. Let $\Gamma(p_1, p_2, p_3)$ be the abstract group

$$\langle \gamma_1, \gamma_2, \gamma_3 \mid \gamma_k^2 = (\gamma_{k-1}\gamma_{k+1})^{p_k} = 1 \text{ for all } k \in \{1,2,3\} \rangle,$$

where $\gamma_{k+3} = \gamma_k$, and the relation $(\gamma_{k-1}\gamma_{k+1})^{p_k} = 1$ is to omit for $p_k = \infty$. A (p_1, p_2, p_3)-*representation* is a representation of the group $\Gamma(p_1, p_2, p_3)$ into the group $\mathrm{PU}(2,1)$ given by taking the generators γ_k to the generators ι_k. In the ultra-parallel case we define similarly $[\ell_1, \ell_2, \ell_3]$-*triangle groups* and $[\ell_1, \ell_2, \ell_3]$-*representations*.

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For fixed (p_1, p_2, p_3) resp. (ℓ_1, ℓ_2, ℓ_3) the space of complex hyperbolic triangle groups is of real dimension one. There is a canonical path ρ_t of representations. The starting point ρ_0 for the path is the case when the triangle lies in a real slice. Real slices are the second type of totally geodesic subspaces in $H^2_{\mathbb{C}}$.

For (p_1, p_2, p_3)-triangle groups, according to Richard Schwartz [Sch02], the conjectural picture is as follows:

Conjecture. *We assume* $3 \leqslant p_1 \leqslant p_2 \leqslant p_3$. *We consider a* (p_1, p_2, p_3)*-representation mapping* γ_k *to* ι_k *(as described above). Define*

$$w_A = \iota_3\iota_2\iota_3\iota_1 \quad \text{and} \quad w_B = \iota_1\iota_2\iota_3.$$

A (p_1, p_2, p_3)*-representation is a discrete embedding if and only if neither* w_A *nor* w_B *is elliptic. The set of the corresponding parameter values is a closed symmetric interval.*

If the element w_A *becomes elliptic before* w_B*, we say that the triple* (p_1, p_2, p_3) *is of type A, else we say that the triple is of type B.*

The triple (p_1, p_2, p_3) *is of type A if* $p_1 < 10$ *and of type B if* $p_1 > 13$.

If the triple (p_1, p_2, p_3) *is of type A then there is a countable collection of parameters, for which the* (p_1, p_2, p_3)*-representation is infinite and discrete but not injective. If the triple* (p_1, p_2, p_3) *is of type B then there are no such discrete but not injective* (p_1, p_2, p_3)*-representations.*

R. Schwartz proved this conjecture for (∞, ∞, ∞)-groups in [Sch01b], and for (p_1, p_2, p_3)-groups with p_1, p_2, p_3 sufficiently large in [Sch03b]. For sufficiently large p_1, p_2, p_3 the triple (p_1, p_2, p_3) is of type B.

The case of (p, p, ∞)-triangle groups was studied in [WG00] by Justin Wyss-Gallifent. It turns out that the triple (p, p, ∞) is of type A for $p \leqslant 13$ and of type B for $p \geqslant 14$. He also described some of the discrete but not injective $(4, 4, \infty)$-representations.

In general, to prove that a certain (p_1, p_2, p_3)-representation is discrete and injective, it is necessary and sufficient to show that the images of elements of $\Gamma(p_1, p_2, p_3)$ of infinite order are not regular elliptic. Here the traces come in, namely, an element of $\mathrm{SU}(2, 1)$ is not regular elliptic if and only if the value of certain discriminant function (introduced by W. Goldman) on the trace of the element is not negative.

The main results of this paper are formulas for traces of elements in complex hyperbolic triangle groups, a combinatorial trace formula (Theorem 4) and a recursive trace formula (Theorem 7) as well as their applications. These formulas generalise the results of Hanna Sandler [San95] on ideal triangle groups.

We parameterise the one-dimensional space of (p_1, p_2, p_3)-triangle groups by an invariant α of triangles in the complex hyperbolic plane. The combinatorial trace formula expresses the trace of an element of the group as a Laurent polynomial in $e^{i\alpha}$ with coefficients independent of α and computable using a certain combinatorial winding number. Then we give a recursion formula for these Laurent polynomials. We also generalise the trace formulas for the groups generated by complex μ-reflections.

We apply the formulas to prove some discreteness and some non-discreteness results for complex hyperbolic triangle groups.

For instance, we compute the parameter value t_A such that for $|t| > t_A$ the corresponding (p_1, p_2, p_3)-representations are not discrete embeddings because the element w_A is regular elliptic (Proposition 12 and Corollary 13). Furthermore, for a certain subfamily of triangle groups we can partially confirm the conjecture of Schwartz about type A and B. We find a sufficient condition for a triple (p_1, p_2, p_3) to be of type B. This condition implies for example that triples (p, p, ∞) with $p \geqslant 14$ and triples $(p, 2p, 2p)$ with $p \geqslant 12$ are of type B (Proposition 16). We also obtain similar results for ultra-parallel triangle groups.

The paper is organised as follows: In section 2 we recall the basic notions of complex hyperbolic geometry, specially for the complex hyperbolic plane $H^2_{\mathbb{C}}$. In sections 3 and 4 we define angular invariant α, which classifies complex hyperbolic triangles with fixed angles up to isometry, and we compare this invariant with invariants defined for some special cases by U. Brehm, J. Hakim and H. Sandler. Section 5 contains the description of certain combinatorial functions of words, first of all the winding number.

After that we are prepared to state and to prove in sections 6 and 7 our main results, the combinatorial trace formula (thm. 4) and the recursive trace formula (thm. 7). In section 8 we discuss as the first application of the recursive trace formula the resulting formulas for the traces of short words in triangle groups. We prove in section 9 some properties of the traces of elements in a triangle group, which are also used later in section 13. In section 10 we describe the generalisations of our trace formulas for groups generated by μ-reflections (for the definition see section 2).

Sections 11–13 contain applications of the trace formulas. In section 11 we discuss necessary conditions for a triangle group representation to be a discrete embedding. Section 12 deals with a subfamily of (p_1, p_2, p_3)-triangle groups with the property $\frac{\pi}{p_1} = \frac{\pi}{p_2} + \frac{\pi}{p_3}$ resp. of $[\ell_1, \ell_2, \ell_3]$-triangle groups with the property $\ell_3 = \ell_1 + \ell_2$. The triangle groups in this subfamily seem to share a lot of properties of the ideal triangle groups. For this subfamily we can prove more precise statements about type A and B. In section 13 we discuss some arithmetic properties of the traces for certain (finite) family of signatures and parameter values of triangle groups. Finally, in section 14 we summarise some of the known results on complex hyperbolic triangle groups and describe them in terms of our parameter r_1, r_2, r_3, and α (for notation compare section 3).

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2. Complex Hyperbolic Plane

In this section we recall some basic notions of complex hyperbolic geometry. The general references on complex hyperbolic geometry are [Gol99], [Par], and [BH99].

Complex Hyperbolic Plane: Let $\mathbb{C}^{2,1}$ denote the vector space $\mathbb{C}^3$ equipped with the Hermitian form

$$\langle z, w\rangle = z_1\bar{w}_1 + z_2\bar{w}_2 - z_3\bar{w}_3$$

of signature $(2,1)$. We call a vector $z \in \mathbb{C}^{2,1}$ *negative*, *null*, or *positive*, according as $\langle z, z\rangle$ is negative, zero, or positive. Let $P(\mathbb{C}^{2,1})$ denote the projectivisation of $\mathbb{C}^{2,1} - \{0\}$. We denote the image of $z \in \mathbb{C}^{2,1}$ under the projectivisation map by $[z]$. We also write $[z] = [z_1 : z_2 : z_3]$ for $z = (z_1, z_2, z_3) \in \mathbb{C}^{2,1}$. The *complex hyperbolic plane* $H^2_{\mathbb{C}}$ is the projectivisation of the set of negative vectors in $\mathbb{C}^{2,1}$. Its ideal boundary $\partial H^2_{\mathbb{C}}$ is defined as the projectivisation of the set of null vectors in $\mathbb{C}^{2,1}$. The complex hyperbolic plane $H^2_{\mathbb{C}}$ is a Kähler manifold of constant holomorphic sectional curvature. The holomorphic isometry group of $H^2_{\mathbb{C}}$ is the projectivisation $\mathrm{PU}(2,1)$ of the group $\mathrm{SU}(2,1)$ of complex linear transformations, which preserve the Hermitian form.

The **Hermitian cross product:** $\boxtimes : \mathbb{C}^{2,1} \times \mathbb{C}^{2,1} \to \mathbb{C}^{2,1}$ is defined by

$$z \boxtimes w = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} (\bar{z} \times \bar{w}) = \begin{pmatrix} \overline{z_3w_2 - z_2w_3} \\ \overline{z_1w_3 - z_3w_1} \\ \overline{z_1w_2 - z_2w_1} \end{pmatrix}.$$

The Hermitian cross product of two vectors yields a vector perpendicular to both of them

$$\langle z \boxtimes w, z\rangle = \langle z \boxtimes w, w\rangle = 0.$$

Furthermore we have

$$\langle a \boxtimes c, b \boxtimes c\rangle = \overline{\langle a, c\rangle\langle c, b\rangle - \langle a, b\rangle\langle c, c\rangle},$$

in particular

$$\langle a \boxtimes b, a \boxtimes b\rangle = |\langle a, b\rangle|^2 - \langle a, a\rangle\langle b, b\rangle.$$

Totally Geodesic Submanifolds: There are two kinds of totally geodesic submanifolds of dimension 2 in $H^2_{\mathbb{C}}$, *complex slices* (or *complex geodesics*) and *real slices* (or *totally real totally geodesics subspaces*). Complex geodesics are obtained by projectivisation of 2-dimensional complex subspaces of $\mathbb{C}^{2,1}$. Given any two points in $H^2_{\mathbb{C}}$, there is a unique complex geodesic containing them. Any positive vector $c \in \mathbb{C}^{2,1}$ determines a 2-dimensional complex subspace

$$\{z \in \mathbb{C}^{2,1} \mid \langle c, z\rangle = 0\}$$

and a complex geodesic, which is the projectivisation of this subspace. The vector c is called a *polar vector* of the complex geodesic. A polar vector can be normalised to $\langle c, c\rangle = 1$. Conversely, any complex geodesic is represented by a polar vector. A typical example is the complex slice $\{[z : 0 : 1] \in H^2_{\mathbb{C}}\}$ with polar vector $c = (0, 1, 0)^T$. Any complex slice is isometric to this one.

We now describe possible configurations of two complex geodesics (compare Proposition 6.8 in [Par]). For two complex geodesics one of the following situations occur:

- they coincide,

- they intersect in a single point in $H^2_{\mathbb{C}}$,
- their closures in $H^2_{\mathbb{C}} \cup \partial H^2_{\mathbb{C}}$ intersect in a single point in $\partial H^2_{\mathbb{C}}$,
- their closures in $H^2_{\mathbb{C}} \cup \partial H^2_{\mathbb{C}}$ are disjoint.

Let C_1 and C_2 be two complex geodesics with polar vectors c_1 and c_2 respectively. We assume that the polar vectors are normalised

$$\langle c_1, c_1 \rangle = \langle c_2, c_2 \rangle = 1.$$

The complex geodesics C_1 and C_2 intersect in $H^2_{\mathbb{C}}$ if and only if

$$|\langle c_1, c_2 \rangle| < 1.$$

Then $c_1 \boxtimes c_2$ is a negative vector, corresponding to the intersection point. The angle of intersection $\angle(C_1, C_2) \in [0, \pi/2]$ between the complex geodesics C_1 and C_2 can then be defined by

$$\cos \angle(C_1, C_2) = |\langle c_1, c_2 \rangle|.$$

The complex geodesics C_1 and C_2 intersect in $\partial H^2_{\mathbb{C}}$ if and only if

$$|\langle c_1, c_2 \rangle| = 1,$$

in this case $c_1 \boxtimes c_2$ is a null vector, corresponding to the intersection point. The closures of the complex geodesics C_1 and C_2 are disjoint if and only if

$$|\langle c_1, c_2 \rangle| > 1,$$

in which case

$$|\langle c_1, c_2 \rangle| = \cosh(\ell/2),$$

where ℓ is the distance between C_1 and C_2.

Any real slice is isometric to $\{[z : w : 1] \in H^2_{\mathbb{C}} \mid z, w \in \mathbb{R}\}$. Real slices are fixed point sets of *real reflections*, i.e. antiholomorphic isometries conjugate to the map

$$[z : w : 1] \mapsto [\bar{z} : \bar{w} : 1].$$

Complex Reflections: Given a complex geodesic C, there is a unique isometry ι_C in $\mathrm{PU}(2,1)$ of order 2, whose fixed point set is equal to C. We call this isometry the *complex reflection* in C (or the *inversion* on C). The complex reflection in C is represented by an element $\iota_C \in \mathrm{SU}(2,1)$ that is given by

$$\iota_C(z) = -z + 2\frac{\langle z, c \rangle}{\langle c, c \rangle} c,$$

where c is a polar vector of C. If we assume that the polar vector c of C is normalised so that $\langle c, c \rangle = 1$ then we can simplify the formula for the corresponding reflection

$$\iota_C(z) = -z + 2\langle z, c \rangle \cdot c.$$

Complex μ-Reflections: For a unit complex number μ a complex μ-reflection is an element of $\mathrm{PU}(2,1)$ conjugate to the map

$$[z_1 : z_2 : z_3] \mapsto [\mu z_1 : z_2 : z_3].$$

Its fixed point set is a complex geodesic and the μ-reflection rotates around this complex geodesic by the angle $\arg(\mu)$. For $\mu = -1$ we get the usual complex reflections described above. The complex μ-reflection in a complex geodesic C with a polar vector c is given by

$$\iota_C^{\mu}(z) = z + (\mu - 1) \cdot \frac{\langle z, c \rangle}{\langle c, c \rangle} \cdot c.$$

Classification of isometries: An isometry φ of $H^2_{\mathbb{C}}$ is called *elliptic* if it has a fixed point in $H^2_{\mathbb{C}}$. It is called *hyperbolic* (or *loxodromic*) if its *displacement* $d_\varphi = \inf\{d(x,\varphi(x)) \mid x \in H^2_{\mathbb{C}}\}$ is positive. Here d is the complex hyperbolic metric on $H^2_{\mathbb{C}}$. A hyperbolic isometry has two fixed points that lie in $\partial H^2_{\mathbb{C}}$. An isometry, which is neither elliptic nor hyperbolic, is called *parabolic*. It has one fixed point that lies in $\partial H^2_{\mathbb{C}}$. So far this was the usual classification of isometries of non-positive curved spaces. In the case of the complex hyperbolic space we can refine this classification. An elliptic element is called *regular elliptic* if all its eigenvalues are distinct. A parabolic element, which can be written as an element of $\mathrm{U}(2,1)$ with only eigenvalue 1, is called *unipotent*, otherwise the parabolic element is called *ellipto-parabolic* (or *skew-parabolic*).

Using the discriminant function

$$\rho(z) = |z|^4 - 8\,\mathrm{Re}(z^3) + 18|z|^2 - 27$$

we can classify isometries of the complex hyperbolic plane by the traces of the corresponding matrices: An isometry $A \in \mathrm{SU}(2,1)$ is regular elliptic iff $\rho(\operatorname{trace} A) < 0$ and hyperbolic iff $\rho(\operatorname{trace} A) > 0$. If $\rho(\operatorname{trace} A) = 0$ there are three cases. If $(\operatorname{trace} A)^3 = 27$ then A is unipotent. Otherwise, A is either a complex reflection in a complex geodesic or a complex reflection about a point, or A is ellipto-parabolic. The proof can be found in [Gol99], Theorem 6.2.4. Note that for real z the function ρ factors into $\rho(z) = (z+1)(z-3)^3$. This means for $A \in \mathrm{SU}(2,1)$ whose trace is real, that A is regular elliptic iff $\operatorname{trace} A \in (-1,3)$ and hyperbolic iff $\operatorname{trace} A \notin [-1,3]$.

3. Complex Hyperbolic Triangles and α-Invariant

In this section we describe a parameterisation of the space of (complex hyperbolic) triangles in $H^2_{\mathbb{C}}$, i.e. of triples (C_1, C_2, C_3) of complex geodesics, by means of an invariant α. Let c_k be the normalised polar vector of the complex geodesic C_k. Let $r_k = |\langle c_{k-1}, c_{k+1}\rangle|$. If the complex geodesics C_{k-1} and C_{k+1} meet at the angle φ_k, then $r_k = \cos\varphi_k$. If the complex geodesics C_{k-1} and C_{k+1} are ultra-parallel with distance ℓ_k, then $r_k = \cosh(\ell_k/2)$. We define the *angular invariant* α of the triangle (C_1, C_2, C_3) by

$$\alpha = \arg\left(\prod_{k=1}^{3} \langle c_{k-1}, c_{k+1}\rangle\right).$$

The angular invariant is obviously invariant under isometries of $H^2_{\mathbb{C}}$. The complex reflection $\iota_k = \iota_{C_k}$ in the complex geodesic C_k is defined by

$$\iota_k(z) = -z + 2\langle z, c_k\rangle \cdot c_k.$$

(Note that because of the property $\langle c_k, c_k\rangle = 1$ we can simplify the formula for the reflection.)

Remark. If the complex geodesics C_{k-1} and C_{k+1} or their closures intersect, then the vector $v_k = c_{k-1} \boxtimes c_{k+1}$ is a negative resp. null vector, which corresponds to the intersection point, i.e. to a vertex of the complex hyperbolic triangle. The points $[v_k]$ can be also meaningfully interpreted in the case of ultra-parallel geodesics C_{k-1} and C_{k+1}.

We call a complex hyperbolic triangle (C_1, C_2, C_3) a $(\varphi_1, \varphi_2, \varphi_3)$-triangle if the complex geodesics C_{k-1} and C_{k+1} meet at the angle φ_k.

Proposition 1. *A $(\varphi_1, \varphi_2, \varphi_3)$-triangle in $H^2_{\mathbb{C}}$ is determined uniquely up to holomorphic isometry by the triple $(\varphi_1, \varphi_2, \varphi_3)$ and the angular invariant*

$$\alpha = \arg\left(\prod_{k=1}^{3} \langle c_{k-1}, c_{k+1}\rangle\right).$$

For any $\alpha \in [0, 2\pi]$ there exists a $(\varphi_1, \varphi_2, \varphi_3)$-triangle in $H^2_{\mathbb{C}}$ with angular invariant α if and only if

$$\cos\alpha < \frac{r_1^2 + r_2^2 + r_3^2 - 1}{2r_1r_2r_3},$$

where $r_k = \cos\varphi_k$.

Remark. In the case of angular invariant $\alpha = \pi$ all the vertices $[v_1]$, $[v_2]$, and $[v_3]$ of the triangle lie in one real slice. The corresponding triangle group representation stabilises this real slice. The angular invariant α with

$$\cos\alpha = \frac{r_1^2 + r_2^2 + r_3^2 - 1}{2r_1r_2r_3}$$

corresponds to the case that all vertices $[v_1]$, $[v_2]$, and $[v_3]$ of the triangle coincide. The inequality in the last proposition means that the triangle exists for values of parameter α in a symmetric open neighbourhood of π and does not exist for values of parameter α outside of this neighbourhood.

Proof. We can normalise so that

$$\begin{aligned} c_1 &= (z, \xi, \beta), \\ c_2 &= (\gamma, \delta, 0), \\ c_3 &= (0, 1, 0) \end{aligned}$$

with $z \in \mathbb{C}$, $\xi, \beta, \gamma, \delta \in \mathbb{R}$ and $\xi, \gamma, \delta > 0$. First we can assume, applying a holomorphic isometry if necessary, that the intersection point of the complex geodesics C_1 and C_2 is the point $[0 : 0 : 1]$. Then the polar vector c_1 is of the form $c_1 = (a, b, 0)$ with $a, b \in \mathbb{C}$ and $|a|^2 + |b|^2 = 1$. The isometry

$$\begin{pmatrix} b & -a & 0 \\ \bar{a} & \bar{b} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

fixes the point $[0 : 0 : 1]$ and maps the polar vector $c_1 = (a, b, 0)$ to the vector $(0, 1, 0)$. Hence we can assume additionally that $c_1 = (0, 1, 0)$. The point $[0 : 0 : 1]$ is contained in the complex geodesic C_2, hence the polar vector c_2 is of the form $c_2 = (c, d, 0)$. An isometry of the form

$$\begin{pmatrix} e^{ix} & 0 & 0 \\ 0 & e^{-ix} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

with $x \in \mathbb{R}$ fixes the point $[0 : 0 : 1]$. Such an isometry also fixes the point $[0 : 1 : 0]$ and hence the complex geodesic C_1. We can choose the real number x in such a way that the image of the polar vector c_2 under the isometry is a complex multiple of a vector of the form $(\gamma, \delta, 0)$ with $\gamma, \delta \in \mathbb{R}$. Finally, let us assume that $c_1 = (0, 1, 0)$ and $c_2 = (\gamma, \delta, 0)$ with $\gamma, \delta \in \mathbb{R}$. Applying an isometry of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{iy} \end{pmatrix}$$

with $y \in \mathbb{R}$ we can arrange that the polar vector c_3 is a complex multiple of a vector of the form (z, ξ, β) with $\xi, \beta \in \mathbb{R}$ and $z \in \mathbb{C}$.

We compute

$$\langle c_1, c_3 \rangle = \xi, \quad \langle c_2, c_1 \rangle = \gamma \bar{z} + \delta \xi, \quad \langle c_3, c_2 \rangle = \delta.$$

The conditions $|\langle c_{k-1}, c_{k+1} \rangle| = r_k$ and $\langle c_k, c_k \rangle = 1$ imply

$$\begin{aligned} &\delta = r_1, \quad \gamma = s_1, \\ &\xi = r_2, \quad |z| > s_2, \quad \beta = (|z|^2 + r_2^2 - 1)^{1/2}, \\ &z = (r_3 e^{-i\alpha} - r_1 r_2) s_1^{-1}, \end{aligned}$$

where $r_k = \cos(\varphi_k)$ and $s_k = \sin(\varphi_k)$. The inequality $|z| > s_2$ finally implies $|r_3 e^{i\alpha} - r_1 r_2| > s_1 s_2$. Computation shows that this inequality is equivalent to $\cos\alpha < (r_1^2 + r_2^2 + r_3^2 - 1)(2 r_1 r_2 r_3)^{-1}$. □

Remark. The groups with angular invariant α and $2\pi - \alpha$ are conjugate via an anti-holomorphic isometry of $H^2_{\mathbb{C}}$. For this reason we can restrict ourselves to the cases of angular invariant $\alpha \in (0, \pi]$.

Similar statements can be proved also in the cases when some of the complex geodesics are ultra-parallel. For example, for an ultra-parallel $[\ell_1, \ell_2, \ell_3]$-triangle (C_1, C_2, C_3), i.e. in the case that the complex geodesics C_{k-1} and C_{k+1} are ultra-parallel with distance ℓ_k, we obtain by similar reasoning the following result:

Proposition 2. *An ultra-parallel $[\ell_1, \ell_2, \ell_3]$-triangle in $H^2_{\mathbb{C}}$ is determined uniquely up to holomorphic isometry by the triple (ℓ_1, ℓ_2, ℓ_3) and the angular invariant*

$$\alpha = \arg\left(\prod_{k=1}^{3} \langle c_{k-1}, c_{k+1} \rangle\right).$$

For any $\alpha \in [0, 2\pi]$ there exists an ultra-parallel $[\ell_1, \ell_2, \ell_3]$-triangle in $H^2_{\mathbb{C}}$ with angular invariant α if and only if

$$\cos\alpha < \frac{r_1^2 + r_2^2 + r_3^2 - 1}{2 r_1 r_2 r_3},$$

where $r_k = \cosh(\ell_k/2)$.

Examples of explicit parameterisations: In the case $p_3 = \infty$ we consider for $\alpha \in [-\pi, \pi]$ the complex geodesics with normalised polar vectors

$$\begin{aligned} c_1 &= (1, z_2, -z_2), \\ c_2 &= (1, z_1, -z_1), \\ c_3 &= (0, 1, 0) \end{aligned}$$

where $z_1 = \cos(\pi/p_1)e^{-i\alpha/2}$ and $z_2 = \cos(\pi/p_2)e^{i\alpha/2}$. Computing

$$\langle c_1, c_3\rangle = r_2 e^{i\alpha/2}, \quad \langle c_2, c_1\rangle = 1, \quad \langle c_3, c_2\rangle = r_1 e^{i\alpha/2}$$

we see that $\gamma_k \mapsto \iota_{C_k}$ defines a (p_1, p_2, ∞)-representation with α-invariant equal to α. For the sake of completeness we also compute the vertices of the triangle

$$\begin{aligned} [v_1] &= [-r_1 e^{i\alpha/2} : 0 : 1], \\ [v_2] &= [-r_2 e^{-i\alpha/2} : 0 : 1], \\ [v_3] &= [0 : 1 : -1]. \end{aligned}$$

As a special case of this parameterisation is the parameterisation of (p, p, ∞)-groups in [WG00]. Another example of explicit parameterisation is the parameterisation of $(4, 4, 4)$-groups in [Sch03a].

4. Connection of the Angular Invariant α and Other Invariants

In this section we compare our parameterisation of the space of complex hyperbolic triangles with other parameterisations by Ulrich Brehm [Bre90] and Jeffrey Hakim and Hanna Sandler [HS00].

Let (C_1, C_2, C_3) be a complex hyperbolic triangle. Let c_k be the normalised polar vector of the complex geodesic C_k. Assume that for $k = 1, 2, 3$ the complex geodesics C_{k-1} and C_{k+1} or their closures intersect. Let $v_k = c_{k-1} \boxtimes c_{k+1}$.

The *Cartan angular invariant* of three points $[v_1]$, $[v_2]$, and $[v_3]$ in $\partial H^2_{\mathbb{C}}$ was defined by É. Cartan in [Car32] as follows:

$$A([v_1], [v_2], [v_3]) = \arg(-\langle v_1, v_2\rangle\langle v_2, v_3\rangle\langle v_3, v_1\rangle).$$

To compare the Cartan angular invariant and our invariant α we need the following proposition:

Proposition 3. *Let $\langle c_{k-1}, c_{k+1}\rangle = r_k e^{i\theta_k}$. Then we have*

$$\begin{aligned} \langle v_k, v_k\rangle &= r_k^2 - 1, \\ \langle v_{k-1}, v_{k+1}\rangle &= e^{i\theta_k}(r_k - r_{k-1}r_{k+1}e^{-i\alpha}), \\ \prod_{k=1}^{3}\langle v_k, v_{k+1}\rangle &= \prod_{k=1}^{3}\overline{\langle v_{k-1}, v_{k+1}\rangle} \\ &= e^{-i\alpha}\cdot\prod_{k=1}^{3}(r_k - r_{k-1}r_{k+1}e^{i\alpha}). \end{aligned}$$

Proof. The proof of this proposition, based on the fact that

$$[v_k] = [c_{k-1} \boxtimes c_{k+1}]$$

and the formula for $\langle a \boxtimes c, b \boxtimes c\rangle$, is straightforward. □

Using the last proposition we obtain the following formula for the Cartan angular invariant

$$A = \arg(-e^{-i\alpha}(1 - e^{i\alpha})^3) = \frac{\alpha - \pi}{2} \mod 2\pi$$

because of $\arg(1 - e^{i\alpha}) = (\alpha - \pi)/2$.

Ulrich Brehm defined in [Bre90] the *shape invariant* of three points $[v_1]$, $[v_2]$, and $[v_3]$ in $H^2_{\mathbb{C}}$ as

$$\sigma([v_1],[v_2],[v_3]) = \mathrm{Re}\left(\frac{\langle v_1,v_2\rangle\langle v_2,v_3\rangle\langle v_3,v_1\rangle}{\langle v_1,v_1\rangle\langle v_2,v_2\rangle\langle v_3,v_3\rangle}\right).$$

The shape invariant σ is invariant under $\mathrm{PU}(2,1)$, and the triangle $[v_1]$, $[v_2]$, $[v_3]$ is determined uniquely up to isometry by its three angles together with the shape invariant. Using Proposition 3 we obtain the following formula that describes the relation between the Brehm's shape invariant σ and our invariant α:

$$\sigma = \frac{r_1^2r_2^2r_3^2\cos 2\alpha - r_1r_2r_3(r_1^2+r_2^2+r_3^2+1)\cos\alpha + (r_1^2r_2^2+r_2^2r_3^2+r_3^2r_1^2)}{(1-r_1^2)(1-r_2^2)(1-r_3^2)}.$$

In [HS00] Jeffrey Hakim and Hanna Sandler defined such angular invariants also in the cases of triangles with one or two ideal vertices:

$$\eta([v_1],[v_2],[v_3]) = \frac{\langle v_3,v_1\rangle\langle v_1,v_2\rangle}{\langle v_3,v_2\rangle\langle v_1,v_1\rangle}.$$

Using Proposition 3 we compute

$$\eta = e^{-i\alpha}\cdot\frac{(r_2-r_1e^{i\alpha})(1-r_1r_2e^{i\alpha})}{(r_1-r_2e^{i\alpha})(r_1^2-1)}.$$

5. Winding Numbers and Other Combinatorial Functions

In this section we introduce some combinatorial functions of words, which will appear in the combinatorial trace formula.

We first fix some notation. We can identify the elements of the (p_1,p_2,p_3)-triangle group, resp. the elements of the abstract group $\Gamma(p_1,p_2,p_3)$, with finite sequences of numbers 1, 2, and 3. We refer to such sequences as *words*. The sequence $a=(a_1,\dots,a_n)$ corresponds to the element

$$\iota_a := \iota_{a_1}\cdots\iota_{a_n}.$$

We shall simplify notation where it is convenient, for example we write (123) instead of $(1,2,3)$ and ι_{123} instead $\iota_{(123)}$ or $\iota_{(1,2,3)}$. The empty sequence corresponds to the identity element denoted by $\iota_\varnothing$. There is a natural definition of the powers $(a_1,\dots,a_n)^w$ for $w\in\mathbb{Z}$, which is consistent with taking powers of the corresponding elements in the group, for example $(123)^2=(123123)$ and $(123)^{-1}=(321)$. We denote by τ_a the trace of the element ι_a.

We are interested in traces of elements. For $A_1,\dots,A_m\in\mathrm{SU}(2,1)$ we have

$$\mathrm{trace}(A_1\cdots A_m) = \mathrm{trace}(A_m\cdot A_1\cdots A_{m-1}).$$

Since the trace of a product does not change under cyclic permutation of the factors, we consider cyclic words instead of usual (linear) words. A *cyclic word* is the orbit of a linear word under cyclic permutations.

We now introduce the winding number. Let χ denote the unique nontrivial character modulo 3, i.e. if a is any integer then $\chi(a)$ is the unique element of the set of integers $\{-1,0,1\}$, which is congruent to a modulo 3. Define

$$w(a_1,\dots,a_n) = \frac{1}{3}\cdot\sum_{m=1}^{n}\chi(a_{m+1}-a_m),$$

where $a_{n+1} := a_1$. This number is invariant under cyclic permutations.

The integer $w(a_1, \dots, a_n)$ can be interpreted as the winding number of the loop $a_1 \to a_2 \to \cdots \to a_n \to a_1$ along the vertices of a triangle with vertices labeled by 1, 2, and 3 clockwise.

Dealing with winding numbers we often use the following consideration: We start with the following operations on cyclic words: reduction

$$(\cdots, k, k, \cdots) \to (\cdots, k, \cdots)$$

and straightening

$$(\cdots, k, l, k, \cdots) \to (\cdots, k, \cdots).$$

These operations do not change the winding number, and any cyclic word a can be transformed into the cyclic word $(123)^{w(a)}$ using these operations.

We now introduce combinatorial functions u_k that will be used in the combinatorial trace formula.

Let $u_k(a_1, \dots, a_n)$ be the number

$$|\{m \in \{1, \dots, n\} \mid \{a_m, a_{m+1}\} = \{k-1, k+1\}\}|,$$

where $a_{n+1} = a_1$. This number is invariant under cyclic permutations.

Considering again the loop $a_1 \to a_2 \to \cdots \to a_n \to a_1$ along the vertices of a triangle with vertices labeled by 1, 2, and 3 clockwise, the number $u_k(a_1, \dots, a_n)$ says how often the loop goes along the edge with vertices labeled by $k-1$ and $k+1$.

Using the function ψ_k given by $\psi_k(a, b) = 1$ iff

$$\{a, b\} = \{k-1, k+1\}$$

and $\psi_k(a, b) = 0$ otherwise we can describe u_k as

$$u_k(a_1, \dots, a_n) = \sum_{m=1}^{n} \psi_k(a_m, a_{m+1}),$$

where again $a_{n+1} = a_1$.

6. Combinatorial Trace Formula

The first main result of this paper is the following combinatorial formula for traces of elements in complex hyperbolic triangle group.

We first fix some notation. For a word $a = (a_1, \dots, a_n)$ and a subset S of the set $[n] := \{1, \dots, n\}$ we denote by $u_k(S)$ resp. $w(S)$ the values of u_k resp. w for the corresponding subsequences of a, for example

$$u_k(S) = u_k(a_{i_1}, \dots, a_{i_m})$$

for $S = \{i_1, \dots, i_m\}$ with $1 \leqslant i_1 < \cdots < i_m \leqslant n$.

Theorem 4. *Let $a = (a_1, \dots, a_n)$ be a cyclic word, $\iota_a = \iota_{a_1} \cdots \iota_{a_n}$ the corresponding element of the (p_1, p_2, p_3)-group and τ_a the trace of ι_a. Then*

$$\tau_a = (-1)^n \cdot \left(2 + \sum_{S \subset [n]} (-2)^{|S|} \cdot r_1^{u_1(S)} r_2^{u_2(S)} r_3^{u_3(S)} \cdot e^{i\alpha w(S)} \right).$$

Equivalently,

$$\tau_a = (-1)^n \cdot (2 + p_a(e^{i\alpha})),$$

where

$$p_a(z) = \sum_w q_w z^w$$

is the Fourier polynomial with coefficients

$$q_w = \sum_{\substack{S\subset[n]\\ w(S)=w}} (-2)^{|S|} \cdot r_1^{u_1(S)} r_2^{u_2(S)} r_3^{u_3(S)}.$$

In the sums above we allow the possibility that the subset S of $[n] = \{1,\dots,n\}$ is empty.

Proof. We write the complex reflections as $\iota_k = -\operatorname{id} + 2c_k c_k^*$, where c^* is the dual vector to a vector $c \in \mathbb{C}^{2,1}$, i.e. $c^*(z) = \langle z, c\rangle$. To get a formula for traces we expand

$$\begin{aligned}
\iota_a &= \iota_{a_1}\cdots\iota_{a_n}\\
&= (-\operatorname{id} + 2c_{a_1}c_{a_1}^*)\cdots(-\operatorname{id} + 2c_{a_n}c_{a_n}^*)\\
&= \sum_{S=\{i_1,\dots,i_m\}\subset[n]} (-1)^{n-|S|}\cdot 2^{|S|}\cdot (c_{a_{i_1}}c_{a_{i_1}}^*\cdots c_{a_{i_m}}c_{a_{i_m}}^*).
\end{aligned}$$

Let $S = \{i_1,\dots,i_m\}$ be a subset of $[n]$. For ease of notation let $b_j = a_{i_j}$ and $b = (b_1,\dots,b_m)$. We have the following two facts:

$$c_{b_1}c_{b_1}^* c_{b_2}c_{b_2}^* \cdots c_{b_{m-1}}c_{b_{m-1}}^* c_{b_m}c_{b_m}^* = c_{b_1}^*(c_{b_2})\cdots c_{b_{m-1}}^*(c_{b_m})\cdot(c_{b_1}c_{b_m}^*)$$

and

$$\operatorname{trace}(c_{b_1}c_{b_m}^*) = c_{b_m}^*(c_{b_1}).$$

These two facts imply

$$\begin{aligned}
\operatorname{trace}(c_{b_1}c_{b_1}^*\cdots c_{b_m}c_{b_m}^*) &= c_{b_1}^*(c_{b_2})\cdots c_{b_{m-1}}^*(c_{b_m})\cdot c_{b_m}^*(c_{b_1})\\
&= \langle c_{b_2}, c_{b_1}\rangle\cdots\langle c_{b_m}, c_{b_{m-1}}\rangle\cdot\langle c_{b_1}, c_{b_m}\rangle.
\end{aligned}$$

We claim that

$$\langle c_{b_2}, c_{b_1}\rangle\cdots\langle c_{b_m}, c_{b_{m-1}}\rangle\cdot\langle c_{b_1}, c_{b_m}\rangle = r_1^{u_1(b)} r_2^{u_2(b)} r_3^{u_3(b)}\cdot e^{i\alpha w(b)}.$$

It is clear that the absolute value of the complex number on the left hand side of the equation is equal to $r_1^{u_1(b)} r_2^{u_2(b)} r_3^{u_3(b)}$. The statement about the argument of this complex number is clear for the words of the form $(123)^w$ for some $w \in \mathbb{Z}$. Any other cyclic word b can be transformed into the word $(123)^{w(b)}$ using the winding number preserving operations of reduction and straightening as described in section 5. It remains to note that this operations do not change the argument of the complex number on the left hand side of the equation. So we have proved the formula

$$\langle c_{a_{i_2}}, c_{a_{i_1}}\rangle\cdots\langle c_{a_{i_m}}, c_{a_{i_{m-1}}}\rangle\cdot\langle c_{a_{i_1}}, c_{a_{i_m}}\rangle = r_1^{u_1(S)} r_2^{u_2(S)} r_3^{u_3(S)}\cdot e^{i\alpha w(S)}$$

for $S = \{i_1,\dots,i_m\} \subset [n]$. (Note that this formula does not hold for $S = \varnothing$, so we have to handle this case separately.) Using this formula we finally obtain

$$\tau_a = (-1)^n\cdot\left(2 + \sum_{S\subset[n]} (-2)^{|S|}\cdot r_1^{u_1(S)} r_2^{u_2(S)} r_3^{u_3(S)}\cdot e^{i\alpha w(S)}\right). \quad \square$$

Remark. In the case of ideal triangle contained in a complex slice, i.e. for $p_1 = p_2 = p_3 = \infty$ and $\alpha = 0$, it follows from the Binomial Theorem that

$$p_a(e^{i\alpha}) = \sum_k q_k = \sum_{S\subset[n]} (-2)^{|S|} = (1+(-2))^n = (-1)^n.$$

In this case the three reflections ι_1, ι_2, and ι_3 coincide. Hence for a word $a = (a_1, \dots, a_n)$ the element ι_a is equal identity for n even and is a complex reflection for n odd.

7. Recursive Trace Formula

In explicit computations we would rather use the recursive trace formula given in Theorem 7 below, which can be derived from the combinatorial trace formula of the last section. We first state some properties of the winding number and of the combinatorial function u_k. They follow from routine induction arguments.

Proposition 5. *We note that*

$$w(a_1, \dots, a_n) = w(a_1, \dots, a_m) + w(a_m, \dots, a_n) + w(a_1, a_m, a_{m+1}, a_n)$$

for $m = 2, \dots, n-1$. *Also*

$$w(a_1, \dots, a_n, a_{n+1}) = w(a_1, \dots, a_n) + w(a_1, a_n, a_{n+1}),$$

and

$$w(b_1, \dots, b_m, a_{n-2}, a_{n-1}, a_n) = w(b_1, \dots, b_m, a_{n-2}, a_n) + w(a_{n-2}, a_{n-1}, a_n).$$

Proposition 6. *We note that*

$$u_k(a_1, \dots, a_n, a_{n+1}) = u_k(a_1, \dots, a_n) + v_k(a_n, a_{n+1}, a_1)$$

and

$$u_k(b_1, \dots, b_k, a_{n-2}, a_{n-1}, a_n) = u_k(b_1, \dots, b_k, a_{n-2}, a_n) + v_k(a_{n-2}, a_{n-1}, a_n),$$

where

$$v_k(a, b, c) = \psi_k(a, b) + \psi_k(b, c) - \psi_k(a, c).$$

Define *deletion operators* δ, δ', and δ'': For a word $a = (a_1, \dots, a_n)$ let

$$\begin{aligned}
\delta(a) &= (a_1, \dots, a_{n-3}, a_{n-2}, a_{n-1}),\\
\delta'(a) &= (a_1, \dots, a_{n-3}, a_{n-2}, a_n),\\
\delta''(a) &= (a_1, \dots, a_{n-3}, a_{n-1}, a_n).
\end{aligned}$$

Theorem 7. *Let* $a = (a_1, \dots, a_n)$ *be a word of length* $n \geqslant 3$ *and* τ_a *the trace of the corresponding element* ι_a. *Then*

$$\tau_a = -(\tau_{\delta a} + \tau_{\delta'' a} + \tau_{\delta\delta'' a}) + \beta(\tau_{\delta' a} + \tau_{\delta\delta' a} + \tau_{\delta'\delta'' a} + \tau_{\delta\delta'\delta'' a}),$$

where

$$\beta = 2 r_1^{v_1(T)} r_2^{v_2(T)} r_3^{v_3(T)} \cdot e^{i\alpha w(T)} - 1$$

with $T = (a_{n-2}, a_{n-1}, a_n)$.

Proof. The proof is along the lines of the proof of Theorem 3 in [San95]. We consider

$$\begin{aligned}
t_a &:= (-1)^n \tau_a - 2 \\
&= \sum_{S\subset[n]} (-2)^{|S|} \cdot r_1^{u_1(S)} r_2^{u_2(S)} r_3^{u_3(S)} e^{i\alpha w(S)}, \\
t'_a &:= t_a - t_{\delta a} = (-1)^n(\tau_a + \tau_{\delta a}) \\
&= \sum_{n\in S\subset[n]} (-2)^{|S|} \cdot r_1^{u_1(S)} r_2^{u_2(S)} r_3^{u_3(S)} e^{i\alpha w(S)}, \\
t''_a &:= t'_a - t'_{\delta' a} = (-1)^n(\tau_a + \tau_{\delta a} + \tau_{\delta' a} + \tau_{\delta\delta' a}) \\
&= \sum_{n-1,n\in S\subset[n]} (-2)^{|S|} \cdot r_1^{u_1(S)} r_2^{u_2(S)} r_3^{u_3(S)} e^{i\alpha w(S)}, \\
t'''_a &:= t''_a - t''_{\delta'' a} \\
&= (-1)^n(\tau_a + \tau_{\delta a} + \tau_{\delta' a} + \tau_{\delta\delta' a} + \tau_{\delta'' a} + \tau_{\delta\delta'' a} + \tau_{\delta'\delta'' a} + \tau_{\delta\delta'\delta'' a}) \\
&= \sum_{n-2,n-1,n\in S\subset[n]} (-2)^{|S|} \cdot r_1^{u_1(S)} r_2^{u_2(S)} r_3^{u_3(S)} e^{i\alpha w(S)}.
\end{aligned}$$

For $S \subset [n-3]$ let $\hat{S} := S \cup \{n-2, n-1, n\}$ and $S' := S \cup \{n-2, n\}$. By Propositions 5 and 6 it holds for $S \subset [n-3]$

$$w(\hat{S}) = w(S') + w(T) \quad \text{and} \quad u_k(\hat{S}) = u_k(S') + v_k(T).$$

This implies

$$\begin{aligned}
t'''_a &= \sum_{n-2,n-1,n\in S\subset[n]} (-2)^{|S|} \cdot r_1^{u_1(S)} r_2^{u_2(S)} r_3^{u_3(S)} e^{i\alpha w(S)} \\
&= \sum_{S\subset[n-3]} (-2)^{|S|+3} \cdot r_1^{u_1(\hat{S})} r_2^{u_2(\hat{S})} r_3^{u_3(\hat{S})} e^{i\alpha w(\hat{S})} \\
&= -2 \cdot r_1^{v_1(T)} r_2^{v_2(T)} r_3^{v_3(T)} e^{i\alpha w(T)} \cdot \sum_{S\subset[n-3]} (-2)^{|S|+2} \cdot r_1^{u_1(S')} r_2^{u_2(S')} r_3^{u_3(S')} e^{i\alpha w(S')} \\
&= -(\beta+1) \cdot t''_{\delta' a}.
\end{aligned}$$

From this equation and the definition of t'''_a we obtain

$$t''_a = t''_{\delta'' a} + t'''_a = t''_{\delta'' a} - (\beta+1) \cdot t''_{\delta' a}.$$

On the other hand t''_a can be expressed in terms of the traces as

$$\begin{aligned}
&(-1)^n(\tau_a + \tau_{\delta a} + \tau_{\delta' a} + \tau_{\delta\delta' a}) \\
&\quad = (-1)^{n-1}(\tau_{\delta'' a} + \tau_{\delta\delta'' a} + \tau_{\delta'\delta'' a} + \tau_{\delta\delta'\delta'' a}) \\
&\quad\quad - (\beta+1) \cdot (-1)^{n-1}(\tau_{\delta' a} + \tau_{\delta\delta' a} + \tau_{\delta'\delta' a} + \tau_{\delta\delta'\delta' a}).
\end{aligned}$$

Using the relation $\delta'\delta' = \delta'\delta''$, we have

$$\begin{aligned}\tau_a &= -(\tau_{\delta''a} + \tau_{\delta\delta''a} + \tau_{\delta'\delta''a} + \tau_{\delta\delta'\delta''a} + \tau_{\delta a} + \tau_{\delta' a} + \tau_{\delta\delta' a}) \\ &\quad + (\beta+1)\cdot(\tau_{\delta' a} + \tau_{\delta\delta' a} + \tau_{\delta'\delta'' a} + \tau_{\delta\delta'\delta'' a}) \\ &= -(\tau_{\delta''a} + \tau_{\delta\delta''a} + \tau_{\delta a}) + \beta\cdot(\tau_{\delta' a} + \tau_{\delta\delta' a} + \tau_{\delta'\delta'' a} + \tau_{\delta\delta'\delta'' a}).\end{aligned}$$ □

In order to use this formula to compute traces recursively, it is necessary to know the traces of all elements associated to words of length 0, 1, and 2:

$$\begin{aligned}&\tau_{\varnothing} = 3, \quad \tau_k = -1, \\ &\tau_{k-1,k+1} = \tau_{k+1,k-1} = 4r_k^2 - 1.\end{aligned}$$

8. Examples

We compute the traces for some special elements of complex hyperbolic triangle groups, in particular for $w_A = \iota_{1323}$ and $w_B = \iota_{123}$.

We have

$$\tau_{123} = -(\tau_{12} + \tau_{23} + \tau_2) + \beta\cdot(\tau_{13} + \tau_1 + \tau_3 + \tau_{\varnothing}),$$

where

$$\beta = 2r_1^{v_1(1,2,3)} r_2^{v_2(1,2,3)} r_3^{v_3(1,2,3)} e^{i\alpha w(1,2,3)} - 1 = 2r_1 r_2^{-1} r_3 e^{i\alpha} - 1.$$

Hence

$$\begin{aligned}\tau_{123} &= -((4r_3^2 - 1) + (4r_1^2 - 1) + (-1)) + (2r_1 r_2^{-1} r_3 e^{i\alpha} - 1)\cdot((4r_2^2 - 1) + 1) \\ &= 8r_1 r_2 r_3 e^{i\alpha} - (4(r_1^2 + r_2^2 + r_3^2) - 3).\end{aligned}$$

Furthermore we have

$$\tau_{2321} = -(\tau_{232} + \tau_{221} + \tau_{22}) + \beta\cdot(\tau_{231} + \tau_{23} + \tau_{21} + \tau_2),$$

where

$$\beta = 2r_1^{v_1(3,2,1)} r_2^{v_2(3,2,1)} r_3^{v_3(3,2,1)} e^{i\alpha w(3,2,1)} - 1 = 2r_1 r_2^{-1} r_3 e^{-i\alpha} - 1.$$

It holds $\iota_{231} = \iota_1(\iota_{123})\iota_1^{-1}$ and therefore $\tau_{231} = \tau_{123}$. Because of $\tau_{232} + \tau_{221} + \tau_{22} = \tau_3 + \tau_1 + \tau_{\varnothing} = 1$ and

$$\begin{aligned}&\tau_{231} + \tau_{23} + \tau_{21} + \tau_2 \\ &= (8r_1 r_2 r_3 e^{i\alpha} - (4r_1^2 + 4r_2^2 + 4r_3^2 - 3)) + (4r_1^2 - 1) + (4r_3^2 - 1) + (-1) \\ &= 8r_1 r_2 r_3 e^{i\alpha} - 4r_2^2 = 4r_2(2r_1 r_3 e^{i\alpha} - r_2)\end{aligned}$$

we obtain

$$\begin{aligned}\tau_{2321} &= -1 + (2r_1r_2^{-1}r_3e^{-i\alpha} - 1)\cdot 4r_2(2r_1r_3e^{i\alpha} - r_2)\\ &= 4(2r_1r_3e^{-i\alpha} - r_2)(2r_1r_3e^{i\alpha} - r_2) - 1\\ &= 4\cdot |2r_1r_3e^{i\alpha} - r_2|^2 - 1\\ &= (16r_1^2r_3^2 + 4r_2^2 - 1) - 16r_1r_2r_3\cos\alpha.\end{aligned}$$

Similarly, for any k we get

$$\begin{aligned}\sigma_k := \tau_{k,k-1,k,k+1} &= \tau_{k,k+1,k,k-1}\\ &= 4\cdot |2r_{k-1}r_{k+1}e^{i\alpha} - r_k|^2 - 1\\ &= (16r_{k-1}^2r_{k+1}^2 + 4r_k^2 - 1) - 16r_1r_2r_3\cos\alpha.\end{aligned}$$

9. Some Properties of the Traces

In this section we study some properties of our trace formulas. Proposition 8 will be also used later in section 13.

Proposition 8. *The coefficients q_w in the combinatorial trace formula satisfy*

$$q_w \in (8r_1r_2r_3)^{|w|}\cdot \mathbb{Z}[4r_1^2, 4r_2^2, 4r_3^2].$$

Proof. The coefficient q_w is a sum of the terms of the form

$$\pm 2^{|b|}\cdot r_1^{u_1(b)}\cdot r_2^{u_2(b)}\cdot r_3^{u_3(b)}$$

for some word b of length $|b|$ and winding number $w(b) = w$. For the words $b = (123)^w$ with $w \in \mathbb{Z}$ we have

$$2^{|b|}\cdot r_1^{u_1(b)}\cdot r_2^{u_2(b)}\cdot r_3^{u_3(b)} = 2^{3|w|}\cdot r_1^{|w|}\cdot r_2^{|w|}\cdot r_3^{|w|} = (8r_1r_2r_3)^{|w|}.$$

Any other word b can be transformed into the word $(123)^{w(b)}$ using the winding number preserving operations described in section 5. But the number

$$\pm 2^{|b|}\cdot r_1^{u_1(b)}\cdot r_2^{u_2(b)}\cdot r_3^{u_3(b)}$$

is divided by 2 under reduction and by $4r_k^2$ for some $k \in \{1, 2, 3\}$ under straightening. Altogether this implies

$$\pm 2^{|b|}\cdot r_1^{u_1(b)}\cdot r_2^{u_2(b)}\cdot r_3^{u_3(b)} \in (8r_1r_2r_3)^{|w(b)|}\cdot \mathbb{Z}[4r_1^2, 4r_2^2, 4r_3^2]$$

and hence $q_w \in (8r_1r_2r_3)^{|w|}\cdot \mathbb{Z}[4r_1^2, 4r_2^2, 4r_3^2]$. □

Theorem 9. *Let $a = (a_1, \ldots, a_n)$ be a cyclic word of length n, and let τ_a be the trace of the element $\iota_a = \iota_{a_1}\cdots\iota_{a_n}$.*

There is a polynomial s_a in $\mathbb{Z}[4r_1^2, 4r_2^2, 4r_3^2][z, \bar{z}]$ of degree at most $n/2$ in $4r_k^2$ and at most $n/3$ in z and $\bar{z}$, independent of the angular invariant α, such that $\tau_a = s_a(\tau_{123}, \bar{\tau}_{123})$.

We can also interpret this result as follows: There is a polynomial S_a in $\mathbb{Z}[t_1, t_2, t_3, z, \bar{z}]$ of degree at most $n/2$ in t_k and at most $n/3$ in z and $\bar{z}$, independent of the angular invariant α, such that

$$\tau_a = S_a(\tau_{12}, \tau_{23}, \tau_{13}, \tau_{123}, \bar{\tau}_{123}),$$

where

$$\tau_{k-1,k+1} = 4r_k^2 - 1 = 2\cos\frac{2\pi}{p_k} + 1$$

and

$$\tau_{123} = (8r_1r_2r_3)\cdot e^{i\alpha} - (4r_1^2 + 4r_2^2 + 4r_3^2 - 3).$$

Proof. Let $R := \mathbb{Z}[4r_1^2, 4r_2^2, 4r_3^2]$. The trace $\tau = \tau_{123}$ is equal to

$$\tau = (8r_1r_2r_3)\cdot e^{i\alpha} - c,$$

where

$$c = 4r_1^2 + 4r_2^2 + 4r_3^2 - 3 \in R.$$

Therefore

$$e^{i\alpha w} = (8r_1r_2r_3)^{-|w|}\cdot(\tau + c)^{|w|}$$

for positive w and

$$e^{i\alpha w} = (8r_1r_2r_3)^{-|w|}\cdot(\bar{\tau} + c)^{|w|}$$

for negative w. In both cases we obtain

$$e^{i\alpha w} \in (8r_1r_2r_3)^{-|w|}\cdot R[\tau, \bar{\tau}].$$

On the other hand the coefficients q_w in the combinatorial trace formula satisfy $q_w \in (8r_1r_2r_3)^{|w|}\cdot R$ by Proposition 8, hence

$$q_w \cdot e^{i\alpha w} \in R[\tau, \bar{\tau}].$$

The last formulation of the result is due to the fact that

$$4r_k^2 = \tau_{k-1,k+1} + 1. \quad \square$$

Remark. Compare the result of Theorem 9 with the following result about subgroups of $\mathrm{SL}(2,\mathbb{R})$ (see for example [Kat92]): Let Γ be a finitely generated subgroup of $\mathrm{SL}(2,\mathbb{R})$. Let $\{S_1,\ldots,S_n\}$ be a set of generators of Γ. Then $\mathrm{trace}(\Gamma) := \{\mathrm{trace}(T) \mid T \in \Gamma\}$ is contained in the ring

$$\mathbb{Z}[\tau_{i_1,\ldots,i_r} \mid \{i_1,\ldots,i_r\} \subset \{1,\ldots,n\}],$$

where $\tau_{i_1,\ldots,i_r} = \mathrm{trace}(S_{i_1}\cdots S_{i_r})$.

For a triangle group Γ in $\mathrm{SL}(2,\mathbb{R})$ generated by reflections $\{\iota_1,\iota_2,\iota_3\}$ in the sides of a real hyperbolic triangle with angles π/p_1, π/p_2, and π/p_3 we have to consider the traces

$$\tau_{i_1,\ldots,i_r} = \mathrm{trace}(\iota_{i_1}\cdots\iota_{i_r}) \quad \text{for} \quad \{1,\ldots,r\} \subset \{1,2,3\}.$$

The traces $\tau_{i_1,\ldots,i_r}$ with $r=1$ resp. $r=3$ are integers, hence we obtain that

$$\mathrm{trace}(\Gamma) \subset \mathbb{Z}[\tau_{12}, \tau_{23}, \tau_{13}],$$

where

$$\tau_{k-1,k+1} = \mathrm{trace}(\iota_{k-1}\iota_{k+1}) = 2\cos\frac{\pi}{p_k}.$$

Remark. In the ideal case, i.e. $p_1 = p_2 = p_3 = \infty$, for $a = (123)^k$ the polynomials s_a are the analogues of the Chebyshev polynomials for $\mathrm{SU}(2,1)$ defined by Hanna Sandler in [San95].

10. Groups Generated by Complex μ-Reflections

In this section we generalise the trace formulas for the case of groups generated by complex μ-reflections.

Let (C_1, C_2, C_3) be a complex hyperbolic triangle. Let c_k be the normalised polar vector of the complex geodesic C_k, and let

$$r_k = |\langle c_{k-1}, c_{k+1}\rangle|.$$

The angular invariant α of the triangle (C_1, C_2, C_3) is

$$\alpha = \arg\left(\prod_{k=1}^{3} \langle c_{k-1}, c_{k+1}\rangle\right).$$

Let μ_1, μ_2, and μ_3 be unit complex numbers. The complex μ_k-reflection $\iota_k = \iota_{C_k}^{\mu_k}$ in the complex geodesic C_k is given by

$$\iota_k(z) = z + (\mu_k - 1)\cdot\langle z, c_k\rangle \cdot c_k.$$

We consider the subgroup generated by the reflections ι_1, ι_2, and ι_3. The formulas for traces can be generalised for this case.

Let

$$n_k(a_1, \dots, a_n) = |\{m \in \{1, \dots, n\} \mid a_m = k\}|.$$

For a word $a = (a_1, \dots, a_n)$ and a subset S of the set $[n] := \{1, \dots, n\}$ we denote by $n_k(S)$, $u_k(S)$, resp. $w(S)$ the values of n_k, u_k, resp. w for the corresponding subsequences of a, for example

$$u_k(S) = u_k(a_{i_1}, \dots, a_{i_m})$$

for $S = \{i_1, \dots, i_m\}$ with $1 \leqslant i_1 < \dots < i_m \leqslant n$.

Theorem 10. *Let $a = (a_1, \dots, a_n)$ be a word, $\iota_a = \iota_{a_1} \cdots \iota_{a_n}$ the corresponding element of the group generated by the three reflections $\iota_k = \iota_{C_k}^{\mu_k}$ and τ_a the trace of ι_a. Then*

$$\tau_a = 2 + \sum_{S\subset[n]} \left(\prod_{k=1}^{3} (\mu_k - 1)^{n_k(S)}\right)\cdot\left(\prod_{k=1}^{3} r_k^{u_k(S)}\right)\cdot e^{i\alpha w(S)}.$$

Proof. We write the complex μ_k-reflection as $\iota_k = \mathrm{id} + (\mu_k - 1)c_k c_k^*$, where c^* is the dual vector to a vector $c \in \mathbb{C}^{2,1}$, i.e. $c^*(z) = \langle z, c\rangle$. In order to get a formula for traces we expand

$$\begin{aligned}\iota_a &= \iota_{a_1}\cdots\iota_{a_n}\\ &= (\mathrm{id} + (\mu_{a_1} - 1)c_{a_1}c_{a_1}^*)\cdots(\mathrm{id} + (\mu_{a_n} - 1)c_{a_n}c_{a_n}^*)\\ &= \sum_{S=\{i_1,\dots,i_m\}\subset[n]} \left(\prod_{k=1}^{3}(\mu_k - 1)^{n_k(S)}\right)\cdot(c_{a_{i_1}}c_{a_{i_1}}^*\cdots c_{a_{i_m}}c_{a_{i_m}}^*).\end{aligned}$$

Now we proceed as in the proof of Theorem 4. ∎

The generalisations of other statements based on the combinatorial trace formula for the case of the groups generated by μ-reflections are straightforward, for instance the generalisation of the recursive trace formula and of Proposition 8.

As an example, we consider Mostow's non-arithmetic complex hyperbolic lattices [Mos80]. Let us denote by $\Gamma(p,\rho)$ a complex hyperbolic triangle group generated by complex μ-reflections of the same finite order $p \in \{3,4,5\}$

$$\mu := \mu_1 = \mu_2 = \mu_3 = e^{\frac{2\pi i}{p}}$$

in the sides of an equiangular triangle with

$$r := r_1 = r_2 = r_3 = \frac{1}{2\sin\frac{\pi}{p}}$$

and the value of the angular invariant α of the form

$$\alpha = \frac{2\pi}{\rho} + \frac{\pi}{p} - \frac{\pi}{2}.$$

Proposition 11. *The traces of elements in the group $\Gamma(p,\rho)$ satisfy*

$$\tau_a \in \mathbb{Q}[e^{\frac{2\pi i}{p}}, e^{\frac{2\pi i}{\rho}}].$$

Proof. Generalising the combinatorial trace formula 4 we obtain the coefficients

$$q_w = \sum_{\substack{S \subset [n] \\ w(S)=w}} \left(\prod_{k=1}^{3} (\mu_k - 1)^{n_k(S)} \right) \cdot \left(\prod_{k=1}^{3} r_k^{u_k(S)} \right).$$

Generalising Proposition 8 we obtain that the coefficients in the combinatorial trace formula satisfy

$$q_w \in ((\mu-1)r)^{3|w|} \cdot \mathbb{Z}[\mu, (\mu-1)^2 r^2].$$

For $\mu = e^{\frac{2\pi i}{p}}$ and $r = \frac{1}{2\sin\frac{\pi}{p}}$ it holds

$$(\mu - 1)r = i \cdot e^{\frac{\pi i}{p}},$$

hence we obtain

$$q_w \in (ie^{\frac{\pi i}{p}})^{3|w|} \cdot \mathbb{Z}[e^{\frac{2\pi i}{p}}]$$

and therefore

$$\tau_a \in \mathbb{Q}[e^{\frac{2\pi i}{p}}, ie^{\frac{\pi i}{p}} e^{i\alpha}].$$

Finally

$$i \cdot e^{-\frac{\pi i}{p}} \cdot e^{i\alpha} = e^{\frac{2\pi i}{\rho}}$$

implies, that the traces of elements in the group $\Gamma(p,\rho)$ satisfy

$$\tau_a \in \mathbb{Q}[e^{\frac{2\pi i}{p}}, e^{\frac{2\pi i}{\rho}}]. \quad \square$$

Mostow proves this result in Lemma 17.2.1 in [Mos80] using implicitly a trace formula similar to our combinatorial trace formula for the special case of the groups $\Gamma(p,\rho)$.

11. Applications: Non-Discreteness of Triangle Groups

In this section we describe some necessary conditions for a triangle group representation to be a discrete embedding. For the computations in this and the next section the parameterisation of complex hyperbolic (p_1, p_2, p_3)-triangle groups by

$$t = \left(\tan\frac{\alpha}{2}\right)^{-1} = \sqrt{\frac{1+\cos\alpha}{1-\cos\alpha}}$$

is more suitable than the parameterisation by the angular invariant α itself. Recall that a (p_1, p_2, p_3)-triangle group exists for $t \in (-t_\infty, t_\infty)$, where

$$t_\infty = \sqrt{\frac{1+c_\infty}{1-c_\infty}} \quad \text{and} \quad c_\infty = \frac{r_1^2+r_2^2+r_3^2-1}{2r_1r_2r_3}.$$

(For $c_\infty > 1$ resp. $c_\infty < -1$ we set $t_\infty = +\infty$ resp. $t_\infty = -\infty$.)

Let $\phi : \Gamma(p_1, p_2, p_3) \to \mathrm{PU}(2,1)$ be a complex hyperbolic triangle group representation and $G := \phi(\Gamma(p_1, p_2, p_3))$ the corresponding complex hyperbolic triangle group. Assume that γ is an element of infinite order in $\Gamma(p_1, p_2, p_3)$ and that its image $\phi(\gamma)$ in G is regular elliptic. Two cases may occur: either $\phi(\gamma)$ is of finite order, in which case ϕ is not injective, or $\phi(\gamma)$ is of infinite order, in which case ϕ is not discrete because the subgroup of G generated by $\phi(\gamma)$ is not discrete. In particular, if a (p_1, p_2, p_3)-representation ϕ is a discrete embedding, then the elements ι_{123} and $\iota_{k,k-1,k,k+1}$ are not regular elliptic.

We have shown that the trace of the element $w_A = \iota_{3231}$ is the real number

$$\sigma_3 = (16r_1^2r_2^2 + 4r_3^2 - 1) - 16r_1r_2r_3\cos\alpha.$$

To check if the element w_A is regular elliptic we use the discriminant function ρ described in section 2. As explained there, an element A in $\mathrm{SU}(2,1)$ whose trace is real is regular elliptic iff $\operatorname{trace} A \in (-1, 3)$. The inequality $\sigma_k > -1$ holds for any $k = 1, 2, 3$ and $\alpha \in \mathbb{R}$ except in the case $2r_1r_2 = r_3$ and $\alpha \in 2\pi \cdot \mathbb{Z}$ since

$$\begin{aligned}\sigma_k &= (16r_{k-1}^2r_{k+1}^2 + 4r_k^2 - 1) - 16r_1r_2r_3\cos\alpha\\ &\geqslant 16r_{k-1}^2r_{k+1}^2 + 4r_k^2 - 1 - 16r_1r_2r_3\\ &= 4(2r_{k-1}r_{k+1} - r_k)^2 - 1 > -1.\end{aligned}$$

It remains to resolve the inequality $\sigma_3 < 3$ with respect to $\cos\alpha$. We obtain the following proposition

Proposition 12. *The element $w_A = \iota_{3231}$ in a (p_1, p_2, p_3)-triangle group is regular elliptic iff $|t| > t_A$, where*

$$t_A = \sqrt{\frac{1+c_A}{1-c_A}} \quad \text{and} \quad c_A = \frac{4r_1^2r_2^2+r_3^2-1}{4r_1r_2r_3}.$$

(For $c_A > 1$ resp. $c_A < -1$ we set $t_A = +\infty$ resp. $t_A = -\infty$.)

This proposition implies

Corollary 13. *The (p_1, p_2, p_3)-triangle group representation with $|t| > t_A$ is not a discrete embedding, because the element w_A is regular elliptic.*

The following proposition says that the element w_A becomes elliptic in the interval of possible triangle groups.

Proposition 14. *For* $(p_1, p_2, p_3) \neq (3, 3, 3)$ *we have* $c_A < c_\infty$ *and* $t_A < t_\infty$.

Proof. We obtain

$$\begin{aligned} 4r_1r_2r_3 \cdot (c_\infty - c_A) &= r_3^2 - (2r_1^2 - 1)(2r_2^2 - 1) \\ &= \cos^2\left(\frac{\pi}{p_3}\right) - \cos\left(\frac{2\pi}{p_1}\right)\cos\left(\frac{2\pi}{p_2}\right). \end{aligned}$$

For $p_3 \geqslant p_2 \geqslant p_1 \geqslant 4$ this implies $c_\infty > c_A$ because of

$$\cos\left(\frac{\pi}{p_3}\right) > \cos\left(\frac{2\pi}{p_2}\right) \geqslant \cos\left(\frac{2\pi}{p_1}\right) \geqslant 0.$$

In the case $p_1 = 3$ the computations are straightforward. □

Remark. Because of $\sigma_k - \sigma_{k+1} = 4(r_k^2 - r_{k+1}^2)(1 - 4r_{k-1}^2)$ it holds $\sigma_1 \geqslant \sigma_2 \geqslant \sigma_3$. This means that among the elements $\iota_{k,k-1,k,k+1}$ the element $w_A = \iota_{3231}$ is the first one to become elliptic as $t > 0$ tends to t_∞ and therefore there is no need to control the ellipticity of the elements ι_{1312} and ι_{2123}.

To check the ellipticity of the element $w_B = \iota_{123}$ is computationally a much more involved task. A tedious but straightforward computation shows that $\rho(\mathrm{trace}(w_B))$ is of the form

$$\rho(\mathrm{trace}(w_B)) = \frac{f_B(t)}{(t^2 + 1)^3},$$

where f_B is an even polynomial of degree 6. Moreover, it holds $f_B(0) > 0$ for $r_1, r_2, r_3 > 1/2$, and the coefficient of t^6 vanishes if $c_\infty = 1$. The polynomial f_B can be computed explicitly, but we shall not use these formulas, so we omit them.

12. Triangle Groups with $r_1^2 + r_2^2 + r_3^2 = 1 + 2r_1r_2r_3$

In this section we study triangle groups corresponding to the triples (r_1, r_2, r_3) satisfying the equation

$$r_1^2 + r_2^2 + r_3^2 = 1 + 2r_1r_2r_3. \qquad (*)$$

These triangle groups seem to share a lot of properties with the ideal triangle groups. Possibly some of the methods used for ideal case can be used in this case too.

The equation $(*)$ is equivalent to $c_\infty = 1$ and $t_\infty = +\infty$. This implies that all real numbers occur as values of parameter t for some $(\varphi_1, \varphi_2, \varphi_3)$-triangle in $H^2_{\mathbb{C}}$.

We first consider the case of triangle group corresponding to a complex hyperbolic triangle with all three vertices in $H^2_{\mathbb{C}} \cup \partial H^2_{\mathbb{C}}$. In this case $r_k = \cos\varphi_k$, where φ_1, φ_2, and φ_3 are the angles of the triangle. We assume $r_1 \leqslant r_2 \leqslant r_3$ and hence $\varphi_1 \geqslant \varphi_2 \geqslant \varphi_3 \geqslant 0$. The equation $(*)$ is then equivalent to

$$\varphi_1 = \varphi_2 + \varphi_3.$$

The signature of a non-ideal triangle group which belongs to the family $(*)$ is of the form

$$(p_1, p_2, p_3) = (abp, a(a + b)p, b(a + b)p)$$

for some positive integer numbers a, b, and p with $a \leqslant b$. Some special cases are $(p, 2p, 2p)$-groups, $(2p, 3p, 6p)$-groups etc. If we allow also ideal vertices, we obtain

(p,p,∞)-groups studied by J. Wyss-Gallifent (compare (4) in section 14) and among them the ideal triangle groups. A computation shows that

$$c_A = 1 - \frac{\sin^2(\varphi_1+\varphi_2)}{4r_1r_2r_3} < 1.$$

In the case of an ultra-parallel complex hyperbolic $[\ell_1,\ell_2,\ell_3]$-triangle group we similarly obtain $\ell_3=\ell_1+\ell_2$ and

$$c_A = 1 + \frac{\sinh^2(\ell_1-\ell_2)}{4r_1r_2r_3} > 1.$$

Some special cases are $[\ell,\ell,2\ell]$-groups studied by J. Wyss-Gallifent (compare (5) in section 14), $[\ell,2\ell,3\ell]$-groups etc.

The computation of

$$\rho(\mathrm{trace}(w_B)) = \frac{f_B(t)}{(t^2+1)^3}$$

for the triples (r_1,r_2,r_3) satisfying the equation $(*)$ is simpler. We obtain

$$\frac{f_B(t)}{1024R} = (1-R)t^4 + (64R^3-80R^2+11R+2)t^2 + (64R^3+48R^2+12R+1),$$

where $R=r_1r_2r_3$. The polynomial f_B has no roots if $R<7/8$. For $R\geqslant 7/8$ the roots of the polynomial f_B can be computed explicitly as $\pm t_B^{\pm}$, where

$$t_B^{\pm} = \sqrt{\frac{2+11R-80R^2+64R^3 \pm R\sqrt{(8R-7)^3(8R+1)}}{2(R-1)}}.$$

More precisely,

(1) for $R<7/8$ we have $f_B(t)>0$ for all $t\in\mathbb{R}$,
(2) for $R=7/8$ we have $f_B(t)>0$ for all $t\in\mathbb{R}\backslash\{\pm t_B^{\pm}\}$,
(3) for $7/8<R<1$ we have $f_B(t)>0$ for $|t|<t_B^-$ or $|t|>t_B^+$ and $f_B(t)<0$ for $t_B^-<|t|<t_B^+$,
(4) for $R>1$ we have $f_B(t)>0$ for $|t|<t_B^+$ and $f_B(t)<0$ for $|t|>t_B^+$.

In the ideal triangle group case we have $R=1$ and hence (compare [GP92])

$$\rho(\tau_{123})\cdot(t^2+1)^3 = 1024\cdot(125-3t^2).$$

In the ultra-parallel case the element w_A remains hyperbolic for all $t\in\mathbb{R}$ and the element w_B goes elliptic for $t=t_B^+$, hence

Proposition 15. *Any ultra-parallel $[\ell_1,\ell_2,\ell_3]$-triangle group such that*

$$r_1^2+r_2^2+r_3^2 = 1+2r_1r_2r_3,$$

i.e. any ultra-parallel $[\ell_1,\ell_2,\ell_1+\ell_2]$-triangle group is of type B.

In the non-ideal case we obtain the following result

Proposition 16. *A (p_1,p_2,p_3)-triangle group such that*

$$r_1^2+r_2^2+r_3^2 = 1+2r_1r_2r_3$$

and

$$r_1r_2r_3 \geqslant \frac{13+\sqrt{297}}{32} \approx 0.9448$$

is of type B. For instance

- *the triples* (p, p, ∞) *with* $p \geqslant 14$,
- *the triples* $(p, 2p, 2p)$ *with* $p \geqslant 12$,
- *the triples* $(p, 3p/2, 3p)$ *with* $p \geqslant 12$,
- *the triples* $(p, 4p/3, 4p)$ *with* $p \geqslant 12$

are of type B.

Proof. The element w_A goes elliptic for $t = t_A$. The element w_B goes elliptic for $t = t_B^-$ since $7/8 < R < 1$. We recall that $R = r_1 r_2 r_3$. We are going to prove that $t_A > t_B^-$. We have

$$c_A - R = \frac{(4r_1^2 r_2^2 - 1)(1 - r_3^2)}{4R} \geqslant 0$$

and hence $c_A \geqslant R$. This implies $t_A \geqslant t_A^\circ$, where

$$t_A^\circ = \sqrt{\frac{1+R}{1-R}}.$$

(t_A° is the value of t_A for $r_1 = r_2 = \sqrt{R}$ and $r_3 = 1$.) We obtain

$$f_B(t_A^\circ) = -\frac{2048R}{1-R} \cdot (16R^2 - 13R - 2).$$

For $R \geqslant (13 + \sqrt{297})/32$ it holds $16R^2 - 13R - 2 > 0$. This implies $f_B(t_A^\circ) < 0$ and hence $t_A^\circ \geqslant t_B^-$. The inequalities $t_A \geqslant t_A^\circ > t_B^-$ imply $t_A > t_B^-$. □

Remark. The condition $r_1 r_2 r_3 \geqslant (13 + \sqrt{297})/32$ is sufficient but not necessary for triples satisfying $(*)$ to be of type B. Experimental results of Richard Schwartz (see the conjectural census for [Sch]) suggest that this condition is also sufficient for other triples (p_1, p_2, p_3), in particular for triples of the form $(10, p_2, p_3)$ with $p_2 \leqslant 25$, $(11, p_2, p_3)$ with $p_2 \leqslant 17$, $(12, p_2, p_3)$ with $p_2 \leqslant 14$, and $(13, 13, p_3)$.

13. Applications: Arithmetic Properties of Traces

Let $G(p_1, p_2, p_3; n)$ be the (p_1, p_2, p_3)-triangle group such that ι_{3132} is a rotation by $2\pi/n$. Then $\tau_{3132} = 1 + 2\cos(2\pi/n)$ (compare Lemma 3.18 in [Par]). On the other hand

$$\tau_{3132} = \sigma_3 = (16r_1^2 r_2^2 + 4r_3^2 - 1) - 16 r_1 r_2 r_3 \cos\alpha$$

and hence

$$\cos\alpha = (8r_1 r_2 r_3)^{-1} \cdot \left((8r_1^2 r_2^2 + 2r_3^2 - 1) - \cos\frac{2\pi}{n} \right).$$

Here $n = \infty$ is allowed and means $\tau_{3132} = 3$.

Proposition 17. *Let τ be the trace of an element in the group $G(p_1, p_2, p_3; n)$, then $2\operatorname{Re}(\tau)$ and $|\tau|^2$ both belong to the ring*

$$\mathbb{Z}\left[2\cos\left(\frac{2\pi}{p_1}\right), 2\cos\left(\frac{2\pi}{p_2}\right), 2\cos\left(\frac{2\pi}{p_3}\right), 2\cos\left(\frac{2\pi}{n}\right)\right].$$

Proof. Because of $2\cos(2\pi/p_k) = 4r_k^2 - 2$ the ring

$$\mathbb{Z}\left[2\cos(2\pi/p_1), 2\cos(2\pi/p_2), 2\cos(2\pi/p_3), 2\cos(2\pi/n)\right]$$

coincide with $R\,[2\cos(2\pi/n)]$, where $R = \mathbb{Z}[4r_1^2, 4r_2^2, 4r_3^2]$. For the trace τ of an element of $G(p_1,p_2,p_3;n)$ it holds according to the combinatorial trace formula $\tau = (-1)^{|a|}\cdot(2+\xi)$, where $|a|$ is the length of a and

$$\xi = \sum_{w\in\mathbb{Z}} q_w \cdot e^{i\alpha w}.$$

We have $2\,\mathrm{Re}(\tau) = (-1)^{|a|}\cdot(4+2\,\mathrm{Re}(\xi))$ and $|\tau|^2 = 4+4\,\mathrm{Re}(\xi)+|\xi|^2$, hence is is sufficient to prove that $2\,\mathrm{Re}(\xi)$ and $|\xi|^2$ belong to the ring $R[2\cos(2\pi/n)]$. We have

$$2\,\mathrm{Re}(\xi) = \sum_{w\in\mathbb{Z}} q_w\cdot(2\cos(\alpha w))$$

and

$$|\xi|^2 = \xi\cdot\bar{\xi} = \sum_{w\in\mathbb{Z}} q_w^2 + \sum_{\substack{w,w'\in\mathbb{Z}\\ w\neq w'}} q_w\cdot q_{w'}\cdot(2\cos(\alpha(w-w'))).$$

But $2\cos(\alpha w)$ is a polynomial of degree $|w|$ in $2\cos\alpha$ with integer coefficients, even for even $|w|$ and odd for odd $|w|$ (Chebyshev polynomials, compare for example [San95]). This implies

$$2\cos(\alpha w) \in (8r_1r_2r_3)^{-|w|}\cdot R[2\cos(2\pi/n)]$$

because of

$$2\cos\alpha \in (8r_1r_2r_3)^{-1}\cdot R[2\cos(2\pi/n)]$$

and hence

$$\begin{aligned}(2\cos\alpha)^{|w|-2j} &\in (8r_1r_2r_3)^{-|w|}\cdot(4r_1^2\cdot 4r_2^2\cdot 4r_3^2)^j\cdot R[2\cos(2\pi/n)]\\ &\subset (8r_1r_2r_3)^{-|w|}\cdot R[2\cos(2\pi/n)].\end{aligned}$$

On the other hand the coefficients q_w satisfy $q_w \in (8r_1r_2r_3)^{|w|}\cdot R$ by Proposition 8. This finishes the proof. □

Corollary 18. *Let τ be the trace of an element in $G(p_1,p_2,p_3;n)$ with*

$$\{p_1,p_2,p_3,n\}\subset\{3,4,6,\infty,q\},$$

where q is any natural number, then $2\,\mathrm{Re}(\tau)$ and $|\tau|^2$ both belong to the ring $\mathbb{Z}[2\cos(2\pi/q)]$.

Proof. For $a\in\{3,4,6,\infty\}$ we have $2\cos(2\pi/a)\in\mathbb{Z}$. □

14. History and Geography

In this section we describe some of the results in the area of complex hyperbolic triangle groups and specify there position in the space of triples (r_1,r_2,r_3).

(1) ideal triangle case: $r_1=r_2=r_3=1$, ($p_k=\infty$, $\varphi_k=0$), type B.

Ideal triangle group representations were first studied in [GP92] by William Goldman and John Parker. They proved that an ideal triangle group representation is a discrete embedding for $\cos\alpha<\frac{17}{18}$ and is not a discrete embedding for $\cos\alpha>\frac{61}{64}$, and they conjectured that an ideal triangle group representation is a discrete embedding for $\cos\alpha\leqslant\frac{61}{64}$. This conjecture was confirmed by Richard E. Schwartz in [Sch01b]. He proved that an ideal triangle group representation is a discrete embedding if and only if $\cos\alpha\leqslant\frac{61}{64}$. In [Sch01a] he also used the so called last ideal triangle group, i.e. the group with ι_{123} parabolic, i.e. with $\cos\alpha = \frac{61}{64}$, to construct the first example of a complete hyperbolic 3-manifold with a spherical CR-structure. Hanna Sandler suggested in [San95] to study formulas for the traces of elements in an ideal triangle group in order to prove the Goldman-Parker conjecture.

(2) $(4,4,4)$-groups: $r_1 = r_2 = r_3 = 2^{-1/2}$, $(p_k = 4,\ \varphi_k = \pi/4)$, type A.

Richard E. Schwartz studied in [Sch03a] the groups $G(4,4,4;n)$ and used the group $G(4,4,4;7)$, i.e. the $(4,4,4)$-group with $\cos\alpha = 2^{-3/2}\cdot(2-\cos(2\pi/7))$ to construct the first example of a compact hyperbolic 3-manifold with a spherical CR-structure.

(3) $(4,4,\infty)$-groups: $r_1 = r_2 = 2^{-1/2}$, $r_3 = 1$, type A.

Justin Wyss-Gallifent studied $(4,4,\infty)$-groups in [WG00]. He proved that a $(4,4,\infty)$-group is not discrete for $\cos\alpha > 1/2$ and that the groups $G(4,4,\infty;n)$ are discrete for $n = 3,4,\infty$.

(4) (p,p,∞)-groups: $r_1 = r_2$, $r_3 = 1$.

The (p,p,∞)-groups were studied also by Justin Wyss-Gallifent in [WG00]. He proved that a triple (p,p,∞) is of type A for $p \leqslant 13$ and of type B for $p \geqslant 14$.

(5) ultra-parallel $[\ell,\ell,2\ell]$-groups: $r_1 = r_2 = r$, $r_3 = 2r^2 - 1$.

Justin Wyss-Gallifent studied in [WG00] ultra-parallel $[\ell,\ell,2\ell]$-groups and obtained partial results similar to the results in [GP92] on ideal triangle groups.

(6) Mostow's non-arithmetic complex hyperbolic lattices: $\mu_1 = \mu_2 = \mu_3 = e^{\frac{2\pi i}{p}}$, $p \in \{3,4,5\}$, $r_1 = r_2 = r_3 = \frac{1}{2\sin\frac{\pi}{p}}$, $\alpha = \frac{2\pi}{\rho} + \frac{\pi}{p} - \frac{\pi}{2}$, $\rho \in \mathbb{Z}$.

G. D. Mostow studied in [Mos80] some triangle groups generated by three μ-reflections of the same order $p \in \{3,4,5\}$ in the sides of a equiangular triangle with angles related to the order p and with special values of the angular invariant α and obtained the first examples of non-arithmetic complex hyperbolic lattices.

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