NON-DISCRETE COMPLEX HYPERBOLIC TRIANGLE GROUPS OF TYPE (m, m, ∞)

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Abstract. In this note we prove that a complex hyperbolic triangle group of type (m, m, ∞), i.e. a group of isometries of the complex hyperbolic plane, generated by complex reflexions in three complex geodesics meeting at angles π/m, π/m and 0, is not discrete if the product of the three generators is regular elliptic.

We study representations of real hyperbolic triangle groups, i.e. groups generated by reflexions in the sides of triangles in $H^2_{\mathbb{R}}$, in the holomorphic isometry group $\mathrm{PU}(2,1)$ of the complex hyperbolic plane $H^2_{\mathbb{C}}$.

For the basic notions of complex hyperbolic geometry, especially for the complex hyperbolic plane $H^2_{\mathbb{C}}$, see for example section 2 in [Pra05]. The general references on complex hyperbolic geometry are [Gol99, Par03].

We use the following terminology: A *complex hyperbolic triangle* is a triple (C_1, C_2, C_3) of complex geodesics in $H^2_{\mathbb{C}}$. For a triple (p_1, p_2, p_3), where each of the numbers p_k can be either a positive integer or equal to ∞, we say that a triangle (C_1, C_2, C_3) is a (p_1, p_2, p_3)-*triangle* if the complex geodesics C_{k-1} and C_{k+1} meet at the angle π/p_k when p_k is finite resp. at the angle 0 when p_k is equal to ∞. A subgroup of $\mathrm{PU}(2,1)$ generated by complex reflexions ι_k in the sides C_k of a complex hyperbolic (p_1, p_2, p_3)-triangle (C_1, C_2, C_3) we call a (p_1, p_2, p_3)-*triangle group*.

We prove in this paper the following result:

Theorem. *Let m be an integer, $m \geqslant 3$, then an (m, m, ∞)-triangle group is not discrete if the product of the three generators is regular elliptic.*

Remark. For ideal triangle groups, i.e. (∞, ∞, ∞)-triangle groups, this statement was proved in [Sch01]. The statement for (m, m, ∞)-triangle groups was formulated in [WG00] (Lemma 3.4.0.19), but the proof there had a gap.

Remark. Using our result and a complex hyperbolic version of Shimizu's lemma, Shigeyasu Kamiya, John R. Parker and James M. Thompson have identified large classes of (m, m, ∞)-groups which are not discrete, see [Kam07], [KPT09].

For a given triple (p_1, p_2, p_3) the space of (p_1, p_2, p_3)-triangle groups is of real dimension one. We now describe a parameterisation of the space of complex hyperbolic triangles in $H^2_{\mathbb{C}}$ by means of an angular invariant α. See section 3 in [Pra05]

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for details. Let (C_1, C_2, C_3) be a complex hyperbolic triangle. Let c_k be the normalised polar vector of the complex geodesic C_k. We define the *angular invariant* α of the triangle (C_1, C_2, C_3) as

$$\alpha = \arg\left(\prod_{k=1}^{3}\langle c_{k-1}, c_{k+1}\rangle\right).$$

A complex hyperbolic triangle in $H^2_{\mathbb{C}}$ is determined uniquely up to isometry by the three angles and the angular invariant α (compare proposition 1 in [Pra05]). For any $\alpha \in [0, 2\pi]$ a (p_1, p_2, p_3)-triangle with the angular invariant α exists if and only if

$$\cos\alpha < \frac{r_1^2 + r_2^2 + r_3^2 - 1}{2r_1r_2r_3},$$

where $r_k = \cos(\pi/p_k)$. In the case $(p_1, p_2, p_3) = (m, m, \infty)$ we have

$$\frac{r_1^2 + r_2^2 + r_3^2 - 1}{2r_1r_2r_3} = 1,$$

thus for every $\alpha \in (0, 2\pi)$ there exists an (m, m, ∞)-triangle with the angular invariant α. The angular invariant $\alpha = 0$ corresponds to the case where all three complex geodesics meet at one point.

Proof of the theorem: Let (C_1, C_2, C_3) be an (m, m, ∞)-triangle. Let $\iota_k = \iota_{C_k}$ be the complex reflexion in the complex geodesic C_k. Let us assume that the element $\iota_1\iota_2\iota_3$ is regular elliptic. If the element $\iota_1\iota_2\iota_3$ is of infinite order, then the cyclic group generated by this element is not discrete, hence the (m, m, ∞)-triangle group generated by $\iota_1, \iota_2, \iota_3$ is not discrete. It remains to show that the element $\iota_1\iota_2\iota_3$ cannot be regular elliptic of finite order.

We shall assume that the element $\iota_1\iota_2\iota_3$ is regular elliptic of finite order and show that this assumption leads to a contradiction. Let τ be the trace of the corresponding matrix in $\mathrm{SU}(2, 1)$. Since the order of the element is finite, the eigenvalues of this matrix are roots of unity. Their product is equal to 1. Hence

$$\tau = \omega_n^{k_1} + \omega_n^{k_2} + \omega_n^{k_3}$$

for some $k_1, k_2, k_3 \in \mathbb{Z}$ with $k_1 + k_2 + k_3 = 0$. Here $\omega_n = \exp(2\pi i/n)$ and n is taken as small as possible. On the other hand, for a (p_1, p_2, p_3)-triangle group the trace τ can be computed (see section 8 in [Pra05]) as

$$\tau = 8r_1r_2r_3e^{i\alpha} - (4(r_1^2 + r_2^2 + r_3^2) - 3),$$

where $r_k = \cos(\pi/p_k)$. For (m, m, ∞)-groups we have $r_1 = r_2 = r$ and $r_3 = 1$, where $r = \cos(\pi/m)$. Hence

$$\tau = (8r^2)e^{i\alpha} - (8r^2 + 1).$$

This equation implies that the complex number τ lies on the circle with centre at $-(8r^2 + 1)$ and radius $8r^2$, or, in other words, τ satisfies the equation

$$(\tau + (8r^2 + 1)) \cdot (\bar{\tau} + (8r^2 + 1)) = |\tau + (8r^2 + 1)|^2 = (8r^2)^2.$$

Moreover, since $\alpha \in (0, 2\pi)$, we have $e^{i\alpha} \neq 1$ and therefore $\tau = 8r^2(e^{i\alpha} - 1) - 1 \neq -1$. Thus

$$\mathrm{Re}(\tau) < -1.$$

Let N be the least common multiple of n and $2m$. Let σ_k be the homomorphism of $\mathbb{Q}[\omega_N]$ given by $\sigma_k(\omega_N) = \omega_N^k$. For k relatively prime to n the restriction of σ_k to $\mathbb{Q}[\omega_n]$ is a Galois automorphism.

Lemma 1. *Let $\tau = \omega_n^{k_1} + \omega_n^{k_2} + \omega_n^{k_3}$ be the trace of the matrix of $\iota_1\iota_2\iota_3$, where n is taken as small as possible. Then $\sigma_k(\tau)$ satisfies the equation*

$$|\sigma_k(\tau) + \sigma_k(8r^2) + 1| = \sigma_k(8r^2).$$

This implies in particular

$$\mathrm{Re}(\sigma_k(\tau)) \leqslant -1.$$

Proof. We have

$$\tau \in \mathbb{Q}[\omega_n] \subset \mathbb{Q}[\omega_N]$$

and

$$2r = 2\cos\left(\frac{\pi}{m}\right) = \omega_{2m} + \bar{\omega}_{2m} \in \mathbb{Q}[\omega_{2m}] \subset \mathbb{Q}[\omega_N],$$

hence the equation $|\tau + (8r^2+1)| = 8r^2$ is defined in $\mathbb{Q}[\omega_N]$. The homomorphism σ_k commutes with complex conjugation and hence maps real numbers to real numbers. Applying the homomorphism σ_k to the equation

$$(\tau + (8r^2+1)) \cdot (\bar{\tau} + (8r^2+1)) = (8r^2)^2$$

we obtain

$$(\sigma_k(\tau) + \sigma_k(8r^2+1))(\sigma_k(\bar{\tau}) + \sigma_k(8r^2+1)) = (\sigma_k(8r^2))^2.$$

This equation is equivalent to

$$|\sigma_k(\tau) + \sigma_k(8r^2) + 1|^2 = (\sigma_k(8r^2))^2.$$

Since $\sigma_k(2r)$ is a real number, the number $\sigma_k(8r^2) = 2(\sigma_k(2r))^2$ is a non-negative real number. Hence $\sigma_k(\tau)$ satisfies the equation

$$|\sigma_k(\tau) + \sigma_k(8r^2) + 1| = \sigma_k(8r^2).$$

This equation means that the complex number $\sigma_k(\tau)$ lies on the circle with centre at $-(\sigma_k(8r^2) + 1) < 0$ and radius $\sigma_k(8r^2) \geqslant 0$. This implies in particular

$$\mathrm{Re}(\sigma_k(\tau)) \leqslant -1. \quad \square$$

Lemma 2. *Let $\tau = \omega_n^{k_1} + \omega_n^{k_2} + \omega_n^{k_3}$ be the trace of the matrix of $\iota_1\iota_2\iota_3$, where n is taken as small as possible. For $i \in \{1, 2, 3\}$ let*

$$d_i = \frac{n}{\gcd(k_i, n)},$$

where gcd *is the greatest common divisor. Then*

$$\frac{1}{\varphi(d_1)} + \frac{1}{\varphi(d_2)} + \frac{1}{\varphi(d_3)} > 1.$$

Proof. According to Lemma 1,

$$\mathrm{Re}(\sigma_k(\tau)) \leqslant -1$$

for any homomorphism σ_k. Summing over all $k \in \{1, \ldots, n-1\}$ relatively prime to n we obtain

$$\mathrm{Re}\left(\sum_{\substack{1 \leqslant k < n \\ (k,n)=1}} \sigma_k(\tau)\right) \leqslant -\varphi(n).$$

Here φ is the Euler φ-function. Taking into account the fact that $\mathrm{Re}(\tau) < -1$ we obtain

$$\mathrm{Re}\left(\sum_{\substack{1\leqslant k<n\\(k,n)=1}} \sigma_k(\tau)\right) < -\varphi(n).$$

Thus

$$\Big|\sum_{\substack{1\leqslant k<n\\(k,n)=1}} \sigma_k(\tau)\Big| > \varphi(n).$$

The root of unity $\omega_n^{k_i}$ is a primitive d_i-th root of unity. The sum of all d_i-th primitive roots of unity is in $\{-1,0,1\}$, and hence is bounded by 1. The map $(\mathbb{Z}/n\mathbb{Z})^* \to (\mathbb{Z}/d_i\mathbb{Z})^*$ induced by the map $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/d_i\mathbb{Z}$ is surjective, and the preimage of any element in $(\mathbb{Z}/d_i\mathbb{Z})^*$ consists of $\varphi(n)/\varphi(d_i)$ elements. Hence we obtain the inequality

$$\Big|\sum_{\substack{1\leqslant k<n\\(k,n)=1}} \sigma_k(\omega_n^{k_i})\Big| \leqslant \frac{1}{\varphi(d_i)}\cdot\varphi(n)$$

for $i\in\{1,2,3\}$. From the inequalities

$$\begin{aligned}\varphi(n) &< \Big|\sum_{\substack{1\leqslant k<n\\(k,n)=1}} \sigma_k(\tau)\Big| \\ &= \Big|\sum_{\substack{1\leqslant k<n\\(k,n)=1}} \sigma_k(\omega_n^{k_1}+\omega_n^{k_2}+\omega_n^{k_3})\Big| \\ &\leqslant \left(\frac{1}{\varphi(d_1)}+\frac{1}{\varphi(d_2)}+\frac{1}{\varphi(d_3)}\right)\cdot\varphi(n)\end{aligned}$$

it follows

$$\frac{1}{\varphi(d_1)}+\frac{1}{\varphi(d_2)}+\frac{1}{\varphi(d_3)} > 1.$$

□

The inequality

$$\frac{1}{\varphi(d_1)}+\frac{1}{\varphi(d_2)}+\frac{1}{\varphi(d_3)} > 1$$

implies that the triple $(\varphi(d_1),\varphi(d_2),\varphi(d_3))$ is equal (up to permutation) to one of the triples

$$(1,?,?),\quad (2,2,?),\quad (2,3,3),\quad (2,3,4),\quad (2,3,5).$$

But for the Euler φ-function we have $\varphi(x)=1$ for $x\in\{1,2\}$, $\varphi(x)=2$ for $x\in\{3,4,6\}$ and $\varphi(x)\geqslant 4$ for all other positive integers x.

- The triples $(2,3,3)$, $(2,3,4)$, $(2,3,5)$ cannot occur since $\varphi(x)\neq 3$ for any integer x.
- Let $\varphi(d_i)=1$ for some $i\in\{1,2,3\}$. Without loss of generality we can assume that $\varphi(d_1)=1$. Then $d_1\in\{1,2\}$, therefore $(k_1,n)\in\{n/2,n\}$ and $k_1\equiv 0, n/2 \bmod n$. Hence

  $$\omega_n^{k_1}\in\{1,-1\}.$$

  If $k_1\equiv 0$ then $k_2+k_3\equiv 0$. Let $k=k_2$, then $k_3\equiv -k$ and

  $$\tau=\omega_n^{k_1}+\omega_n^{k_2}+\omega_n^{k_3}=1+\omega_n^{k}+\omega_n^{-k}=1+2\cos(2\pi k/n)$$

and $\mathrm{Re}(\tau) = 1 + 2\cos(2\pi k/n) \geqslant -1$ in contradiction to $\mathrm{Re}(\tau) < -1$. If $k_1 \equiv n/2$ then $k_2 + k_3 \equiv -n/2$. Let $k = k_2$, then $k_3 \equiv -k - n/2$ and

$$\tau = \omega_n^{k_1} + \omega_n^{k_2} + \omega_n^{k_3} = -1 + \omega_n^k - \omega_n^{-k} = -1 + 2i\sin(2\pi k/n)$$

and $\mathrm{Re}(\tau) = -1$ in contradiction to $\mathrm{Re}(\tau) < -1$.

- If $\varphi(d_i) = \varphi(d_j) = 2$ for $i, j \in \{1, 2, 3\}$, $i \neq j$, then $d_i, d_j \in \{3, 4, 6\}$. Therefore (k_i, n) and (k_j, n) are in $\{n/6, n/4, n/3\}$ and $k_i, k_j \equiv \pm n/6, \pm n/4, \pm n/3 \bmod n$. Hence
$$\omega_n^{k_i}, \omega_n^{k_j} \in \{\alpha^{\pm 2}, \alpha^{\pm 3}, \alpha^{\pm 4}\},$$
where $\alpha = \omega_{12} = \exp(2\pi i/12)$. Thus
$$\tau = \alpha^p + \alpha^q + \alpha^r, \quad p + q + r = 0, \quad p, q \in \{\pm 2, \pm 3, \pm 4\}.$$
Using $\mathrm{Re}(\tau) < -1$ and $\mathrm{Re}(\alpha^r) \geqslant -1$ we obtain
$$\mathrm{Re}(\alpha^p + \alpha^q) = \mathrm{Re}(\tau) - \mathrm{Re}(\alpha^r) < -1 + 1 = 0.$$
Since $\mathrm{Re}(\alpha^{\pm 2}) = \frac{1}{2}$, $\mathrm{Re}(\alpha^{\pm 3}) = 0$ and $\mathrm{Re}(\alpha^{\pm 4}) = -\frac{1}{2}$, we can only have $\mathrm{Re}(\alpha^p + \alpha^q) < 0$ if $\alpha^p + \alpha^q = \alpha^{\pm 3} + \alpha^{\pm 4}$ or $\alpha^p + \alpha^q = \alpha^{\pm 4} + \alpha^{\pm 4}$. For these cases we easily check that $\mathrm{Re}(\alpha^p + \alpha^q + \alpha^{-p-q}) < -1$ holds only if $\alpha^p = \alpha^q = \alpha^4$ or $\alpha^p = \alpha^q = \alpha^{-4}$, i.e. if $\tau = 3\alpha^{\pm 4}$, but then a suitable homomorphism σ_k has the property $\mathrm{Re}(\sigma_k(\tau)) > -1$ in contradiction to Lemma 1.

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