HIGHER SPIN KLEIN SURFACES

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ABSTRACT. We find all m-spin structures on Klein surfaces of genus larger than one. An m-spin structure on a Riemann surface P is a complex line bundle on P whose m-th tensor power is the cotangent bundle of P. A Klein surface can be described by a pair (P,τ), where P is a Riemann surface and τ is an anti-holomorphic involution on P. An m-spin structure on a Klein surface (P,τ) is an m-spin structure on the Riemann surface P which is preserved under the action of the anti-holomorphic involution τ. We determine the conditions for the existence and give a complete description of all real m-spin structures on a Klein surface. In particular we compute the number of m-spin structures on a Klein surface (P,τ) in terms of its natural topological invariants.

1. Introduction

Under an *m-spin Riemann surface* we understand a compact Riemann surface P with a complex line bundle $e: L \to P$ such that the m-th tensor power $e^{\otimes m}: L^{\otimes m} \to P$ is isomorphic to the cotangent bundle of P (compare with [Jar]). This is a natural generalisation of the classical ($m=2$) algebraic curves with Theta characterstics studied by Riemann [R]. The moduli spaces of m-spin Riemann surfaces have been studied because of their connections with integrable systems [Wit], [FSZ]

The invariants of an m-spin Riemann surface (P,e) are given by the genus g of P and the Arf invariant $\delta \in \{0,1\}$. The Arf invariant is determined by the parity of the dimension of the space of sections of the m-spin bundle, see [Ati], [Mum]. For a given Riemann surface of genus g, the number of corresponding m-spin Riemann surfaces is m^{2g}. For odd m the Arf invariant is always $\delta=0$. For even m, the number of m-spin Riemann surfaces with $\delta=1$ and $\delta=0$ is $m^{g-1}(m^g-1)$ and $m^{g-1}(m^g+1)$ respectively, see [Jar], [NP05], [NP09].

A Klein surface is a generalisation of a Riemann surface in the case of non-orientable surfaces or surfaces with boundary. A Klein surface is a quotient P/τ, where P is a compact Riemann surface and $\tau: P \to P$ is an anti-holomorphic involution on P. The category of such pairs (P,τ) is isomorphic to the category of real algebraic curves, see [AG].

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The boundary of the surface P/τ corresponds to the set of fixed points of the involution τ and to the set of real points of the corresponding real algebraic curve. If not empty, the boundary of P/τ decomposes into pairwise disjoint simple closed smooth contours, called *ovals*, see [Nat90a]. (The second kind of contours invariant under the involution τ, called *twists*, are invariant contours which are not pointwise fixed by τ.) The topological type of a Klein surface (P,τ) is determined by the genus g of P, the number k of connected components of the boundary of P/τ and the orientability ($\varepsilon=1$) or non-orientability ($\varepsilon=0$) of P/τ. The invariants (g,k,ε) of a Klein surface (P,τ) satisfy the conditions $0\leqslant k\leqslant g$ for $\varepsilon=0$, $1\leqslant k\leqslant g+1$ and $k\equiv g+1 \bmod 2$ for $\varepsilon=1$, see [Wei]. Moreover, the space of all Klein surfaces with the invariants (g,k,ε) is connected, has dimension $3g-3$ for $g>1$ and is $K(\pi,1)$, see [Nat75, Nat78].

Under an m-spin Klein surface we understand a Klein surface (P,τ) with an m-spin structure $(P, e: L\to P)$ and an anti-holomorphic involution $\beta: L\to L$ such that $e\circ\beta=\tau\circ e$. Recent work [OT] shows the connection between real 2-spin Klein surfaces and Abelian Yang-Mills theory. Combining the topological invariants (g,k,ε) for the Klein surface (P,τ) and (g,m,δ) for the m-spin surface (P,e) we obtain the topological invariants $(g,k,\varepsilon,m,\delta)$ of (P,τ,e,β). In this paper we prove that for any Klein surface (P,τ) of type (g,k,ε) with $g\geqslant 2$ the number $N(g,k,\varepsilon,m,\delta)$ of m-spin Klein surfaces (P,τ,e,β) with the Arf invariant δ only depends on the invariants $(g,k,\varepsilon,m,\delta)$. Moreover, we compute the number $N=N(g,k,\varepsilon,m,\delta)$:

- For odd m we prove $N=m^g$ if $g\equiv 1 \bmod m$, $\delta=0$ and $N=0$ otherwise.
- For even m, $\varepsilon=0$ and $k=0$ we prove $N=\frac{m^g}{2}$ if $g\equiv 1 \bmod \frac{m}{2}$ and $N=0$ otherwise.
- For even m, $\varepsilon=0$ and $k\geqslant 1$ we prove $N=m^g\cdot 2^{k-2}$ if $g\equiv 1 \bmod \frac{m}{2}$ and $N=0$ otherwise.
- For $m\equiv 0 \bmod 4$ and $\varepsilon=1$ we prove $N=m^g\cdot 2^{k-2}$ if $g\equiv 1 \bmod \frac{m}{2}$ and $N=0$ otherwise.
- For $m\equiv 2 \bmod 4$, $\varepsilon=1$ and $\delta=0$ we prove $N=\frac{m^g}{2}(2^{k-1}+1)$ if $g\equiv 1 \bmod \frac{m}{2}$ and $N=0$ otherwise.
- For $m\equiv 2 \bmod 4$, $\varepsilon=1$ and $\delta=1$ we prove $N=\frac{m^g}{2}(2^{k-1}-1)$ if $g\equiv 1 \bmod \frac{m}{2}$ and $N=0$ otherwise.

For $m=2$ similar results were obtained in [Nat90b, Nat04]. The cases when P is a sphere or a torus require different methods.

Our investigation of m-spin Klein surfaces is based on m-Arf functions. An m-Arf function is a function on the set $\pi_1^0(P)$ of oriented simple contours on P with values in $\mathbb{Z}/m\mathbb{Z}$ which satisfies certain geometric-algebraic properties. It can also be interpreted as the monodromy of a natural connection on the m-spin bundle. According to [NP09], m-Arf functions are in 1-to-1 correspondence with m-spin Riemann surfaces.

In sections 2 and 3 we extend the constructions from [NP09] to Klein surfaces. We prove that m-spin Klein surfaces correspond to m-Arf functions which satisfy the conditions

- $\sigma(\tau c)=-\sigma(c)$ for all $c\in\pi_1^0(P)$;
- $\sigma(c)=0$ for any twist $c\in\pi_1^0(P)$.

In section 4 we prove our main theorems.

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2. Coverings of the Group of Automorphisms of the Hyperbolic Plane

2.1. **Standard Covering.** Let $G = \mathrm{Aut}(\mathbb{H})$ be the full isometry group of the hyperbolic plane $\mathbb{H}$. Here our model of the hyperbolic plane is the upper half-plane in $\mathbb{C}$. This group has two connected components, the group $G^+ = \mathrm{Aut}_+(\mathbb{H})$ of all orientation-preserving isometries of $\mathbb{H}$ and the group $G^- = \mathrm{Aut}_-(\mathbb{H})$ of all orientation-reversing isometries of $\mathbb{H}$. Let e_G be the identity element in G. Let $j \in G^-$ be the reflection in the imaginary axis, $j(z) = -\bar{z}$. Then $G^- = j \cdot G^+$.

Definition 2.1. Let $\pi : G_m \to G$ be the Lie group m-fold covering of G given by $G_m = G_m^+ \cup G_m^-$ with

$$G_m^+ = \left\{ (g, \delta) \in G^+ \times \mathrm{Hol}(\mathbb{H}, \mathbb{C}^*) \;\middle|\; \delta^m = \frac{d}{dz} g \right\},$$

$$G_m^- = \left\{ (g, \delta) \in G^- \times \overline{\mathrm{Hol}}(\mathbb{H}, \mathbb{C}^*) \;\middle|\; \delta^m = \frac{d}{d\bar{z}} g \right\}$$

with the product of elements (g_1, δ_1) and (g_2, δ_2) in G_m given by

$$(g_2, \delta_2) \cdot (g_1, \delta_1) = \begin{cases} (g_2 \circ g_1, (\delta_2 \circ g_1) \cdot \delta_1) & \text{if} \quad (g_2, \delta_2) \in G_m^+, \\ (g_2 \circ g_1, (\delta_2 \circ g_1) \cdot \bar{\delta}_1) & \text{if} \quad (g_2, \delta_2) \in G_m^-. \end{cases}$$

The identity element of G_m is $e_{G_m} = (e_G, 1)$, where the second component is the constant function $z \mapsto 1$.

The centre of $G^+ = \mathrm{Aut}_+(\mathbb{H})$ is trivial, $Z(G^+) = \{e_G\}$. Hence the centre of G_m^+ is contained in $\pi^{-1}(e_G)$, the set of elements of the form $(e_G, \exp(2\pi i k/m))$, $k = 0, 1, \ldots, m-1$, where the second component is the constant function $z \mapsto \exp(2\pi i k/m)$. From the group law it follows that all such elements belong to the centre, hence the centre is cyclic of order m:

$$Z(G_m^+) = \pi^{-1}(e_G) \cong \mathbb{Z}/m\mathbb{Z}.$$

Then $U = (e_G, \exp(2\pi i/m))$ is a generator of the centre,

$$Z(G_m^+) = \langle U \rangle = \{e_{G_m}, U, U^2, \ldots, U^{m-1}\}.$$

Let $J \in G_m^-$ be a pre-image of the reflection j. Then $G_m^+ = J \cdot G_m^-$.

Proposition 2.1. *For the pre-image $J \in G_m^-$ of j we have $J^2 = e_{G_m}$.*

Proof. The element J must of the form $J = (j, \delta)$ with $\delta^m = \frac{d}{d\bar{z}} j = -1$, i.e. $\delta : \mathbb{H} \to \mathbb{C}^*$ is a constant function with $\delta^m = -1$. Hence $J^2 = (j, \delta) \cdot (j, \delta) = (j \circ j, (\delta \circ j) \cdot \bar{\delta}) = (e_G, |\delta|^2) = (e_G, 1) = e_{G_m}$. □

Remark. The Lie group G^+ is connected with infinite cyclic fundamental group, hence the Lie group m-fold covering G_m^+ of G^+ is unique up to an isomorphism. The Lie group $G = G^+ \cup G^-$ is not connected and could have several non-isomorphic Lie group m-fold coverings. In fact we will see (compare with remark after Proposition 2.3) that for odd m there is only one m-fold covering up to isomorphy, while for even m there are two non-isomorphic m-fold coverings with pre-images of all reflections having order 2 or 4 respectively. We are using the former covering which we described explicitly in Definition 2.1.

Elements of G^+ can be classified with respect to the fixed point behavior of their action on $\mathbb{H}$. An element is called *hyperbolic* if it has two fixed points, which lie on the boundary $\partial\mathbb{H} = \mathbb{R} \cup \{\infty\}$ of $\mathbb{H}$. A hyperbolic element with fixed points α, β in $\mathbb{R}$ is of the form

$$\tau_{\alpha,\beta}(\lambda) : z \mapsto \frac{(\lambda\alpha - \beta)z - (\lambda - 1)\alpha\beta}{(\lambda - 1)z + (\alpha - \lambda\beta)},$$

where $\lambda > 0$. One of the fixed points of a hyperbolic element is attracting, the other fixed point is repelling. The *axis* of a hyperbolic element g is the geodesic between the fixed points of g, oriented from the repelling fixed point to the attracting fixed point. The axis of a hyperbolic element is preserved by the element. The map $\lambda \mapsto \tau_{\alpha,\beta}(\lambda)$ defines a homomorphism $\mathbb{R}_+ \to G$ (with respect to the multiplicative structure on $\mathbb{R}_+$). We have $(\tau_{\alpha,\beta}(\lambda))^{-1} = \tau_{\alpha,\beta}(\lambda^{-1}) = \tau_{\beta,\alpha}(\lambda)$.

An element is called *parabolic* if it has one fixed point, which is on the boundary $\partial\mathbb{H}$. A parabolic element with real fixed point α is of the form

$$\pi_\alpha(\lambda) : z \mapsto \frac{(1 - \lambda\alpha)z + \lambda\alpha^2}{-\lambda z + (1 + \lambda\alpha)}.$$

The map $\lambda \mapsto \pi_\alpha(\lambda)$ defines a homomorphism $\mathbb{R} \to G$ (with respect to the additive structure on $\mathbb{R}$). We have $(\pi_\alpha(\lambda))^{-1} = \pi_\alpha(-\lambda)$.

An element that is neither hyperbolic nor parabolic is called *elliptic*. It has one fixed point that is in $\mathbb{H}$. Given a base-point $x \in \mathbb{H}$ and a real number φ, let $\rho_x(\varphi) \in G$ denote the rotation through angle φ counter-clockwise about the point x. Any elliptic element is of the form $\rho_x(\varphi)$, where x is the fixed point. Thus we obtain a 2π-periodic homomorphism $\rho_x : \mathbb{R} \to G$ (with respect to the additive structure on $\mathbb{R}$).

Elements of G_m^+ can be classified with respect to the fixed point behavior of action on $\mathbb{H}$ of their image in G^+. We say that an element of G_m is *hyperbolic*, *parabolic* resp. *elliptic* if its image in G^+ has this property.

The homomorphisms

$$\tau_{\alpha,\beta} : \mathbb{R}_+ \to G, \quad \pi_\alpha : \mathbb{R} \to G, \quad \text{resp.} \quad \rho_x : \mathbb{R} \to G$$

define one-parameter-subgroups in the group G. Each of these homomorphisms lifts to a unique homomorphism into the m-fold cover:

$$T_{\alpha,\beta} : \mathbb{R}_+ \to G_m, \quad P_\alpha : \mathbb{R} \to G_m \quad \text{resp.} \quad R_x : \mathbb{R} \to G_m.$$

The elements $T_{\alpha,\beta}(\lambda)$, $P_\alpha(\lambda)$ and $R_x(\xi)$ are hyperbolic, parabolic and elliptic resp.

A simple computation shows that for $x = i \in \mathbb{H}$ we obtain

$$R_x(2\pi) = (e_G, \exp(2\pi i/m)) = U.$$

Hence

$$R_x(2\pi k) = R_x(2\pi)^k = U^k$$

for $x = i \in \mathbb{H}$ and any integer k. Since $\rho_x(2\pi k) = \mathrm{id}$ for any integer k, it follows that the lifted element $R_x(2\pi k)$ belongs to $\pi^{-1}(e_G) = Z(G_m^+)$. Note that the element $R_x(2\pi k)$ depends continuously on x. But the fibre $\pi^{-1}(e_G)$ is discrete, so the element $R_x(2\pi k)$ must remain constant, thus it does not depend on x. We obtain $R_x(2\pi k) = U^k$ for any integer k.

The following identities are easy to check geometrically:

Proposition 2.2. *We have* $j\tau_{\alpha,\beta}(\lambda)j^{-1} = \tau_{-\alpha,-\beta}(\lambda)$, $j\pi_\alpha(\lambda)j^{-1} = \pi_{-\alpha}(-\lambda)$, $j\rho_x(t)j^{-1} = \rho_{-\bar{x}}(-t)$. *In particular* $j\tau_{0,\infty}(\lambda)j^{-1} = \tau_{0,\infty}(\lambda)$, $j\pi_0(\lambda)j^{-1} = \pi_0(-\lambda)$, $j\rho_i(t)j^{-1} = \rho_i(-t)$.

Lifting these identities into G_m we obtain the following proposition:

Proposition 2.3. *1) We have*

$$\begin{aligned} &JT_{\alpha,\beta}(\lambda)J^{-1} = J^{-1}T_{\alpha,\beta}(\lambda)J = T_{-\alpha,-\beta}(\lambda), \\ &JP_\alpha(\lambda)J^{-1} = J^{-1}P\alpha(\lambda)J = P_{-\alpha}(-\lambda), \\ &JR_x(t)J^{-1} = J^{-1}R_x(t)J = R_{-\bar{x}}(-t). \end{aligned}$$

2) In particular

$$\begin{aligned} &JT_{0,\infty}(\lambda)J^{-1} = J^{-1}T_{0,\infty}(\lambda)J = T_{0,\infty}(\lambda), \\ &JP_0(\lambda)J^{-1} = J^{-1}P_0(\lambda)J = P_0(-\lambda), \\ &JR_i(t)J^{-1} = J^{-1}R_i(t)J = R_i(-t), \\ &JUJ^{-1} = J^{-1}UJ = U^{-1}. \end{aligned}$$

Proof. 1) The identity $j\rho_x(t)j^{-1} = \rho_{-\bar{x}}(-t)$ implies that the paths $t \mapsto JR_x(t)J^{-1}$ and $t \mapsto R_{-\bar{x}}(-t)$ in G_m have the same projection in G and coincide at $t = 0$, thus

$$JR_x(t)J^{-1} = R_{-\bar{x}}(-t).$$

The proofs of the other identities are similar.

2) The proofs are straightforward. For the last identity recall that $U = R_i(2\pi)$ and $U^{-1} = R_i(-2\pi)$.

□

Remark. In Proposition 2.1 we proved that $J^2 = e$ using the explicit description of G_m in Definition 2.1 (compare also with the remark after Proposition 2.1). Note that our proof of Proposition 2.3 works for any Lie group m-fold covering of G, not just for the one described in 2.1. If we forget about Proposition 2.1, we can use Proposition 2.3 to derive some information about J^2. For the reflection j we have $j^2 = e$, hence J^2 is in the pre-image of e, i.e. $J^2 = U^q$ for some integer q. The identities $JUJ^{-1} = U^{-1}$ and $J^{-1}UJ = U^{-1}$ imply $JU = U^{-1}J$ and $UJ = JU^{-1}$. We have $J^3 = J^2J = U^qJ$ and, using $UJ = JU^{-1}$, we obtain $J^3 = U^qJ = JU^{-q}$. On the other hand $J^3 = JJ^2 = JU^q$. Thus $JU^{-q} = JU^q$ and therefore $U^{2q} = e$. For odd m this is only possible for $q \equiv 0 \bmod m$, hence $J^2 = e$, while for even m we could have $q \equiv 0 \bmod m$ and hence $J^2 = e$ or $q \equiv m/2 \bmod m$ and hence $J^2 = U^{m/2}$, $J^4 = e$.

2.2. **Level function.**

Definition 2.2. Let Δ be the set of all elliptic elements of order 2 in G^+. Let Ξ be the complement of the set Δ in G^+, i.e. $\Xi = G^+ \backslash \Delta$. There exists a homeomorhism $G^+ \to \mathbb{S}^1 \times \mathbb{C}$ such that Δ corresponds to $\{*\} \times \mathbb{C}$ and $\Xi = G^+ \backslash \Delta$

corresponds to $(\mathbb{S}^1 \backslash \{*\}) \times \mathbb{C}$ (see, for example, [JN]). From this description it follows in particular that the subset Ξ is simply connected. The pre-image $\tilde{\Xi} \subset G_m^+$ of Ξ consists of m connected components, each of which is homeomorphic to Ξ. Each connected component of the subset $\tilde{\Xi}$ contains one and only one element of

$$\pi^{-1}(e_G) = Z(G_m^+) = \{e_{G_m}, U, \ldots, U^{m-1}\}.$$

Let $\tilde{\Xi}_k$ be the connected component of $\tilde{\Xi}$ that contains U^k. Let a be an element of G_m^+. For $a \in \tilde{\Xi}_k$ we set $s_m(a) = k$. Any $a \notin \tilde{\Xi}$ can be written as $a = R_x(\pi) \cdot U^k$ for some $x \in \mathbb{H}$ and some integer k. We set $s_m(R_x(\pi) \cdot U^k) = k$ for integer k. We call the function $s_m : G_m^+ \to \mathbb{Z}/m\mathbb{Z}$ the *level function*. We say that a is *at the level* k if $s_m(a) = k$.

Remark. Any hyperbolic or parabolic element in G_m^+ is of the form $T_{\alpha,\beta}(\lambda) \cdot U^k$ or $P_\alpha(\lambda) \cdot U^k$ resp. For elements written in this form we have

$$s_m(T_{\alpha,\beta}(\lambda) \cdot U^k) = k, \quad s_m(P_\alpha(\lambda) \cdot U^k) = k.$$

Proposition 2.4. *For any elements A and B in G_m^+ we have*

$$s_m(BAB^{-1}) = s_m(A).$$

Proposition 2.5. *We have $s_m(JCJ) = -s_m(C)$ for any hyperbolic or parabolic element C in G_m^+.*

Proof. Hyperbolic and parabolic elements C of G_m^+ are of the form $T_{\alpha,\beta}(\lambda) \cdot U^k$ and $P_\alpha(\lambda) \cdot U^k$ respectively. According to Proposition 2.3 we have $JT_{\alpha,\beta}(\lambda)J = T_{-\alpha,-\beta}(\lambda)$, $JP_\alpha(\lambda)J = P_{-\alpha}(-\lambda)$ and $JUJ = U^{-1}$, hence $J(T_{\alpha,\beta}(\lambda) \cdot U^k)J = T_{-\alpha,-\beta}(\lambda) \cdot U^{-k}$ and $J(P_\alpha(\lambda) \cdot U^k)J = P_{-\alpha}(-\lambda) \cdot U^{-k}$. □

Proposition 2.6. *We have $s_m(FCF^{-1}) = -s_m(C)$ for any hyperbolic or parabolic element C in G_m^+ and any element F in G_m^-.*

Proof. We can write the element $F \in G_m^-$ as $F = A \cdot J$ for some $A \in G_m^+$, hence $FCF^{-1} = A(JCJ)A^{-1}$. According to Proposition 2.4 we have $s_m(A(JCJ)A^{-1}) = s_m(JCJ)$ and according to Proposition 2.5 we have $s_m(JCJ) = -s_m(C)$. □

3. Higher Spin on Klein Surfaces

3.1. Higher Spin Bundles on Riemann Surfaces and Lifts of Fuchsian Groups.

Definition 3.1. Let $L \to P$ be complex line bundle over a hyperbolic Riemann surface P. Let Γ be a torsionfree Fuchsian group such that $P = \mathbb{H}/\Gamma$. Let $E \to \mathbb{H}$ be the induced complex line bundle over $\mathbb{H}$. Let $E \simeq \mathbb{H} \times \mathbb{C}$ be a trivialization of the bundle E. With respect to this trivialization the action of Γ on E is given by

$$g \cdot (z, t) = (g(z), \delta(g, z) \cdot t),$$

where $\delta : \Gamma \times \mathbb{H} \to \mathbb{C}^*$ is a map such that the function $\delta_g : \mathbb{H} \to \mathbb{H}$ given be $\delta_g(z) = \delta(g, z)$ is holomorphic for any $g \in \Gamma$ and

$$\delta_{g_2 \cdot g_1} = (\delta_{g_2} \circ g_1) \cdot \delta_{g_1}$$

for any $g_1, g_2 \in \Gamma$. The map δ is called the *transition map* of the bundle $L \to P$ with respect to the given trivialization.

Remark. In particular, if L is the cotangent bundle of the surface P, then the transition map can be chosen so that $\delta_g = (g')^{-1}$. If L is the tangent bundle of the surface P, then the transition map can be chosen so that $\delta_g = g'$. Let $L_1 \to P$, $L_2 \to P$ be two complex line bundles over a Riemann surface P, and let δ_1 resp. δ_2 be their transition maps, then $\delta_1 \cdot \delta_2$ is a transition map of the bundle $L_1 \otimes L_2 \to P$. In particular, if δ is the transition map of the bundle $L \to P$, then δ^m is a transition map of the bundle $L^m = L \otimes \cdots \otimes L \to P$ (with respect to the induced trivialization).

Definition 3.2. An *m-spin structure* on a Riemann surface P is a transition map δ of a complex line bundle $L \to P$ that satisfies the condition $\delta_g^m = (g')^{-1}$, i.e. the induced transition map δ^m of the bundle $L^m \to P$ coincides with the transition map of the cotangent bundle of P.

Remark. A complex line bundle $L \to P$ is said to be *m-spin* if the bundle $L^m \to P$ is isomorphic to the cotangent bundle of P. For a compact Riemann surface P there is a 1-1-correspondence between m-spin structures on P and m-spin bundles over P.

Remark. For $m = 2$ we obtain the classical notion of a spin bundle.

Definition 3.3. Let Γ be a Fuchsian group. A *lift of the Fuchsian group* Γ into G_m^+ is a subgroup Γ^* of G_m^+ such that the restriction of the covering map $G_m^+ \to G^+$ to Γ^* is an isomorphism $\Gamma^* \to \Gamma$.

The following result was proved in [NP05, NP09]:

Theorem 3.1. *Let Γ be a Fuchsian group without elliptic elements. There is a 1-1-correspondence between the lifts of Γ into the m-fold cover of $\mathrm{Aut}_+(\mathbb{H})$ and m-spin bundles on the Riemann surface $\mathbb{H}/\Gamma$.*

We will sketch the proof here: A lift of Γ is of the form

$$\Gamma^* = \{(g, \delta_g) \bigm| g \in \Gamma,\ \delta_g \in \mathrm{Hol}(\mathbb{H}, \mathbb{C}^*),\ \delta_g^m = \frac{d}{dz} g\}.$$

The corresponding m-spin bundle $e_{\Gamma^*} : L_{\Gamma^*} \to P = \mathbb{H}/\Gamma$ is of the form

$$L_{\Gamma^*} = (\mathbb{H} \times \mathbb{C})/\Gamma^* \to \mathbb{H}/\Gamma = P,$$

where the action of Γ^* on $\mathbb{H} \times \mathbb{C}$ is given by

$$(g, \delta_g) \cdot (z, x) = (g(z), \delta_g(z) \cdot x).$$

Every m-spin bundle on $P = \mathbb{H}/\Gamma$ is obtained as e_{Γ^*} for some lift Γ^* of Γ.

Remark. A more general correspondence between a Fuchsian group Γ (with or without elliptic elements) and m-spin bundles on the orbifold $\mathbb{H}/\Gamma$ was established in [NP13].

3.2. **Lifts of Fuchsian Groups and Arf Functions.** Lifts of a Fuchsian group Γ into G_m^+ can be described by means of associated m-Arf functions, certain functions on the space of homotopy classes of simple contours on $P = \mathbb{H}/\Gamma$ with values in $\mathbb{Z}/m\mathbb{Z}$ described by simple geometric properties.

Definition 3.4. Let Γ be a Fuchsian group that consists of hyperbolic elements. Let the corresponding Riemann surface $P = \mathbb{H}/\Gamma$ be a compact surface with finitely many holes. Let $p \in P$. Let $\pi_1(P) = \pi_1(P, p)$ be the fundamental group of P. We

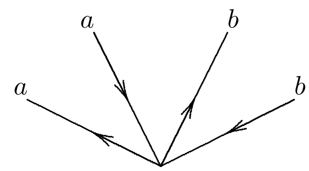

Figure 1: $\sigma(ab) = \sigma(a) + \sigma(b) - 1$

denote by $\pi_1^0(P)$ the set of all non-trivial elements of $\pi_1(P)$ that can be represented by simple contours. An *m-Arf function* is a function

$$\sigma : \pi_1^0(P) \to \mathbb{Z}/m\mathbb{Z}$$

satisfying the following conditions

1. $\sigma(bab^{-1}) = \sigma(a)$ for any elements $a, b \in \pi_1^0(P)$,
2. $\sigma(a^{-1}) = -\sigma(a)$ for any element $a \in \pi_1^0(P)$,
3. $\sigma(ab) = \sigma(a) + \sigma(b)$ for any elements a and b which can be represented by a pair of simple contours in P intersecting in exactly one point p with $\langle a, b\rangle \neq 0$,
4. $\sigma(ab) = \sigma(a) + \sigma(b) - 1$ for any elements $a, b \in \pi_1^0(P)$ such that the element ab is in $\pi_1^0(P)$ and the elements a and b can be represented by a pair of simple contours in P intersecting in exactly one point p with $\langle a, b\rangle = 0$ and placed in a neighbourhood of the point p as shown in Figure 1.

Remark. In the case $m = 2$ there is a 1-1-correspondence between the 2-Arf functions in the sense of Definition 3.4 and Arf functions in the sense of [Nat04], Chapter 1, Section 7 and [Nat91]. Namely, a function $\sigma : \pi_1^0(P) \to \mathbb{Z}/2\mathbb{Z}$ is a 2-Arf function if and only if $\omega = 1 - \sigma$ is an Arf function in the sense of [Nat04].

Higher Arf functions were introduced in [NP05, NP09], where the following result was shown:

Theorem 3.2. *There is a 1-1-correspondence between the lifts of* Γ *into* G_m^+ *and* m*-Arf functions on* $P = \mathbb{H}/\Gamma$.

We will sketch the construction here: Let $\Psi : \mathbb{H} \to P$ be the natural projection. Choose $q \in \Psi^{-1}(p)$ and let $\Phi : \Gamma \to \pi_1(P)$ be the induced isomorphism. Let Γ^* be a lift of Γ in G_m. Let s_m be the level function introduced in section 2.2.

Let us consider a function $\hat{\sigma}_{\Gamma^*} : \pi_1(P) \to \mathbb{Z}/m\mathbb{Z}$ such that the following diagram commutes

$$\begin{tikzcd} \Gamma \arrow[r, "\cong"] \arrow[d, "\Phi"'] & \Gamma^* \arrow[d, "s_m|_{\Gamma^*}"] \\ \pi_1(P) \arrow[r, "\hat{\sigma}_{\Gamma^*}"] & \mathbb{Z}/m\mathbb{Z} \end{tikzcd}$$

Then the function $\sigma_{\Gamma^*} = \hat{\sigma}_{\Gamma^*}|_{\pi_1^0(P)}$ is an m-Arf function, the *m-Arf function associated to the lift* Γ^*. Every m-Arf function is obtained as σ_{Γ^*} for some lift Γ^* of Γ.

The composite mapping $e_{\Gamma^*} \mapsto \Gamma^* \mapsto \sigma_{\Gamma^*}$ establishes a 1-1-correspondence between m-spin bundles on $P = \mathbb{H}/\Gamma$, lifts of the Fuchsian group Γ and m-Arf functions on P.

3.3. **Klein Surfaces and Real Fuchsian Groups.**

Definition 3.5. *Klein surface* (or *non-singular real algebraic curve*) is a topological surface with a maximal atlas whose transition maps are *dianalytic*, i.e. either holomorphic or anti-holomorphic. A *homomorphism* between Klein surfaces is a continuous mapping which is dianalytic in local charts.

For more information on Klein surfaces, see [AG, Nat90a].

Let us consider pairs (P, τ), where P is a compact Riemann surface and $\tau : P \to P$ is an anti-holomorphic involution on P. For each such pair (P, τ) the quotient $P/\langle\tau\rangle$ is a Klein surface. Each isomorphism class of Klein surfaces contains a surface of the form $P/\langle\tau\rangle$. Moreover, two such quotients $P_1/\langle\tau_1\rangle$ and $P_2/\langle\tau_2\rangle$ are isomorphic as Klein surfaces if and only if there exists a biholomorphic map $\psi : P_1 \to P_2$ such that $\psi \circ \tau_1 = \tau_2 \circ \psi$, in which case we say that the pairs (P_1, τ_1) and (P_2, τ_2) are *isomorphic*. Hence from now on we will consider pairs (P, τ) up to isomorphism instead of Klein surfaces.

The category of such pairs (P, τ) is isomorphic to the category of real algebraic curves (see [AG]), where fixed points of τ (i.e. boundary points of the corresponding Klein surface) correspond to real points of the real algebraic curve.

For example a non-singular plane real algebraic curve given by the equation $F(x, y) = 0$ is the set of real points of such a pair (P, τ), where P is the normalisation and compactification of the surface $\{(x, y) \in \mathbb{C}^2 \mid F(x, y) = 0\}$ and τ is given by the complex conjugation, $\tau(x, y) = (\bar{x}, \bar{y})$.

All Klein surfaces can be constructed from real Fuchsian groups, a special kind of non-Euclidean crystallographic groups.

Definition 3.6. A *non-Euclidean crystallographic group* or *NEC group* is a discrete subgroup of $\mathrm{Aut}(\mathbb{H})$, see [Macb]. We consider a simple kind of NEC groups, the real Fuchsian groups. A *real Fuchsian group* is a NEC group $\hat{\Gamma}$ such that the intersection $\hat{\Gamma}^+ = \hat{\Gamma} \cap \mathrm{Aut}_+(\mathbb{H})$ is a Fuchsian group consisting of hyperbolic automorphisms, $\hat{\Gamma} \neq \hat{\Gamma}^+$ and the quotient $P = \mathbb{H}/\hat{\Gamma}^+$ is a compact surface.

Let $\hat{\Gamma}$ be a real Fuchsian group. Let $\hat{\Gamma}^\pm = \hat{\Gamma} \cap \mathrm{Aut}_\pm(\mathbb{H})$, $P_{\hat{\Gamma}} = \mathbb{H}/\hat{\Gamma}^+$ and let $\Phi : \mathbb{H} \to P_{\hat{\Gamma}}$ be the natural projection. Then for any automorphism $g \in \hat{\Gamma}^-$, the map $\tau_{\hat{\Gamma}} = \Phi \circ g \circ \Phi^{-1}$ is an anti-holomorphic involution of $P_{\hat{\Gamma}}$. Thus the real Fuchsian group $\hat{\Gamma}$ induces a Klein surface $[\hat{\Gamma}] = (P_{\hat{\Gamma}}, \tau_{\hat{\Gamma}})$. It is not hard to see that any Klein surface is obtained in this way (see [Nat04, Nat75, Nat78]).

Proposition 3.3. *Let $\hat{\Gamma}$ be a real Fuchsian group and $[\hat{\Gamma}] = (P_{\hat{\Gamma}}, \tau_{\hat{\Gamma}})$ the corresponding Klein surface as defined above. The anti-holomorphic involution $\tau = \tau_{\hat{\Gamma}}$ on $P_{\hat{\Gamma}}$ induces an involution $\tau = (\tau_{\hat{\Gamma}})_*$ on $\pi_1(P_{\hat{\Gamma}}) \cong \hat{\Gamma}^+$. The induced involution satisfies*

$$\tau(f) = gfg^{-1}$$

for every $f \in \hat{\Gamma}^+$ and $g \in \hat{\Gamma}^-$.

Proof. Let $g \in \hat{\Gamma}^-$. An element $f \in \hat{\Gamma}^+$ corresponds to the contour $[\Phi(\ell_f)] \in \pi_1(P_{\hat{\Gamma}})$, where ℓ_f is the axis of f and $\Phi : \mathbb{H} \to P_{\hat{\Gamma}}$ is the natural projection. The image of f under the induced involution τ corresponds to the contour

$$[\tau_{\hat{\Gamma}}(\Phi(\ell_f))] = [(\Phi \circ g \circ \Phi^{-1})(\Phi(\ell_f))] = [\Phi(g(\ell_f))].$$

It is easy to see geometrically that $g(\ell_f)$ is the axis of gfg^{-1}, hence $\tau(f) = gfg^{-1}$. $\square$

3.4. From Lifts of Real Fuchsian Groups to Higher Spin Bundles on Klein Surfaces.

Definition 3.7. An *m-spin bundle on a Klein surface* (P,τ) is a pair $(e : L \to P, \beta)$, where $e : L \to P$ is an m-spin bundle on P and $\beta : L \to L$ is an anti-holomorphic involution on L such that $e \circ \beta = \tau \circ e$.

Definition 3.8. Two m-spin bundles $(e_1 : L_1 \to P_1, \beta_1)$ and $(e_2 : L_2 \to P_2, \beta_2)$ on Klein surfaces (P_1, τ_1) and (P_2, τ_2) are *isomorphic* if there exist biholomorphic maps $\varphi_L : L_1 \to L_2$ and $\varphi_P : P_1 \to P_2$ such that the obvious diagrams commute: $e_2 \circ \varphi_L = \varphi_P \circ e_1$, $\beta_2 \circ \varphi_L = \varphi_L \circ \beta_1$ and $\tau_2 \circ \varphi_P = \varphi_P \circ \tau_1$.

Definition 3.9. A *lift of a real Fuchsian group* $\hat\Gamma$ into G_m is a subgroup $\hat\Gamma^*$ of G_m such that the projection $\pi|_{\hat\Gamma^*} : \hat\Gamma^* \to \hat\Gamma$ is an isomorphism.

Proposition 3.4. *To any lift of a real Fuchsian group into the m-fold cover G_m of* $\mathrm{Aut}(\mathbb{H})$ *we can associate an m-spin bundle on the corresponding Klein surface.*

Proof. Let $\hat\Gamma^*$ be a lift of a real Fuchsian group $\hat\Gamma$ into the m-fold cover G_m of $\mathrm{Aut}(\mathbb{H})$. Let $\Gamma = \hat\Gamma^+ = \hat\Gamma \cap \mathrm{Aut}_+(\mathbb{H})$ be the corresponding Fuchsian group and $\Gamma^* = \hat\Gamma^* \cap G_m^+$ be the corresponding lift of Γ. Let $P = \mathbb{H}/\Gamma$, $L_{\Gamma^*} = (\mathbb{H} \times \mathbb{C})/\Gamma^*$ and let $e_{\Gamma_*} : L_{\Gamma^*} \to P$ be the corresponding m-spin bundle as in Theorem 3.1. Let us choose some $(g, \delta_g) \in \hat\Gamma^* \cap G_m^-$ and consider the mapping $(z,x) \mapsto (g(z), \delta_g(z) \cdot \bar{x})$.

If (z', x') and (z,x) correspond to the same point in $L_{\Gamma^*} = (\mathbb{H} \times \mathbb{C})/\Gamma^*$, then $(g(z'), \delta_g(z') \cdot \bar{x}')$ and $(g(z), \delta_g(z) \cdot \bar{x})$ correspond to the same point in L_{Γ^*}. Thus the mapping $(z,x) \mapsto (g(z), \delta_g(z) \cdot \bar{x})$ induces a map $\beta_{\hat\Gamma^*} : L_{\Gamma^*} \to L_{\Gamma^*}$.

If we choose different $(g_1, \delta_{g_1}), (g_2, \delta_{g_2}) \in \hat\Gamma^* \cap G_m^-$, then $(g_1(z), \delta_{g_1}(z) \cdot \bar{x})$ and $(g_2(z), \delta_{g_2}(z) \cdot \bar{x})$ correspond to the same point in L_{Γ^*}. Thus the map $\beta_{\hat\Gamma^*}$ does not depend on the choice of the element $g \in \hat\Gamma^* \cap G_m^-$.

If we apply $\beta_{\hat\Gamma^*}$ twice we get

$$\begin{aligned}
(z,x) &\mapsto (g(z), \delta_g(z) \cdot \bar{x}) \\
&\mapsto (g(g(z)), \delta_g(g(z)) \cdot \overline{\delta_g(z) \cdot \bar{x}}) \\
&= ((g \circ g)(z), ((\delta_g \circ g) \cdot \bar{\delta}_g)(z) \cdot x) \\
&= ((g \circ g)(z), \delta_{g \circ g}(z) \cdot x) \\
&= (g \circ g) \cdot (z,x).
\end{aligned}$$

We have $g \circ g \in \hat\Gamma^*$ since $g \in \hat\Gamma^*$ and we have $g \circ g \in G_m^+$ for any $g \in G_m$, hence $g \circ g \in \hat\Gamma^* \cap G_m^+ = \Gamma^*$. Thus (z,x) and $(g \circ g) \cdot (z,x)$ are equal modulo the action of Γ^*. We have therefore shown that $\beta_{\hat\Gamma^*}$ is indeed an involution. We can now associate with the lift $\hat\Gamma^*$ of the real Fuchsian group $\hat\Gamma$ the m-spin bundle $e_{\hat\Gamma^*} := (e_{\Gamma^*}, \beta_{\hat\Gamma^*})$. $\square$

Proposition 3.5. *To any m-spin bundle on the Klein surface (P,τ) we can associate a lift of a real Fuchsian group into the m-fold cover G_m of* $\mathrm{Aut}(\mathbb{H})$.

Proof. Any m-spin bundle on (P,τ) is obtained as e_{Γ^*} for some lift Γ^* of Γ into G_m. Let $(e : L \to P, \beta : L \to L)$ be an m-spin bundle on (P,τ). We have $e \circ \beta = \tau \circ e$.

Consider a lift $\tilde{\beta}$ of $\beta : L \to L$ to the universal cover $\tilde{L} = \mathbb{H} \times \mathbb{C}$ of L. Let $\tilde{e}$ be the projection $\mathbb{H} \times \mathbb{C} \to P$. The map $\tilde{\beta}$ is bi-anti-holomorphic, invariant under Γ^* and with the property $\tilde{e} \circ \tilde{\beta} = \tau \circ \tilde{e}$, hence $\tilde{\beta}$ is of the form

$$\tilde{\beta}(z, x) = (g(z), f(z, x)),$$

where g is some element of $\hat{\Gamma}^-$ and f is some anti-holomorphic map. For a fixed z the map $x \mapsto f(z, x)$ is a bi-anti-holomorphic map $\mathbb{C} \to \mathbb{C}$, hence $f(z, x) = a(z) \cdot \bar{x} + b(z)$, where $a : \mathbb{H} \to \mathbb{C}^*$ and $b : \mathbb{H} \to \mathbb{C}$ are holomorphic functions. Since β is a bundle map, it preserves the zero section of L, hence $b(z) = 0$ for all z. Thus $\tilde{\beta}$ is of the form

$$\tilde{\beta}(z, x) = (g(z), a(z) \cdot \bar{x}),$$

where $a : \mathbb{H} \to \mathbb{C}^*$ is a holomorphic function. Considering the m-fold tensor products, we obtain an anti-holomorphic involution given by

$$\tilde{\beta}^{\otimes m}(z, x) = (g(z), a^m(z) \cdot \bar{x})$$

on the cotangent bundle of P, hence

$$a^m = \frac{d}{d\bar{z}} g.$$

Therefore $\tilde{g} = (g, \delta_g)$ with $\delta_g = a$ defines a lift of the element g into G_m^-. The fact that the map $\tilde{\beta}$ is invariant under the action of Γ^* on $\mathbb{H} \times \mathbb{C}$ implies that the element $\tilde{g} = (g, \delta_g)$ normalises the lift Γ^*, i.e. $\tilde{g} \cdot \Gamma^* \cdot \tilde{g}^{-1} = \Gamma^*$. The fact that β is an involution implies that the element $\tilde{g} = (g, \delta_g)$ is of order two. The fact that $\tilde{g} \cdot \Gamma^* \cdot \tilde{g}^{-1} = \Gamma^*$ and $\tilde{g}^2 = \tilde{e}$ implies that the subgroup of G_m generated by Γ^* and $\tilde{g}$ is a lift of $\hat{\Gamma}$ into G_m. □

3.5. **Lifts of Real Fuchsian Groups and Real Arf Functions.** A lift $\hat{\Gamma}^*$ of a real Fuchsian group $\hat{\Gamma}$ into the m-fold cover G_m of $\mathrm{Aut}(\mathbb{H})$ induces a lift $\Gamma^* = \hat{\Gamma}^* \cap G_m^+$ of the Fuchsian group $\Gamma = \hat{\Gamma} \cap G^+$ into G_m^+, whence an m-Arf function $\sigma_{\hat{\Gamma}^*}$ on $\mathbb{H}/\Gamma$. Let us study the special properties of such m-Arf functions.

Lemma 3.6. *Let $\hat{\Gamma}$ be a real Fuchsian group, $\Gamma = \hat{\Gamma}^+ = \hat{\Gamma} \cap G^+$ the corresponding Fuchsian group, $[\hat{\Gamma}] = (P = \mathbb{H}/\Gamma, \tau)$ the corresponding Klein surface and $\hat{\Gamma}^*$ a lift of $\hat{\Gamma}$. Then the induced m-Arf function $\sigma = \sigma_{\Gamma^*}$ on P has the following property: $\sigma(\tau c) = -\sigma(c)$ for any $c \in \pi_1^0(P)$.*

Proof. The anti-holomorphic involution on P is given by $\tau = \Phi \circ f \circ \Phi^{-1}$, where $f \in \hat{\Gamma}^- = \hat{\Gamma} \cap G^-$ and Φ is the natural projection $\mathbb{H} \to P$. The induced involution on $\pi_1^0(P) \cong \Gamma \cong \Gamma^*$ is given by conjugation by an element of $(\hat{\Gamma}^*)^- = \hat{\Gamma}^* \cap G_m^-$, which according to Proposition 2.6 changes the sign of s_m, hence $\sigma(\tau c) = -\sigma(c)$ for all $c \in \pi_1^0(P)$. □

Definition 3.10. We call an m-Arf function on a Klein surface (P, τ) *compatible* (with the involution τ) if $\sigma(\tau c) = -\sigma(c)$ for any $c \in \pi_1^0(P)$.

To understand the structure of a Klein surface (P, τ), we look at the contours which are invariant under the involution τ. There are two kinds of invariant contours, depending on whether the restriction of τ to the invariant contour is identity or a "half-turn".

Definition 3.11. Let (P,τ) be a Klein surface. The set of fixed points of the involution τ is called the *set of real points* of (P,τ) and denoted by P^τ. The set P^τ decomposes into pairwise disjoint simple closed smooth contours, called *ovals*.

Definition 3.12. A *twist* (or *twisted oval*) is a simple contour in P which is invariant under the involution τ but does not contain any fixed points of τ.

Remark. A twisted oval is not an oval, however the corresponding element of $H_1(P)$ is a fixed point of the induced involution and the corresponding element of $\pi_1(P)$ is preserved up to conjugation by the induced involution.

Lemma 3.7. *Let σ be a compatible m-Arf function on a Klein surface (P,τ). If m is odd, the σ vanishes on all ovals and all twists. If m is even, then $\sigma(c)$ is either equal to 0 or to $m/2$ for any oval and any twist c.*

Proof. For any invariant contour $c\in\pi_1^0(P)\cong\Gamma$, whether oval or twist, we have $\tau c=c$ and therefore $\sigma(\tau c)=\sigma(c)$. On the other hand σ is compatible, hence $\sigma(\tau c)=-\sigma(c)$ for all c. Therefore $2\sigma(c)=0$ modulo m. For odd m this implies $\sigma(c)=0$, while for even m we can have either $\sigma(c)=0$ or $\sigma(c)=m/2$. $\square$

Not all compatible m-Arf functions correspond to lifts of real Fuchsian groups. We will prove now that if an m-Arf function corresponds to a lift of a real Fuchsian group, then stronger conditions on the twists than Lemma 3.7 are satisfied.

For a hyperbolic automorphism $c\in\mathrm{Aut}_+(\mathbb{H})$ let $\bar c$ be the reflection whose mirror coincides with the axis of c, let $\sqrt{c}$ be the hyperbolic automorphism such that $(\sqrt{c})^2=c$ and let $\tilde c=\bar c\sqrt{c}$. The discussion summarised in section 2.2 of [Nat04] (compare with Theorem 4.2 for more details) implies

Lemma 3.8. *If $c\in\hat\Gamma$ is a hyperbolic element that corresponds to an oval on $P=\mathbb{H}/\hat\Gamma$, then $\hat\Gamma$ contains the reflection $\bar c$. If $c\in\hat\Gamma$ is a hyperbolic element that corresponds to a twist on $P=\mathbb{H}/\hat\Gamma$, then $\hat\Gamma$ contains the element $\tilde c=\bar c\sqrt{c}$.*

Lemma 3.9. *Let $\hat\Gamma$ be a real Fuchsian group, $\Gamma=\hat\Gamma^+=\hat\Gamma\cap G^+$ the corresponding Fuchsian group, $[\hat\Gamma]=(P=\mathbb{H}/\Gamma,\tau)$ the corresponding Klein surface and $\hat\Gamma^*$ a lift of $\hat\Gamma$. Then the induced m-Arf function $\sigma=\sigma_{\Gamma^*}$ on P vanishes on all twists.*

Proof. We need to show that the case m even, c a twist, $\sigma(c)=m/2$ is not possible. Let c be a hyperbolic element in $\Gamma\cong\pi_1^0(P)$ which corresponds to a twist. According to Lemma 3.8 the group $\hat\Gamma$ contains the element $\tilde c=\bar c\sqrt{c}$. Let $C\in(\hat\Gamma^*)^+$ and $\tilde C\in(\hat\Gamma^*)^-$ be the lifts of c and $\tilde c$ resp. Without loss of generality we can assume that $c=\tau_{0,\infty}(\lambda)$, so that $\tilde c=j\cdot\tau_{0,\infty}(\lambda/2)$. Then the lift of $\tilde c$ in $\hat\Gamma^*$ is of the form $JT_{0,\infty}(\lambda/2)\cdot U^q$ for some integer q. Using identities from Proposition 2.3 we obtain that

$$\begin{aligned}(JT_{0,\infty}(\lambda/2)U^q)^2&=(JT_{0,\infty}(\lambda/2)U^q)(JT_{0,\infty}(\lambda/2)U^q)\\&=(JT_{0,\infty}(\lambda/2)U^q)(U^{-q}JT_{0,\infty}(\lambda/2))\\&=JT_{0,\infty}(\lambda/2)JT_{0,\infty}(\lambda/2)=JJT_{0,\infty}(\lambda/2)T_{0,\infty}(\lambda/2)\\&=T_{0,\infty}(\lambda/2)T_{0,\infty}(\lambda/2)=T_{0,\infty}(\lambda).\end{aligned}$$

The element $T_{0,\infty}(\lambda)$ is therefore in $\hat\Gamma^*$ and is a pre-image of $(\tilde c)^2=c$, hence $T_{0,\infty}(\lambda)$ is the lift of c in $\hat\Gamma^*$. We obtain that $\sigma(c)=s_m(T_{0,\infty})=0$. $\square$

Definition 3.13. A *real m-Arf function* on a Klein surface (P,τ) is an m-Arf function on P such that

(i) σ is compatible with τ, i.e. $\sigma(\tau c) = -\sigma(c)$ for any $c \in \pi_1^0(P)$.
(ii) σ vanishes on all twists.

Lemmas 3.6 and 3.9 imply:

Theorem 3.10. *Let $\hat{\Gamma}^*$ be a lift of a real Fuchsian group $\hat{\Gamma}$. Then the induced m-Arf function $\sigma = \sigma_{\hat{\Gamma}^*}$ is a real m-Arf function on the Klein surface $[\hat{\Gamma}]$.*

Remark. We slightly change terminology here. In [Nat04, Nat94] 2-Arf functions satisfying (i) were called real Arf functions, while 2-Arf functions satisfying (i) and (ii) were called non-special real Arf functions.

Remark. One can show that in the presence of ovals ($P^\tau \neq \varnothing$) compatibility with τ implies property (ii). However, if $P^\tau = \varnothing$ and m is even, there exist compatible Arf functions which assume the value $m/2$ on all twists. We will not be interested in these Arf functions since they do not come from real Fuchsian groups.

Definition 3.14. Two lifts $(\hat{\Gamma}^*)_1$ and $(\hat{\Gamma}^*)_2$ of a real Fuchsian group $\hat{\Gamma}$ are *similar* if $(\hat{\Gamma}^*)_1^- = (\hat{\Gamma}^*)_2^- \cdot U^q$ for some integer q.

Remark. Note that for similar lifts $(\hat{\Gamma}^*)_1$ and $(\hat{\Gamma}^*)_2$ of a real Fuchsian group $\hat{\Gamma}$ we have $(\hat{\Gamma}^*)_1^+ = (\hat{\Gamma}^*)_2^+$.

Theorem 3.11. *Let $\hat{\Gamma}$ be a real Fuchsian group. The mapping that assigns to a lift $\hat{\Gamma}^*$ of $\hat{\Gamma}$ into G_m the m-Arf function of $(\hat{\Gamma}^*)^+$ establishes a 1-1-correspondence between similarity classes of lifts of the real Fuchsian group $\hat{\Gamma}$ and real m-Arf functions on the Klein surface $[\hat{\Gamma}]$.*

Proof. According to Theorem 3.10, we can assign to any lift $\hat{\Gamma}^*$ of $\hat{\Gamma}$ a real m-Arf function $\sigma = \sigma_{\hat{\Gamma}^*}$ on $[\hat{\Gamma}]$.

Let σ be a real m-Arf function on $[\hat{\Gamma}]$. We will show that there exist m lifts of $\hat{\Gamma}$ such that the m-Arf function induced by each of the lifts is equal to σ. Moreover, these m lifts are similar to each other.

According to Theorem 3.2 there exists a unique lift Γ^* into G_m^+ of the group $\Gamma = \hat{\Gamma}^+$ such that the induced m-Arf function is equal to σ. Choose an element f in $\hat{\Gamma}^-$. To uniquely determine a lift of the group $\hat{\Gamma} = \langle \Gamma, f\rangle = \Gamma \cup f \cdot \Gamma$ we need to specify a lift $F \in (\hat{\Gamma}^*)^-$ of f. To ensure that the set $\Gamma^* \cup F \cdot \Gamma^*$ is indeed a group we require $F\Gamma^* F^{-1} = \Gamma^*$ and $F^2 \in \Gamma^*$. According to Proposition 3.3, we have $fgf^{-1} = \tau g$ and hence $\sigma(fgf^{-1}) = \sigma(\tau g)$. The Arf function σ is real, hence $\sigma(\tau g) = -\sigma(g)$. Thus $\sigma(fgf^{-1}) = \sigma(\tau g) = -\sigma(g)$ and therefore $F\Gamma^* F^{-1} = \Gamma^*$.

If P is separating, we can assume without loss of generality that $f = j$. The lifts of f are then of the form $F = J \cdot U^q$. Using identities from Proposition 2.3 we obtain for any integer q that

$$F^2 = (JU^q)^2 = (JU^q)(JU^q) = (JU^q)(U^{-q}J) = J^2 = \tilde{e}.$$

If P is non-separating, we can assume without loss of generality that $f = j\tau_{0,\infty}(\lambda/2)$, where the element $\tau_{0,\infty}(\lambda)$ corresponds to a twist. The lifts of f are then of the form $F = JT_{0,\infty}(\lambda/2)U^q$. Using identities from Proposition 2.3 we obtain as in the proof of Lemma 3.9 that for any integer q

$$F^2 = (JT_{0,\infty}(\lambda/2) \cdot U^q)^2 = T_{0,\infty}(\lambda).$$

Since the Arf function σ is real, we have

$$\sigma(\tau_{0,\infty}(\lambda)) = 0 = s_m(T_{0,\infty}(\lambda)),$$

hence the element $F^2 = T_{0,\infty}(\lambda)$ is in Γ^*. The properties $F\Gamma^*F^{-1} = \Gamma^*$ and $F^2 \in \Gamma^*$ imply that the subgroup of $\hat{\Gamma}^*$ generated by Γ^* and F is a lift of $\hat{\Gamma}$. □

3.6. **From Higher Spin Bundles on Klein Surfaces to Lifts of Real Fuchsian Groups.**

Theorem 3.12. *There is a 1-1-correspondence between m-spin bundles on Klein surfaces and similarity classes of lifts of real Fuchsian groups into the m-fold cover G_m of* $\mathrm{Aut}(\mathbb{H})$.

Proof. We will show that similar lifts of a real Fuchsian group $\hat{\Gamma}$ induce isomorphic m-spin bundles on the corresponding Klein surface. The uniformisations of the anti-holomorphic involutions that correspond to similar lifts of $\hat{\Gamma}$ are of the form $\tilde{\beta}_1(z,x) = (g(z), \delta_g(z)\cdot\bar{x})$ and $\tilde{\beta}_2(z,x) = (g(z), \zeta\cdot\delta_g(z)\cdot\bar{x})$, where $\zeta \in \mathbb{C}$, $\zeta^m = 1$. Then taking $\varphi_L(z,x) = (z, \sqrt{\zeta}\cdot x)$ and $\varphi_P = \mathrm{id}_P$ (see Definition 3.8) we can see that two m-spin bundles on the Klein surface are isomorphic:

$$\begin{aligned}
\varphi_L(\tilde{\beta}_1(z,x)) &= \varphi_L(g(z), \delta_g(z)\cdot\bar{x}) = (g(z), \sqrt{\zeta}\cdot\delta_g(z)\cdot\bar{x}),\\
\tilde{\beta}_2(\varphi_L(z,x)) &= \tilde{\beta}_2(z, \sqrt{\zeta}\cdot x) = (g(z), \zeta\cdot\delta_g(z)\cdot\overline{\sqrt{\zeta}\cdot x})\\
&= (g(z), \zeta\cdot\sqrt{\bar{\zeta}}\cdot\delta_g(z)\cdot\bar{x}) = (g(z), \sqrt{\zeta}\cdot\delta_g(z)\cdot\bar{x}).
\end{aligned}$$

□

Theorems 3.11 and 3.12 immediately imply

Theorem 3.13. *The mapping that assigns to an m-spin bundle on a Klein surface the corresponding real m-Arf function establishes a 1-1-correspondence.*

4. Classification of Real Arf Functions

4.1. **Topological Invariants of Klein Surfaces.**

Definition 4.1. Given two Klein surfaces (P_1,τ_1) and (P_2,τ_2), we say that they are *topologically equivalent* if there exists a homeomorhism $\phi: P_1 \to P_2$ such that $\phi\circ\tau_1 = \tau_2\circ\phi$.

Let (P,τ) be a Klein surface. We say that (P,τ) is *separating* or *of type I* if the set $P\backslash P^\tau$ is not connected, otherwise we say that it is *non-separating* or *of type II*. The *topological type* of (P,τ) is the triple (g,k,ε), where g is the genus of the Riemann surface P, k is the number of connected components of the fixed point set P^τ of τ, $\varepsilon = 0$ if (P,τ) is non-separating and $\varepsilon = 1$ otherwise. We say that a real Fuchsian group $\hat{\Gamma}$ is of topological type (g,k,ε) if the corresponding Klein surface is of topological type (g,k,ε). In this paper we consider hyperbolic surfaces, hence $g \geqslant 2$.

The following result of Weichold [Wei] gives a classification of Klein surfaces up to topological equivalence:

Theorem 4.1. *Two Klein surfaces are topologically equivalent if and only if they are of the same topological type. A triple* (g, k, ε) *is a topological type of some Klein surface if and only if either* $\varepsilon = 1$, $1 \leqslant k \leqslant g+1$, $k \equiv g+1 \bmod 2$ *or* $\varepsilon = 0$, $0 \leqslant k \leqslant g$.

Any separating Klein surface can be obtained by gluing together a Riemann surface with boundary with its copy via the identity map along the boundary components. If we replace the identity map with a half-turn on some of the boundary components, we obtain a non-separating Klein surface. Moreover, all non-separating Klein surfaces are obtained in this way.

We will use the description of generating sets of real Fuchsian groups given in [Nat04, Nat75, Nat78]:

Recall that for a hyperbolic automorphism $c \in \mathrm{Aut}_+(\mathbb{H})$, $\bar{c}$ is the reflection whose mirror coincides with the axis of c, $\sqrt{c}$ is the hyperbolic automorphism such that $(\sqrt{c})^2 = c$ and $\tilde{c} = \bar{c}\sqrt{c}$.

Theorem 4.2. (Generating sets of real Fuchsian groups)

1) *Let* $(g, k, 1)$ *be a topological type of a Klein surface, i.e.* $1 \leqslant k \leqslant g+1$ *and* $k \equiv g+1 \bmod 2$. *Let* $n = k$. *Let* $\tilde{g} = (g+1-n)/2$. *Let*
$$(a_1, b_1, \dots, a_{\tilde{g}}, b_{\tilde{g}}, c_1, \dots, c_n)$$
be a generating set of a Fuchsian group of signature $(\tilde{g}, k)$, *then*
$$(a_1, b_1, \dots, a_{\tilde{g}}, b_{\tilde{g}}, c_1, \dots, c_n, \bar{c}_1, \dots, \bar{c}_n)$$
is a generating set of a real Fuchsian group $\hat{\Gamma}$ *of topological type* $(g, k, 1)$. *Any real Fuchsian group of topological type* $(g, k, 1)$ *is obtained in this way.*
2) *Let* $(g, k, 0)$ *be a topological type of a Klein surface, i.e.* $0 \leqslant k \leqslant g$. *Let us choose* $n \in \{k+1, \dots, g+1\}$ *such that* $n \equiv g+1 \bmod 2$. *Let* $\tilde{g} = (g+1-n)/2$. *Let*
$$(a_1, b_1, \dots, a_{\tilde{g}}, b_{\tilde{g}}, c_1, \dots, c_n)$$
be a generating set of a Fuchsian group of signature $(\tilde{g}, n)$, *then*
$$(a_1, b_1, \dots, a_{\tilde{g}}, b_{\tilde{g}}, c_1, \dots, c_n, \bar{c}_1, \dots, \bar{c}_k, \tilde{c}_{k+1}, \cdots, \tilde{c}_n)$$
is a generating set of a real Fuchsian group $\hat{\Gamma}$ *of topological type* $(g, k, 0)$. *Any real Fuchsian group of topological type* $(g, k, 0)$ *is obtained in this way.*
3) *Let* $\hat{\Gamma}$ *be a real Fuchsian group as in part 1 or 2 and let* (P, τ) *be the corresponding Klein surface. We now think of the elements*
$$(a_1, b_1, \dots, a_{\tilde{g}}, b_{\tilde{g}}, c_1, \dots, c_n)$$
as contours in $\pi_1^0(P)$ *rather than generators of* $\hat{\Gamma}$. *We have* $P^\tau = c_1 \cup \cdots \cup c_k$. *The contours* $c_1, \dots, c_k$ *correspond to ovals, the contours* $c_{k+1}, \dots, c_n$ *correspond to twists. Let* P_1 *and* P_2 *be the connected components of the complement of the contours* $c_1, \dots, c_n$ *in* P. *Each of these components is a surface of genus* $\tilde{g} = (g+1-n)/2$ *with* n *holes. We have* $\tau(P_1) = P_2$. *We will refer to* P_1 *and* P_2 *as decomposition of* (P, τ) *in two halves. (Note that such a decomposition is unique if* (P, τ) *is separating, but is not unique if* (P, τ) *is non-separating since the twists* $c_{k+1}, \dots, c_n$ *can be chosen in different ways.) Then*
$$(a_1, b_1, \dots, a_{\tilde{g}}, b_{\tilde{g}}, c_1, \dots, c_n)$$

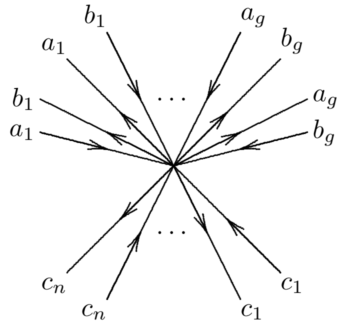

Figure 2: Canonical system of curves

is a generating set of $\pi_1^0(P_1)$*, while its image under* τ *gives a generating set of* $\pi_1^0(P_2)$*. For two invariant contours* c_i *and* c_j*, we say that a contour of the form*

$$r_i \cup (\tau\ell)^{-1} \cup r_j \cup \ell,$$

where ℓ *is a simple path in* P_1 *starting on* c_j *and ending on* c_i*,* r_i *is the path along* c_i *from the end point of* ℓ *to the end point of* $\tau(\ell)$ *and* r_j *is the path along* c_j *from the starting point of* $\tau(\ell)$ *to the starting point of* ℓ*, is a* bridge *between* c_i *and* c_j*. (If* c_i *or* c_j *is an oval, the path* r_i *or* r_j *respectively consists of just one point.) Let* $d_1, \dots, d_{n-1}$ *be contours which only intersect at the base point, such that* d_i *is a bridge between* c_i *and* c_n*. Let* $a'_i = (\tau a_i)^{-1}$ *and* $b'_i = (\tau b_i)^{-1}$ *for* $i = 1, \dots, \tilde{g}$*. Then*

$$(a_1, b_1, \dots, a_{\tilde{g}}, b_{\tilde{g}}, a'_1, b'_1, \dots, a'_{\tilde{g}}, b'_{\tilde{g}}, c_1, \dots, c_{n-1}, d_1, \dots, d_{n-1})$$

is a generating set of $\pi_1^0(P)$*. Note that* $\tau(c_i) = c_i$ *and* $\tau(d_i) = c_i^{|c_i|} d_i^{-1} c_n^{|c_n|}$*, where* $|c_j| = 0$ *if* c_j *is an oval and* $|c_j| = 1$ *if* c_j *is a twist. We will refer to such a generating set as a* symmetric generating set *of type* $(\tilde{g}, k, n)$*.*

4.2. **Topological Invariants of Higher Arf Functions.** In this section we recall the topological invariants of m-Arf functions as described in [NP05, NP09].

Definition 4.2. A *canonical system of curves* on a compact Riemann surface P of genus g with n holes is a set of simply closed curves $\{\tilde{a}_1, \tilde{b}_1, \dots, \tilde{a}_g, \tilde{b}_g, \tilde{c}_1, \dots, \tilde{c}_n\}$ based at a point $p \in P$ with the following properties:

1) The contour $\tilde{c}_i$ encloses a hole in P for $i = 1, \dots, n$.
2) Any two curves only intersect at the point p.
3) A neighbourhood of the point p with the curves is homeomorphic to the one shown in Figure 2.
4) The system of curves cuts the surface P into $n+1$ connected components of which n are homeomorphic to a ring and one is homeomorphic to a disc and has boundary
$$\tilde{a}_1 \tilde{b}_1 \tilde{a}_1^{-1} \tilde{b}_1^{-1} \dots \tilde{a}_g \tilde{b}_g \tilde{a}_g^{-1} \tilde{b}_g^{-1} \tilde{c}_1 \dots \tilde{c}_n.$$

If $\{\tilde{a}_1, \tilde{b}_1, \dots, \tilde{a}_g, \tilde{b}_g, \tilde{c}_1, \dots, \tilde{c}_n\}$ is a canonical system of curves, then we call the corresponding set $\{a_1, b_1, \dots, a_g, b_g, c_1, \dots, c_n\}$ of elements in the fundamental group $\pi_1(P)$ a *standard generating set* or a *standard basis* of $\pi_1(P)$.

Remark. Note that a symmetric generating set as defined in Theorem 4.2 is not a standard generating set in the sense of Definition 4.2, however it is free homotopic to a standard one.

Definition 4.3. Let $\sigma : \pi_1^0(P) \to \mathbb{Z}/m\mathbb{Z}$ be an m-Arf function. For $g = 1$ we define the *Arf invariant* $\delta = \delta(P, \sigma)$ as

$$\delta = \gcd(m, \sigma(a_1), \sigma(b_1), \sigma(c_1) + 1, \dots, \sigma(c_n) + 1),$$

where

$$\{a_1, b_1, c_i \ (i = 1, \dots, n)\}$$

is a standard (or symmetric) generating set of the fundamental group $\pi_1(P)$. For $g \geqslant 2$ and even m we define the *Arf invariant* $\delta = \delta(P, \sigma)$ as $\delta = 0$ if there is a standard (or symmetric) generating set

$$\{a_i, b_i \ (i = 1, \dots, g), c_i \ (i = 1, \dots, n)\}$$

of the fundamental group $\pi_1(P)$ such that

$$\sum_{i=1}^{g} (1 - \sigma(a_i))(1 - \sigma(b_i)) \equiv 0 \bmod 2$$

and as $\delta = 1$ otherwise. For $g \geqslant 2$ and odd m we set $\delta = 0$.

Definition 4.4. For even m and $g \geqslant 2$ we say that an m-Arf function with Arf invariant δ is *even* if $\delta = 0$ and *odd* if $\delta = 1$.

Remark. The Arf invariant δ is a topological invariant of the Arf function σ, i.e. it does not change under self-homeomorphisms of the Riemann surface P.

The following is a special case of our earlier classification result, Theorem 5.3 in [NP09]:

Theorem 4.3. *Let P be a hyperbolic Riemann surface of genus g with n holes. Let $c_1, \dots, c_n$ be contours around the holes as in Definition 4.2. Let σ be an m-Arf function on P and let δ be the m-Arf invariant of σ. Then*
(a) If $g \geqslant 2$ and $m \equiv 1 \bmod 2$ then $\delta = 0$.
(b) If $g \geqslant 2$ and $m \equiv 0 \bmod 2$ and $\sigma(c_i) \equiv 0 \bmod 2$ for some i then $\delta = 0$.
(c) If $g = 1$ then δ is a divisor of $\gcd(m, \sigma(c_1) + 1, \dots, \sigma(c_n) + 1)$.
(d) $\sigma(c_1) + \cdots + \sigma(c_n) \equiv (2 - 2g) - n \bmod m$.

The following result describes the construction of such m-Arf functions. It follows from Lemma 3.7, Lemma 3.9, Theorem 4.9 and the proof of Theorem 5.3 in [NP09].

Theorem 4.4. *Let P be a hyperbolic Riemann surface of genus g with n holes. Then for any standard generating set*

$$(a_1, b_1, \dots, a_g, b_g, c_1, \dots, c_n)$$

of $\pi_1(P)$ and any choice of values $\alpha_1, \beta_1, \dots, \alpha_g, \beta_g, \gamma_1, \dots, \gamma_n$ in $\mathbb{Z}/m\mathbb{Z}$ with

$$\gamma_1 + \cdots + \gamma_n \equiv (2 - 2g) - n \bmod m,$$

there exists an m-Arf function σ on P such that $\sigma(a_i) = \alpha_i$, $\sigma(b_i) = \beta_i$ for $i = 1, \dots, g$ and $\sigma(c_i) = \gamma_i$ if $i = 1, \dots, n$. The Arf invariant δ of this m-Arf function σ satisfies the following conditions:
(a) If $g \geqslant 2$ and $m \equiv 1 \bmod 2$ then $\delta = 0$.
(b) If $g \geqslant 2$ and $m \equiv 0 \bmod 2$ and $\gamma_i \equiv 0 \bmod 2$ for some i then $\delta = 0$.

(c) If $g \geqslant 2$ and $m \equiv 0 \bmod 2$ and $\gamma_1 \equiv \cdots \equiv \gamma_n \equiv 1 \bmod 2$ then $\delta \in \{0,1\}$ and

$$\delta \equiv \sum_{i=1}^{g} (1-\alpha_i)(1-\beta_i) \bmod 2.$$

(d) If $g=1$ then $\delta = \gcd(m, \alpha_1, \beta_1, \gamma_1+1, \dots, \gamma_n+1)$.

Remark. Note that in the case $g \geqslant 2$ the formula for δ in part (c) holds on some standard generating set by definition. The important statement is that, in the case that $m \equiv 0 \bmod 2$ and $\gamma_1 \equiv \cdots \equiv \gamma_n \equiv 1 \bmod 2$, this formula holds for any standard generating set.

In the special case of a compact Riemann surface without holes (i.e. $n=0$) we have

Proposition 4.5. *Let P be a compact Riemann surface of genus $g \geqslant 2$. Assume that $2-2g \equiv 0 \bmod m$. Then for any standard generating set $(a_1, b_1, \dots, a_g, b_g)$ of $\pi_1(P)$ and any choice of values $\alpha_1, \beta_1, \dots, \alpha_g, \beta_g$ in $\mathbb{Z}/m\mathbb{Z}$, there exists an m-Arf function σ on P such that $\sigma(a_i)=\alpha_i$, $\sigma(b_i)=\beta_i$ for $i=1,\dots,g$. The Arf invariant δ of this m-Arf function σ satisfies the following conditions:*

(a) If m is odd then $\delta=0$.

(b) If m is even then $\delta \in \{0,1\}$ and

$$\delta \equiv \sum_{i=1}^{g} (1-\alpha_i)(1-\beta_i) \bmod 2.$$

4.3. **Values of Real Arf Functions on Symmetric Generating Sets.**

Lemma 4.6. *Let (P,τ) be a Klein surface and let σ be an m-Arf function on P. (Here we do not assume that σ is a real Arf function.) Let d be a bridge as defined in Theorem 4.2.*

- *Let (P,τ) be separating. Then*

$$\sigma(\tau d) = -\sigma(d).$$

- *Let (P,τ) be non-separating. Let $c_1, \dots, c_n$ be invariant contours such that the first k correspond to ovals and the next $n-k$ correspond to twists (see Theorem 4.2). Assume that σ vanishes on all twists $c_{k+1}, \dots, c_n$. Then*

$$\sigma(\tau d) = -\sigma(d).$$

Proof. Let $d = r_i \cup (\tau\ell)^{-1} \cup r_j \cup \ell$ be a bridge between c_i and c_j as in Theorem 4.2.

- If c_i and c_j are both ovals we have (see Figure 3):

$$\begin{aligned} d &= (\tau\ell)^{-1} \cup \ell, \\ \tau d &= \ell^{-1} \cup \tau\ell = d^{-1}. \end{aligned}$$

  Now we see that $\sigma(\tau d) = \sigma(d^{-1}) = -\sigma(d)$.
- If c_i is an oval and c_j is a twist we have (see Figure 4):

$$\begin{aligned} d &= (\tau\ell)^{-1} \cup r_j \cup \ell, \\ \tau d &= \ell^{-1} \cup \tau r_j \cup \tau\ell, \\ (\tau d)^{-1} &= (\tau\ell)^{-1} \cup (\tau r_j)^{-1} \cup \ell \\ &= \big((\tau\ell)^{-1} \cup r_j \cup \ell\big) \cup \big(\ell^{-1} \cup r_j^{-1} \cup (\tau r_j)^{-1} \cup \ell\big) \\ &= d \cup c_j. \end{aligned}$$

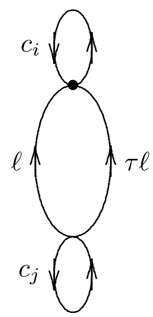

Figure 3: A bridge between ovals c_i and c_j

Using Property 3 of Arf functions we obtain

$$\sigma((\tau d)^{-1}) = \sigma(d) + \sigma(c_j).$$

Using Property 2 of Arf functions we obtain $\sigma((\tau d)^{-1}) = -\sigma(\tau d)$ and therefore

$$\sigma(\tau d) = -\sigma(d) - \sigma(c_j).$$

Now we see that $\sigma(c_j) = 0$ implies $\sigma(\tau d) = -\sigma(d)$.

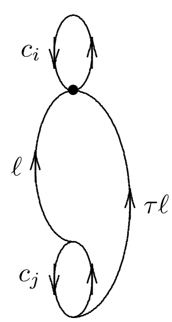

Figure 4: A bridge between an oval c_i and a twist c_j

- If c_i and c_j are both twists we have (see Figure 5):

$$d = r_i \cup (\tau\ell)^{-1} \cup r_j \cup \ell,$$
$$\tau d = \ell^{-1} \cup \tau r_j \cup \tau\ell \cup \tau r_i$$

and

$$
\begin{aligned}
&(\tau d)^{-1} \\
&= (\tau r_i)^{-1} \cup (\tau\ell)^{-1} \cup (\tau r_j)^{-1} \cup \ell \\
&= ((\tau r_i)^{-1} \cup r_i^{-1}) \cup (r_i \cup (\tau\ell)^{-1} \cup r_j \cup \ell) \cup (\ell^{-1} \cup r_j^{-1} \cup (\tau r_j)^{-1} \cup \ell) \\
&= c_i \cup d \cup c_j.
\end{aligned}
$$

Using Property 3 of Arf functions we obtain

$$\sigma((\tau d)^{-1}) = \sigma(c_i) + \sigma(d) + \sigma(c_j).$$

Using Property 2 of Arf functions we obtain $\sigma((\tau d)^{-1}) = -\sigma(\tau d)$ and therefore

$$\sigma(\tau d) = -\sigma(d) - \sigma(c_i) - \sigma(c_j).$$

Now we see that $\sigma(c_i)=\sigma(c_j)=0$ implies $\sigma(\tau d)=-\sigma(d)$.

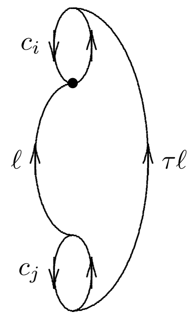

Figure 5: A bridge between two twists c_i and c_j

$\square$

Lemma 4.7. *Let (P,τ) be a Klein surface of type (g,k,ε) and σ an m-Arf function on P. Let $c_1,\dots,c_n$ be invariant contours as in Theorem 4.2, with $c_1,\dots,c_k$ corresponding to ovals and $c_{k+1},\dots,c_n$ corresponding to twists. Let*

$$\mathcal{B}=(a_1,b_1,\dots,a_{\tilde g},b_{\tilde g},a'_1,b'_1,\dots,a'_{\tilde g},b'_{\tilde g},c_1,\dots,c_{n-1},d_1,\dots,d_{n-1})$$

be a symmetric generating set of $\pi_1^0(P)$. Assume that
(i) $\sigma(a_i)=\sigma(a'_i)$, $\sigma(b_i)=\sigma(b'_i)$ for $i=1,\dots,\tilde g$,
(ii) $2\sigma(c_i)=0$ for $i=1,\dots,n-1$.
(iii) $\sigma(c_i)=0$ for $i=k+1,\dots,n$.
Then σ is a real m-Arf function on (P,τ).

Remark. Condition (iii) means that σ vanishes on all twists.

Proof. To be real, the m-Arf function σ must vanish on all twists and be compatible with τ, i.e. satisfy the equation $\sigma(\tau x)=-\sigma(x)$ for all $x\in\pi_1^0(P)$. Condition (iii) implies that σ vanishes on all twists. We will first check the equation $\sigma(\tau x)=-\sigma(x)$ for all x in $\mathcal{B}$.

- $x=a_i,b_i,\ i=1,\dots,\tilde g$: Recall that $a'_i=(\tau a_i)^{-1}$, hence $\tau a_i=(a'_i)^{-1}$ and $\sigma(\tau a_i)=\sigma((a_i)'^{-1})=-\sigma(a'_i)$. Condition (i) implies $\sigma(a'_i)=\sigma(a_i)$, hence $\sigma(\tau a_i)=-\sigma(a'_i)=-\sigma(a_i)$. Similarly $\sigma(\tau b_i)=-\sigma(b_i)$.
- $x=a'_i,b'_i,\ i=1,\dots,\tilde g$: Recall that $a'_i=(\tau a_i)^{-1}$, hence $\tau a'_i=a_i^{-1}$ and $\sigma(\tau a'_i)=\sigma(a_i^{-1})=-\sigma(a_i)$. Condition (i) implies $\sigma(a_i)=\sigma(a'_i)$, hence $\sigma(\tau a'_i)=-\sigma(a_i)=-\sigma(a'_i)$. Similarly $\sigma(\tau b'_i)=-\sigma(b'_i)$.
- $x=c_i,\ i=1,\dots,n-1$: Recall that $\tau c_i=c_i$ for an oval c_i, $i=1,\dots,k$, while τc_i is conjugate to c_i for a twist c_i, $i=k+1,\dots,n-1$. In both cases $\sigma(\tau c_i)=\sigma(c_i)$. Condition (ii) implies $2\sigma(c_i)=0$ for $i=1,\dots,n-1$, hence $\sigma(\tau c_i)=\sigma(c_i)=-\sigma(c_i)$.
- $x=d_i,\ i=1,\dots,n-1$: Condition (iii) implies that σ vanishes on all twists. According to Lemma 4.6, it follows that $\sigma(\tau d_i)=-\sigma(d_i)$ for $i=1,\dots,n-1$.

Consider $\tilde\sigma:\pi_1(P)\to\mathbb{Z}/m\mathbb{Z}$ given by $\tilde\sigma(x)=-\sigma(\tau x)$. The involution τ is an orientation-reversing homeomorphism, so it is easy to check using Definition 3.4 that if σ is an m-Arf function then so is $\tilde\sigma$. We have checked that for any x in $\mathcal{B}$ we have $\sigma(\tau x)=-\sigma(x)$, hence $\tilde\sigma(x)=-\sigma(\tau x)=\sigma(x)$. Thus we have two m-Arf functions,

σ and $\tilde{\sigma}$, which coincide on a generating set. We can conclude that σ and $\tilde{\sigma}$ coincide everywhere on $\pi_1(P)$, i.e. for any $x \in \pi_1(P)$ we have $\sigma(\tau x) = -\tilde{\sigma}(x) = -\sigma(x)$. This shows that the Arf function σ is real. $\square$

4.4. **Classification and Enumeration of Real Arf Functions.** Let (P, τ) be a Klein surface of type (g, k, ε), $g \geqslant 2$. Let $c_1, \dots, c_n$ be invariant contours as in Theorem 4.2. The contours $c_1, \dots, c_k$ correspond to ovals. In the separating case ($\varepsilon = 1$) we have $n = k$. In the non-separating case $\varepsilon = 0$ we have $n > k$ and the contours $c_{k+1}, \dots, c_n$ correspond to twists. Let

$$\mathcal{B} = (a_1, b_1, \dots, a_{\tilde{g}}, b_{\tilde{g}}, a'_1, b'_1, \dots, a'_{\tilde{g}}, b'_{\tilde{g}}, c_1, \dots, c_{n-1}, d_1, \dots, d_{n-1})$$

be a symmetric generating set of $\pi_1^0(P)$. Let P_1 and P_2 be the connected components of the complement of the contours $c_1, \dots, c_n$ in P. Each of these components is a surface of genus $\tilde{g} = (g + 1 - n)/2$ with n holes. We have $\tau(P_1) = P_2$.

Lemma 4.8. *Let σ be a real m-Arf function on (P, τ), then*

$$\sigma(c_1) + \dots + \sigma(c_n) = 1 - g \bmod m$$

and

$$g = 1 \bmod \frac{m}{2} \quad \text{if } m \text{ is even,}$$
$$g = 1 \bmod m \quad \text{if } m \text{ is odd.}$$

Proof. Theorem 4.3, applied to P_1, implies that

$$\sigma(c_1) + \dots + \sigma(c_n) = (2 - 2\tilde{g}) - n = 1 - g \bmod m.$$

Moreover, $\sigma(c_i) \in \{0, m/2\}$ for even m and $\sigma(c_i) = 0$ for odd m completes the proof. $\square$

Theorem 4.9. *Let $\varepsilon = 0$ and let m be even. Recall that in this case $n > k$.*

1) Let σ be a real m-Arf function on (P, τ). Then

$$\sigma(a_i) = \sigma(a'_i) \text{ and } \sigma(b_i) = \sigma(b'_i) \quad \text{for } i = 1, \dots, \tilde{g},$$
$$\sigma(c_1), \dots, \sigma(c_k) \in \{0, m/2\}, \quad \sigma(c_{k+1}) = \dots = \sigma(c_n) = 0,$$
$$\sigma(c_1) + \dots + \sigma(c_k) = 1 - g \bmod m,$$
$$g = 1 \bmod (m/2).$$

2) Let the set of values $\mathcal{V}$ in $(\mathbb{Z}/m\mathbb{Z})^{4\tilde{g}+2n-2}$ be

$$(\alpha_1, \beta_1, \dots, \alpha_{\tilde{g}}, \beta_{\tilde{g}}, \alpha'_1, \beta'_1, \dots, \alpha'_{\tilde{g}}, \beta'_{\tilde{g}}, \gamma_1, \dots, \gamma_{n-1}, \delta_1, \dots, \delta_{n-1}).$$

Assume that

$$\alpha_i = \alpha'_i \text{ and } \beta_i = \beta'_i \quad \text{for } i = 1, \dots, \tilde{g},$$
$$\gamma_1, \dots, \gamma_k \in \{0, m/2\}, \quad \gamma_{k+1} = \dots = \gamma_{n-1} = 0,$$
$$\gamma_1 + \dots + \gamma_k = 1 - g \bmod m.$$

Then there exists a real m-Arf function σ on (P, τ) with values $\mathcal{V}$ on the generating set $\mathcal{B}$. For this m-Arf function we have

$$\sigma(c_n) = 0.$$

3) The number of real m-Arf functions on (P, τ) is

$$m^g \quad \text{for } k = 0 \quad \text{and} \quad m^g \cdot 2^{k-1} \quad \text{for } k \geqslant 1.$$

4) The Arf invariant $\delta \in \{0,1\}$ of a real m-Arf function σ on (P,τ) is given by

$$\delta = \sum_{i=1}^{n-1}(1-\sigma(c_i))(1-\sigma(d_i)) \bmod 2.$$

5) Consider $\gamma_1, \dots, \gamma_{n-1}$ as above. Let

$$\Sigma = \sum_{i=1}^{n-1}(1-\gamma_i)(1-\delta_i).$$

Out of m^{n-1} possible choices for $(\delta_1, \dots, \delta_{n-1}) \in (\mathbb{Z}/m\mathbb{Z})^{n-1}$ there are $m^{n-1}/2$ which give $\Sigma = 0 \bmod 2$ and $m^{n-1}/2$ which give $\Sigma = 1 \bmod 2$.

6) The number of even and odd real m-Arf functions respectively on (P,τ) is

$$\frac{m^g}{2} \quad \text{for } k=0 \quad \text{and} \quad m^g \cdot 2^{k-2} \quad \text{for } k \geqslant 1.$$

Proof. 1) For the real m-Arf function σ on (P,τ) we have $\sigma(c_i) \in \{0, m/2\}$ for $i = 1, \dots, k$ according to Lemma 3.7 and $\sigma(c_i) = 0$ for $i = k+1, \dots, n$ by Definition 3.13. Lemma 4.8 implies $\sigma(c_1) + \dots + \sigma(c_n) = 1 - g \bmod m$ and $g = 1 \bmod (m/2)$. Furthermore $\sigma(c_{k+1}) = \dots = \sigma(c_n) = 0$ implies $\sigma(c_1) + \dots + \sigma(c_n) = \sigma(c_1) + \dots + \sigma(c_k)$.

2) We know that $1 - g = \gamma_1 + \dots + \gamma_k \bmod m$ and $\gamma_1, \dots, \gamma_k$ are multiples of $m/2$, hence $2 - 2g = 0 \bmod m$. This implies, according to Proposition 4.5, that the values $\mathcal{V}$ on $\mathcal{B}$ determine a unique m-Arf function σ on P. According to Lemma 4.7, to show that this m-Arf function σ is real, it is sufficient to show that $\sigma(c_n) = 0$. Lemma 4.8 implies that

$$1 - g = \sigma(c_1) + \dots + \sigma(c_n) = \gamma_1 + \dots + \gamma_{n-1} + \sigma(c_n) \bmod m.$$

On the other hand $\gamma_1 + \dots + \gamma_k = 1 - g \bmod m$ and $\gamma_{k+1} = \dots = \gamma_{n-1} = 0$, hence

$$1 - g = \gamma_1 + \dots + \gamma_{n-1} \bmod m.$$

Comparing these equations we obtain

$$\sigma(c_n) = 0 \bmod m.$$

Thus σ is a real m-Arf function.

3) There are $m^{2\tilde{g}}$ ways to choose $\alpha_i = \alpha'_i$ and $\beta_i = \beta'_i$. For $k \geqslant 1$ there are 2^{k-1} ways to choose $\gamma_1, \dots, \gamma_{k-1}$, while γ_k is determined by the condition $\gamma_1 + \dots + \gamma_k = 1 - g \bmod m$. There is only one way to choose $\gamma_{k+1}, \dots, \gamma_{n-1}$. There are m^{n-1} ways to choose $\delta_1, \dots, \delta_{n-1}$. Hence for $k \geqslant 1$ there are

$$m^{2\tilde{g}+n-1} \cdot 2^{k-1} = m^g \cdot 2^{k-1}$$

different choices of $\mathcal{V}$ and hence different choices for a real m-Arf function σ. For $k = 0$ there are

$$m^{2\tilde{g}+n-1} = m^g$$

different choices of $\mathcal{V}$ and hence different choices for a real m-Arf function σ.

4) According to Proposition 4.5 the Arf invariant $\delta \in \{0,1\}$ of an Arf function σ with values $\mathcal{V}$ on $\mathcal{B}$ is determined by the equation

$$\delta = \sum_{i=1}^{\tilde{g}}(1-\alpha_i)(1-\beta_i) + \sum_{i=1}^{\tilde{g}}(1-\alpha'_i)(1-\beta'_i) + \sum_{i=1}^{n-1}(1-\gamma_i)(1-\delta_i) \bmod 2.$$

Using $\alpha_i = \alpha_i'$ and $\beta_i = \beta_i'$ we obtain

$$\sum_{i=1}^{\tilde{g}}(1-\alpha_i)(1-\beta_i) + \sum_{i=1}^{\tilde{g}}(1-\alpha_i')(1-\beta_i') = 2\sum_{i=1}^{\tilde{g}}(1-\alpha_i)(1-\beta_i) = 0 \bmod 2,$$

hence

$$\delta = \sum_{i=1}^{n-1}(1-\gamma_i)(1-\delta_i) \bmod 2.$$

Note that $(1-\gamma_i)(1-\delta_i) \neq 0 \bmod 2$ if and only if γ_i and δ_i are both even. Hence

$$\delta = \left|\{i \in \{1,\dots,n-1\} \;\middle|\; \gamma_i \text{ and } \delta_i \text{ are even}\}\right| \bmod 2.$$

5) We need to determine, for given γ_i, how many of the m^{n-1} ways to choose δ_i lead to

$$\Sigma = \sum_{i=1}^{n-1}(1-\gamma_i)(1-\delta_i)$$

being even and odd respectively. If there exists $r \in \{1,\dots,n-1\}$ such that γ_r is even, then for any choice of $\delta_1,\dots,\hat{\delta_r},\dots,\delta_{n-1}$ exactly half of the possible choices for δ_r will lead to even Σ and half to odd Σ. What about the case when all γ_i are odd? Recall that $\gamma_i \in \{0, m/2\}$ for $i = 1,\dots,k$ and $\gamma_i = 0$ for $i = k+1,\dots,n-1$, i.e. γ_i is odd if and only if $i \in \{1,\dots,k\}$, $\gamma_i = m/2$, $m = 2 \bmod 4$. Thus the case when all γ_i are odd is only possible if $m = 2 \bmod 4$, $\gamma_1 = \dots = \gamma_k = m/2$ and $n = k+1$. Recall that $n = g+1 \bmod 2$, comparing with $n = k+1$ we obtain $k = g \bmod 2$. Recall that $1-g = \gamma_1 + \dots + \gamma_k$, comparing with $\gamma_1 = \dots = \gamma_k = m/2$ we obtain $1-g = k\cdot m/2 \bmod m$. Using $k = g \bmod 2$ and $m/2 = 1 \bmod 2$ we obtain $1-g = g \bmod 2$. This contradiction shows that $\gamma_1,\dots,\gamma_{n-1}$ cannot all be odd.

6) For $k \geqslant 1$ there are $\frac{1}{2}\cdot(m^{n-1}\cdot 2^{k-1}) = m^{n-1}\cdot 2^{k-2}$ ways to choose the values γ_i, δ_i that lead to even and odd Σ respectively. Therefore the number of real m-Arf functions which are even and odd respectively is

$$m^{2\tilde{g}}\cdot m^{n-1}\cdot 2^{k-2} = m^g \cdot 2^{k-2}.$$

For $k = 0$ there are $\frac{1}{2}\cdot m^{n-1} = m^{n-1}$ ways to choose the values γ_i, δ_i that lead to even and odd Σ respectively. Therefore the number of real m-Arf functions which are even and odd respectively is

$$m^{2\tilde{g}}\cdot \frac{1}{2}m^{n-1} = \frac{m^g}{2}.$$

□

Theorem 4.10. *Let $\varepsilon = 1$ and let m be even. Recall that in this case $n = k$.*

1) Let σ be a real m-Arf function on (P,τ). Then

$$\begin{aligned}
&\sigma(a_i) = \sigma(a_i') \text{ and } \sigma(b_i) = \sigma(b_i') \quad \text{for } i = 1,\dots,\tilde{g},\\
&\sigma(c_1),\dots,\sigma(c_k) \in \{0, m/2\},\\
&\sigma(c_1)+\dots+\sigma(c_k) = 1-g \bmod m,\\
&g = 1 \bmod (m/2).
\end{aligned}$$

2) Assume that

$$1-g=0 \bmod \frac{m}{2}.$$

Let the set of values $\mathcal{V}$ in $(\mathbb{Z}/m\mathbb{Z})^{4\tilde{g}+2k-2}$ be

$$(\alpha_1,\beta_1,\dots,\alpha_{\tilde{g}},\beta_{\tilde{g}},\alpha'_1,\beta'_1,\dots,\alpha'_{\tilde{g}},\beta'_{\tilde{g}},\gamma_1,\dots,\gamma_{k-1},\delta_1,\dots,\delta_{k-1}).$$

Assume that

$$\alpha_i=\alpha'_i \text{ and } \beta_i=\beta'_i \quad \text{for } i=1,\dots,\tilde{g},$$
$$\gamma_1,\dots,\gamma_{k-1}\in\{0,m/2\}.$$

Then there exists a real m-Arf function σ on (P,τ) with

$$\sigma(a_i)=\alpha_1,\ \sigma(b_i)=\beta_i,\ \sigma(a'_i)=\alpha'_i,\ \sigma(b'_i)=\beta'_i,\ \sigma(c_i)=\gamma_i,\ \sigma(d_i)=\delta_i.$$

For this m-Arf function we have

$$\sigma(c_k)=(1-g)-\gamma_1-\dots-\gamma_{k-1} \bmod m.$$

3) The number of real m-Arf functions on (P,τ) is

$$m^g\cdot 2^{k-1}.$$

4) The Arf invariant $\delta\in\{0,1\}$ of a real m-Arf function σ on (P,τ) is given by

$$\delta=\sum_{i=1}^{k-1}(1-\sigma(c_i))(1-\sigma(d_i)) \bmod 2.$$

5) Consider $\gamma_1,\dots,\gamma_{k-1}$ as above. Let

$$\Sigma=\sum_{i=1}^{k-1}(1-\gamma_i)(1-\delta_i).$$

In the case $m=2 \bmod 4$, $\gamma_1=\dots=\gamma_{k-1}=m/2$ any choice of $(\delta_1,\dots,\delta_{k-1})\in(\mathbb{Z}/m\mathbb{Z})^{k-1}$ gives $\Sigma=0 \bmod 2$. In all other cases, out of m^{k-1} possible choices for $(\delta_1,\dots,\delta_{k-1})\in(\mathbb{Z}/m\mathbb{Z})^{k-1}$ there are $m^{k-1}/2$ which give $\Sigma=0 \bmod 2$ and $m^{k-1}/2$ which give $\Sigma=1 \bmod 2$.

6) In the case $m=0 \bmod 4$ the number of even and odd real m-Arf functions respectively is

$$m^g\cdot 2^{k-2}.$$

In the case $m=2 \bmod 4$ the numbers of even and odd real m-Arf functions respectively are

$$m^g\cdot\frac{2^{k-1}+1}{2} \quad \text{and} \quad m^g\cdot\frac{2^{k-1}-1}{2}.$$

Proof. 1) For the real m-Arf function σ on (P,τ) we have $\sigma(c_i)\in\{0,m/2\}$ for $i=1,\dots,k$ according to Lemma 3.7. Lemma 4.8 implies $\sigma(c_1)+\dots+\sigma(c_k)=1-g \bmod m$ and $g=1 \bmod (m/2)$.

2) We know that $1-g=0 \bmod (m/2)$, hence $2-2g=0 \bmod m$. This implies, according to Proposition 4.5, that the values $\mathcal{V}$ on $\mathcal{B}$ determine a unique m-Arf function σ on P. According to Lemma 4.7, to show that this m-Arf function σ is real, it is sufficient to show that $2\sigma(c_k)=0$. Lemma 4.8 implies that

$$1-g=\sigma(c_1)+\dots+\sigma(c_k)=\gamma_1+\dots+\gamma_{k-1}+\sigma(c_k) \bmod m,$$

hence

$$\sigma(c_k)=(1-g)-\gamma_1-\dots-\gamma_{k-1} \bmod m.$$

We know that $1-g$ and $\gamma_1, \dots, \gamma_k$ are multiples of $m/2$, hence $2\sigma(c_k) = 0 \bmod m$. Thus σ is a real m-Arf function.

3) There are $m^{2\tilde{g}}$ ways to choose $\alpha_i = \alpha'_i$ and $\beta_i = \beta'_i$. There are 2^{k-1} ways to choose $\gamma_1, \dots, \gamma_{k-1}$. There are m^{k-1} ways to choose $\delta_1, \dots, \delta_{k-1}$. Hence there are
$$m^{2\tilde{g}+k-1} \cdot 2^{k-1} = m^g \cdot 2^{k-1}$$
different choices of $\mathcal{V}$ and hence different choices for a real m-Arf function σ.

4) The Arf invariant $\delta \in \{0,1\}$ of an Arf function σ with values $\mathcal{V}$ on $\mathcal{B}$ is determined by the equation
$$\delta = \sum_{i=1}^{\tilde{g}}(1-\alpha_i)(1-\beta_i) + \sum_{i=1}^{\tilde{g}}(1-\alpha'_i)(1-\beta'_i) + \sum_{i=1}^{k-1}(1-\gamma_i)(1-\delta_i) \bmod 2.$$
Using $\alpha_i = \alpha'_i$ and $\beta_i = \beta'_i$ we obtain
$$\sum_{i=1}^{\tilde{g}}(1-\alpha_i)(1-\beta_i) + \sum_{i=1}^{\tilde{g}}(1-\alpha'_i)(1-\beta'_i) = 2\sum_{i=1}^{\tilde{g}}(1-\alpha_i)(1-\beta_i) = 0 \bmod 2,$$
hence
$$\delta = \sum_{i=1}^{k-1}(1-\gamma_i)(1-\delta_i) \bmod 2.$$
Note that $(1-\gamma_i)(1-\delta_i) \neq 0 \bmod 2$ only if γ_i and δ_i are both even. Hence
$$\delta = \big|\{i \in \{1, \dots, k-1\} \mid \gamma_i \text{ and } \delta_i \text{ are even}\}\big| \bmod 2.$$

5) We need to determine, for given γ_i, how many of the m^{k-1} ways to choose δ_i lead to $\Sigma = \sum_{i=1}^{k-1}(1-\sigma(c_i))(1-\sigma(d_i))$ being even and odd respectively. If there exists $r \in \{1, \dots, k-1\}$ such that γ_r is even, then for any choice of $\delta_1, \dots, \hat{\delta}_r, \dots, \delta_{k-1}$ exactly half of the possible choices for δ_r will lead to Σ being even and odd respectively. What about the case when all γ_i are odd? Recall that $\gamma_i \in \{0, m/2\}$ for $i = 1, \dots, k-1$, i.e. γ_i is odd if and only if $\gamma_i = m/2$ and $m = 2 \bmod 4$. Thus the case when all γ_i are odd is only possible if $m = 2 \bmod 4$ and $\gamma_1 = \dots = \gamma_{k-1} = m/2$. In this case Σ is even.

6) Let $m = 0 \bmod 4$. For any of the 2^{k-1} ways to choose $\gamma_1, \dots, \gamma_{k-1}$ the number of ways to choose $\delta_1, \dots, \delta_{k-1}$ so that Σ is even and odd respectively is $m^{k-1}/2$. Thus the number of even and odd real m-Arf functions is
$$m^{2\tilde{g}} \cdot 2^{k-1} \cdot \frac{m^{k-1}}{2} = m^g \cdot 2^{k-2}$$
respectively. Let $m = 2 \bmod 4$. For any of the $2^{k-1}-1$ ways to choose
$$(\gamma_1, \dots, \gamma_{k-1}) \neq (m/2, \dots, m/2)$$
the number of ways to choose $\delta_1, \dots, \delta_{k-1}$ so that Σ is even and odd respectively is $m^{k-1}/2$. For $(\gamma_1, \dots, \gamma_{k-1}) = (m/2, \dots, m/2)$ any of the m^{k-1} choices of $\delta_1, \dots, \delta_{k-1}$ gives even Σ. Therefore the number of choices of γ_i and δ_i that give odd Σ is
$$(2^{k-1}-1)\frac{m^{k-1}}{2}$$

and the number of choices of γ_i and δ_i that give even Σ is

$$(2^{k-1}-1)\frac{m^{k-1}}{2}+m^{k-1}=(2^{k-1}+1)\cdot m^{k-1}2.$$

Thus the number of even and odd real m-Arf functions is

$$m^{2\tilde{g}}\cdot(2^{k-1}\pm 1)\cdot\frac{m^{k-1}}{2}=m^{2\tilde{g}+k-1}\cdot\frac{2^{k-1}\pm 1}{2}=m^{g}\cdot\frac{2^{k-1}\pm 1}{2}.$$

respectively.

□

Theorem 4.11. *Let m be odd.*

1) Let σ be a real m-Arf function on (P,τ). Then

$$\begin{aligned}&\sigma(a_i)=\sigma(a_i') \text{ and } \sigma(b_i)=\sigma(b_i') \quad \text{for } i=1,\dots,\tilde{g},\\ &\sigma(c_1)=\dots=\sigma(c_n)=0,\\ &g=1 \bmod m.\end{aligned}$$

2) Assume that

$$g=1 \bmod m.$$

Let the set of values $\mathcal{V}$ in $(\mathbb{Z}/m\mathbb{Z})^{4\tilde{g}+2n-2}$ be

$$(\alpha_1,\beta_1,\dots,\alpha_{\tilde{g}},\beta_{\tilde{g}},\alpha_1',\beta_1',\dots,\alpha_{\tilde{g}}',\beta_{\tilde{g}}',\gamma_1,\dots,\gamma_{n-1},\delta_1,\dots,\delta_{n-1}).$$

Assume that

$$\begin{aligned}&\alpha_i=\alpha_i' \text{ and } \beta_i=\beta_i' \quad \text{for } i=1,\dots,\tilde{g},\\ &\gamma_1=\dots=\gamma_{n-1}=0.\end{aligned}$$

Then there exists a real m-Arf function σ on (P,τ) with

$$\sigma(a_i)=\alpha_1,\ \sigma(b_i)=\beta_i,\ \sigma(a_i')=\alpha_i',\ \sigma(b_i')=\beta_i',\ \sigma(c_i)=\gamma_i,\ \sigma(d_i)=\delta_i.$$

For this Arf function we have $\sigma(c_n)=0$.

3) The number of real m-Arf functions on (P,τ) is m^g.

Proof. 1) For the real m-Arf function σ on (P,τ) we have $\sigma(c_i)=0$ for $i=1,\dots,k$ according to Lemma 3.7 and $\sigma(c_i)=0$ for $i=k+1,\dots,n$ by Definition 3.13. Lemma 4.8 implies $\sigma(c_1)+\dots+\sigma(c_n)=1-g \bmod m$ and $g=1 \bmod m$.

2) The condition $g=1 \bmod m$ implies $2-2g=0 \bmod m$, hence, according to Proposition 4.5, the values $\mathcal{V}$ on $\mathcal{B}$ determine a unique m-Arf function σ on P. According to Lemma 4.7, to show that this m-Arf function σ is real, it is sufficient to show that $\sigma(c_n)=0$. Lemma 4.8 implies that

$$\sigma(c_1)+\dots+\sigma(c_n)=1-g \bmod m.$$

On the other hand $\gamma_1=\dots=\gamma_{n-1}=0$, hence

$$\sigma(c_1)+\dots+\sigma(c_n)=\gamma_1+\dots+\gamma_{n-1}+\sigma(c_n)=\sigma(c_n) \bmod m.$$

Comparing these equations we obtain

$$\sigma(c_n)=1-g \bmod m.$$

The condition $g=1 \bmod m$ implies $\sigma(c_n)=0 \bmod m$. Thus σ is a real m-Arf function.

3) There are $m^{2\tilde{g}}$ way to choose $\alpha_i = \alpha_i'$ and $\beta_i = \beta_i'$, only one way to choose $\gamma_1, \ldots, \gamma_{n-1}$ and m^{n-1} ways to choose $\delta_1, \ldots, \delta_{n-1}$, hence there are

$$m^{2\tilde{g}+n-1} = m^g$$

different choices of $\mathcal{V}$ and hence different choices for a real m-Arf function σ.

□

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