ON THE LINK SPACE OF A \(\mathbb{Q}\)-GORENSTEIN QUASI-HOMOGENEOUS SURFACE SINGULARITY

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ABSTRACT. In this paper we prove the following theorem: Let \(M\) be the link space of a quasi-homogeneous hyperbolic \(\mathbb{Q}\)-Gorenstein surface singularity. Then \(M\) is diffeomorphic to a coset space \(\hat{\Gamma}_1 \backslash \hat{G}/\hat{\Gamma}_2\), where \(\hat{G}\) is the 3-dimensional Lie group \(\text{PSL}(2,\mathbb{R})\), while \(\hat{\Gamma}_1\) and \(\hat{\Gamma}_2\) are discrete subgroups of \(\hat{G}\), \(\hat{\Gamma}_1\) is co-compact and \(\hat{\Gamma}_2\) is cyclic. Conversely, if \(M\) is diffeomorphic to a coset space as above, then \(M\) is diffeomorphic to the link space of a quasi-homogeneous hyperbolic \(\mathbb{Q}\)-Gorenstein singularity. We also prove the following characterisation of quasi-homogeneous \(\mathbb{Q}\)-Gorenstein surface singularities: A quasi-homogeneous surface singularity is \(\mathbb{Q}\)-Gorenstein of index \(r\) if and only if for the corresponding automorphy factor \((U, \Gamma, L)\) some tensor power of the complex line bundle \(L\) is \(\Gamma\)-equivariantly isomorphic to some tensor power of the tangent bundle of the Riemannian surface \(U\).

1. INTRODUCTION

Graded affine coordinate rings of quasi-homogeneous surface singularities can be identified with graded rings of generalised automorphic forms. The description in terms of automorphy factors was found in 1975–77 by Dolgachev, Milnor, Neumann and Pinkham [Dol75, Dol77, Mil75, Neu77, Pin77].

For some special classes of quasi-homogeneous surface singularities as Gorenstein and \(\mathbb{Q}\)-Gorenstein singularities one can obtain more precise descriptions of the corresponding automorphy factors.

In Theorem 3 we obtain a characterisation of hyperbolic and spherical \(\mathbb{Q}\)-Gorenstein quasi-homogeneous surface singularities in terms of their automorphy factors. This characterisation leads to a description of their links as quotients of certain 3-dimensional Lie groups by discrete subgroups. More precisely, we prove the following statement.

**Theorem 1.** The link space of a hyperbolic \(\mathbb{Q}\)-Gorenstein quasi-homogeneous surface singularity of level \(m\) and index \(r\) is diffeomorphic to a biquotient

\[\hat{\Gamma}_1 \backslash \hat{G}/\hat{\Gamma}_2,\]

where \(\hat{G}\) is the universal cover \(\hat{\text{PSL}}(2,\mathbb{R})\) of the 3-dimensional Lie group \(\text{PSL}(2,\mathbb{R})\), while \(\hat{\Gamma}_1\) and \(\hat{\Gamma}_2\) are discrete subgroups of level \(m\) in \(\hat{G}\), \(\hat{\Gamma}_1\) is co-compact, and the image of \(\hat{\Gamma}_2\) in \(\text{PSL}(2,\mathbb{R})\) is a cyclic subgroup of order \(r\). Conversely, any biquotient as above is diffeomorphic to the link space of a quasi-homogeneous hyperbolic \(\mathbb{Q}\)-Gorenstein singularity.

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These statements are generalisations of the results of Dolgachev [Dol83] on Gorenstein quasi-homogeneous surface singularities. The Gorenstein quasi-homogeneous surface singularities correspond to the case of a trivial group $\Gamma_2$.

Similar statements are also true in the case of Euclidean automorphy factors and corresponding singularities. This case was already discussed by Dolgachev in [Dol83]. The Euclidean $\mathbb{Q}$-Gorenstein quasi-homogeneous surface singularities are rational singularities, which are quotients of Gorenstein Euclidean singularities by actions of finite groups.

The description of the link space of a hyperbolic Gorenstein quasi-homogeneous surface singularity as a quotient of the Lie group $\text{PSL}(2, \mathbb{R})$ by the action of a discrete subgroup was the motivation for the study in [Pra03], [BPR03] of a certain construction of fundamental domains for such actions. This construction leads to interesting results on the combinatorial geometry of the link spaces of Gorenstein quasi-homogeneous surface singularities.

We expect that our construction of fundamental domains can be generalised in order to study the combinatorial geometry of the link spaces of $\mathbb{Q}$-Gorenstein quasi-homogeneous surface singularities. We shall discuss the combinatorial geometry of the link spaces in the $\mathbb{Q}$-Gorenstein case in an ongoing paper.

The paper is organised as follows: In section 2 we explain the description of quasi-homogeneous surface singularities via automorphy factors. In section 3 we define $\mathbb{Q}$-Gorenstein quasi-homogeneous surface singularities and introduce our characterisation of the corresponding automorphy factors (Theorem 3). Then in section 4 we prove some technical results needed to prove this characterisation. After that we prove Theorem 3 in section 5. Finally we prove Theorem 1 in section 6.

Notation: In this paper we use $\mathbb{R}_+$ for $\{x \in \mathbb{R} \mid x > 0\}$. We denote by $L^*$ the associated $\mathbb{C}^*$-bundle of a complex line bundle $L$, while $L^\vee$ is the dual bundle of $L$.

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2. Automorphy Factors

In this section we recall the results of Dolgachev, Milnor, Neumann and Pinkham [Dol75, Dol77, Mil75, Neu77, Pin77] on the graded affine coordinate rings, which correspond to quasi-homogeneous surface singularities.

**Definition.** A (negative unramified) automorphy factor $(U, \Gamma, L)$ is a complex line bundle $L$ over a simply connected Riemann surface $U$ together with a discrete cocompact subgroup $\Gamma \subset \text{Aut}(U)$ acting compatibly on $U$ and on the line bundle $L$, such that the following two conditions are satisfied:

1) The action of $\Gamma$ is free on $L^*$, the complement of the zero-section in $L$.
2) Let $\Gamma' \triangleleft \Gamma$ be a normal subgroup of finite index, which acts freely on $U$, and let $\tilde{L} \to C$ be the complex line bundle $\tilde{L} = L/\Gamma'$ over the compact Riemann surface $C = U/\Gamma'$. Then $\tilde{L}$ is a negative line bundle.

A simply connected Riemann surface $U$ can be $\mathbb{C}P^1$, $\mathbb{C}$, or $H$, the real hyperbolic plane. We call the corresponding automorphy factor and the corresponding singularity spherical, Euclidean, resp. hyperbolic.
Remark. There always exists a normal freely acting subgroup of $\Gamma$ of finite index. In the hyperbolic case the existence follows from the theorem of Fox-Bundgaard-Nielsen. If the second assumption in the last definition holds for some normal freely acting subgroup of finite index, then it holds for any such subgroup.

The simplest examples of such a complex line bundle with group action are the cotangent bundle of the complex projective plane $U = \mathbb{C}P^1$ and the tangent bundle of the hyperbolic plane $U = H$ equipped with the canonical action of a subgroup $\Gamma \subset \text{Aut}(U)$.

Let $(U, \Gamma, L)$ be a negative unramified automorphy factor. Since the bundle $\tilde{L} = L / \Gamma'$ is negative, one can contract the zero section of $\tilde{L}$ to get a complex surface with one isolated singularity corresponding to the zero section. There is a canonical action of the group $\Gamma / \Gamma'$ on this surface. The quotient is a complex surface $X(U, \Gamma, L)$ with an isolated singular point $o$, which depends only on the automorphy factor $(U, \Gamma, L)$.

The following theorem summarises the results of Dolgachev, Milnor, Neumann, and Pinkham:

**Theorem 2.** The surface $X(U, \Gamma, L)$ associated to a negative unramified automorphy factor $(U, \Gamma, L)$ is a quasi-homogeneous affine algebraic surface with a normal isolated singularity. Its affine coordinate ring is the graded $\mathbb{C}$-algebra of generalised $\Gamma$-invariant automorphic forms

$$A = \bigoplus_{m \geq 0} H^0(U, L^{-m})^\Gamma.$$

All normal isolated quasi-homogeneous surface singularities $(X, x)$ are obtained in this way, and the automorphy factors with $(X(U, \Gamma, L), o)$ isomorphic to $(X, x)$ are uniquely determined by $(X, x)$ up to isomorphism.

3. Q-GORENSTEIN QUASI-HOMOGENEOUS SURFACE SINGULARITIES

In this section we recall the definition of Q-Gorenstein singularities and the characterisation of the corresponding automorphy factors.

A normal isolated singularity of dimension $n$ is Gorenstein if and only if there is a nowhere vanishing $n$-form on a punctured neighbourhood of the singular point. For example all isolated singularities of complete intersections are Gorenstein.

A natural generalisation of Gorenstein singularities are the Q-Gorenstein singularities (compare [Rei87, Ish87, Ish00]). A normal isolated singularity of dimension at least 2 is Q-Gorenstein if there is a natural number $r$ such that the divisor $r \cdot K_X$ is defined on a punctured neighbourhood of the singular point by a function. Here $K_X$ is the canonical divisor of $X$. The smallest such number $r$ is called the index of the singularity. A normal isolated surface singularity is Gorenstein if and only if it is Q-Gorenstein of index 1.

In section 5 we prove the following characterisation of Q-Gorenstein quasi-homogeneous surface singularities in terms of the corresponding automorphy factors:

**Theorem 3.** A quasi-homogeneous surface singularity is Q-Gorenstein of index $r$ if and only if for the corresponding automorphy factor $(U, \Gamma, L)$ there is an integer $m$ (called the level or the exponent of the automorphy factor) without common divisors with $r$ and a $\Gamma$-invariant isomorphism $L^m \cong \mathcal{T}_U$. 
Let \((U, \Gamma, L)\) be a negative unramified automorphy factor of level \(m\) and index \(r\), which corresponds to a \(\mathbb{Q}\)-Gorenstein singularity. The isomorphism \(L^m \cong T^*_U\) induces an isomorphism \(\tilde{L}^m \cong T^*_C\). The bundle \(\tilde{L}\) is negative. A simple computation with Chern numbers shows that the possible values of the exponent are \(m = -1\) or \(m = -2\) for \(U = \mathbb{CP}^1\), whereas \(m = 0\) for \(U = \mathbb{C}\) and \(m \in \mathbb{N}\) for \(U = H\).

4. The associated bundle of the quotient bundle

Let \((U, \Gamma, L)\) be a spherical or hyperbolic negative unramified automorphy factor. As in the definition let \(\Gamma' \triangleleft \Gamma\) be a normal subgroup of \(\Gamma\) acting freely on \(U\), and let \(p : \tilde{L} \to C\) be the complex line bundle with total space \(\tilde{L} = L/\Gamma'\) and base \(C = U/\Gamma'\). In this section we consider the associated \(\mathbb{C}^*\)-bundle of the bundle \(p\), i.e. \(p|_{\tilde{L}^*} : \tilde{L}^* \to C\). For ease of notation we set \(W := \tilde{L}^*\) and \(q := p|_{W}\).

We first present some technical lemmas, which will be used later to determine \(\Omega^{2,r}(W) := (\Omega^2(W))^\otimes r\).

**Lemma 4.** The following \(\mathcal{O}_C\)-algebras are isomorphic

\[
q_*(\mathcal{O}_W) \cong \bigoplus_{m \in \mathbb{Z}} \mathcal{O}_C(\tilde{L}^m).
\]

**Lemma 5.** We have \(\Omega^2(W) \cong q^*(\Omega^1_C)\).

**Lemma 6.** If the bundle \(\tilde{L}\) is non-trivial and the sheaf \(\Omega^{2,r}(W)\) is trivial, then there exists up to complex multiples only one nowhere vanishing section in \(\Omega^{2,r}(W)\).

We postpone the proofs of these lemmas until the end of this section and discuss first the main result of the section, the description of \((\Gamma/\Gamma')\)-invariant sections in \(\Omega^{2,r}(W)\).

**Proposition 7.** The sheaf \(\Omega^{2,r}(W)\) is trivial if and only if there exists an integer \(m\) and an isomorphism \(\tilde{L}^m \cong T^*_C\).

Assume that \(\Omega^{2,r}(W)\) is trivial and let \(m\) be the integer such that \(\tilde{L}^m \cong T^*_C\). Then the global nowhere vanishing sections in \(\Omega^{2,r}(W)\) are \((\Gamma/\Gamma')\)-invariant if and only if the isomorphism \(\tilde{L}^m \cong T^*_C\) is \((\Gamma/\Gamma')\)-equivariant.

**Proof.** 1) We first prove that \(\Omega^{2,r}(W) \cong \mathcal{O}_W\) implies \(\tilde{L}^m \cong T^*_C\) for some \(m \in \mathbb{Z}\).

Assume that \(\Omega^{2,r}(W) \cong \mathcal{O}_W\). This implies on the one hand using lemma 4 that

\[
q_*(\Omega^{2,r}(W)) \cong q_*(\mathcal{O}_W) \cong \bigoplus_{i \in \mathbb{Z}} \mathcal{O}_C(\tilde{L}^i).
\]

On the other hand we have using lemma 5

\[
q_*(\Omega^{2,r}(W)) \cong q_*(q^*(\Omega^1_C)^\otimes r).
\]

Now we obtain for \(\Omega^{1,r}_C := (\Omega^1_C)^\otimes r\)

\[
q_*(q^*(\Omega^1_C)^\otimes r) \cong q_*(q^*(\Omega^{1,r}_C))
\]

because \(q^*\) is compatible with tensor products. The projection formula implies

\[
q_*(q^*(\Omega^{1,r}_C)) \cong q_*(\mathcal{O}_W \otimes_{\mathcal{O}_C} q^*(\Omega^{1,r}_C)) \cong q_*(\mathcal{O}_W) \otimes_{\mathcal{O}_C} \Omega^{1,r}_C.
\]
Finally using lemma 4 again we obtain
\[ q_*(\Omega^{2,r}(W)) \cong q_*(\mathcal{O}_W) \otimes_{\mathcal{O}_C} \Omega^{1,r}_C \]
\[ \cong \bigoplus_{m \in \mathbb{Z}} \mathcal{O}_C(\hat{L}^m) \otimes_{\mathcal{O}_C} \Omega^{1,r}_C \]
\[ \cong \bigoplus_{m \in \mathbb{Z}} \mathcal{O}_C(\hat{L}^m \otimes T_C^{-r}). \]

Comparing both equations for \( q_*(\Omega^{2,r}(W)) \) we obtain
\[ \bigoplus_{i \in \mathbb{Z}} \mathcal{O}_C(\hat{L}^i) \cong \bigoplus_{m \in \mathbb{Z}} \mathcal{O}_C(\hat{L}^m \otimes T_C^{-r}), \]
hence
\[ \mathcal{O}_C(\hat{L}^m \otimes T_C^{-r}) \cong \mathcal{O}_C(\hat{L}^0) \cong \mathcal{O}_C \]
for some \( m \in \mathbb{Z} \). This implies \( \hat{L}^m \cong T_C^r \).

2) We now assume that \( \hat{L} \cong T_C^r \), i.e. \( m = 1 \). The Riemann surface \( U \) is then the real hyperbolic plane \( H/G \). We study the tangent bundle \( T_C \) of the surface \( C = H/G \). We define a 2-form on \((T_C)^*\) in local coordinates by \( \eta = \frac{1}{\sqrt{g}} (dz \wedge dt) \). Using the fact that any change of coordinates is of the form \((z, t) \mapsto (\varphi(z), \varphi'(z) \cdot t)\) we can verify that this local definition gives rise to a global nowhere vanishing 2-form on \((T_C)^*\), and that this 2-form is invariant under an action of \( g \in \Gamma \) if and only if the action is given by \((z, t) \mapsto (g(z), g'(z) \cdot t)\), i.e. the action coincides with the canonical action of \( g \) on \( T_C \). The 2-form on \((T_C)^*\) induces a nowhere vanishing section \( \eta \) in \( \Omega^{2,\tau}(\hat{L}^0)^* \), which is invariant under an action of \( g \in \Gamma \) if and only if the action is given in local coordinates by \((z, t) \mapsto (g(z), (g'(z))^{-1} \cdot t)\), i.e. the action coincides with the canonical action of \( g \) on \( T_C^r \). Hence if the isomorphism \( \hat{L} \cong T_C^r \) is \((\Gamma/G)^*\)-equivariant, there exists a \((\Gamma/G)^*\)-invariant nowhere vanishing section in \( \Omega^{2,\tau}(W) \).

3) We now assume that \( \hat{L} \cong T_C^{-r} \cong (T_C^r)^* \), i.e. \( m = -1 \). The Riemann surface \( U \) is then the complex projective line \( \mathbb{CP}^1 \). We study the cotangent bundle \( T_C^* \) of the surface \( C = \mathbb{CP}^1/G \). We define a 2-form on \((T_C^*)^*\) in local coordinates by \( \eta = dz \wedge dt \). Using the fact that any change of coordinates is of the form \((z, t) \mapsto (\varphi(z), \frac{1}{\varphi'(z)} \cdot t)\) we can verify that this local definition gives rise to a global nowhere vanishing 2-form on \((T_C^*)^*\). We continue in the proof as for \( m = 1 \) and obtain a nowhere vanishing section \( \eta \) in \( \Omega^{2,\tau}(\hat{L}^0)^* \), which is invariant under an action of \( g \in \Gamma \) if and only if the action coincides with the canonical action of \( g \) on \( T_C^{-r} \). Hence if the isomorphism \( \hat{L} \cong T_C^{-r} \) is \((\Gamma/G)^*\)-equivariant, there exists a \((\Gamma/G)^*\)-invariant nowhere vanishing section in \( \Omega^{2,\tau}(W) \).

4) As the next step of the proof we consider the case \( \hat{L}^m \cong T_C^r \) with \( m \neq 0 \). Let \( \eta \) be the nowhere vanishing section in \( \Omega^{2,\tau}(\hat{L}^m)^* \), i.e. in \( \Omega^{2,\tau}(\hat{L}^0)^* \), constructed in subsections 2 and 3. We consider the covering \( \tau : \hat{L} \rightarrow \hat{L}^m \). The pull-back \( \tau^*(\eta) \) of the section \( \eta \) under the covering \( \tau \) is a nowhere vanishing section in \( \Omega^{2,\tau}(\hat{L}) = \Omega^{2,\tau}(W) \). If the isomorphism \( \hat{L}^m \cong T_C^r \) is \((\Gamma/G)^*\)-equivariant, the induced section in \( \Omega^{2,\tau}(W) \) is \((\Gamma/G)^*\)-invariant.

5) We now assume that \( \Omega^{2,\tau}(W) \) is trivial and that there exists a nowhere vanishing section \( \omega \) in \( \Omega^{2,\tau}(W) \), which is \((\Gamma/G)^*\)-invariant. Then in particular there exists an integer \( m \) such that \( \hat{L}^m \cong T_C^r \). Let \( \eta \) be the nowhere vanishing section in
\( \Omega^2, (T^*_{\mathbb{C}})^* \) constructed before. By lemma 6 the sections \( \omega \) and \( \eta \) are complex multiples of each other, hence the isomorphism \( \tilde{L}^m \cong T^*_{\mathbb{C}} \) is \((\Gamma / \Gamma')\)-equivariant. \( \square \)

Now it remains to show the technical lemmas, which we have used in the proof of lemma 7. We first prove lemma 4:

**Proof.** We have to prove that the following \( \mathcal{O}_C \)-algebras are isomorphic
\[
q_* (\mathcal{O}_W) \cong \bigoplus_{m \in \mathbb{Z}} \mathcal{O}_C (\tilde{L}^m).
\]

Let us consider a local trivialisation \( \varphi : \tilde{L}|_V \to V \times \mathbb{C} \) of the complex line bundle \( \tilde{L} \) over an open affine subset \( V \subset C \). This trivialisation induces trivialisations \( \varphi : W|_V \to V \times \mathbb{C}^* \) of the \( \mathbb{C}^* \)-bundle \( W \to C \) and \( \varphi^{\otimes m} : \tilde{L}^m|_V \to V \times \mathbb{C}^{\otimes m} \) of the complex line bundle \( \tilde{L}^m = \tilde{L}^{\otimes m} \to C \). Then we obtain
\[
q_* (\mathcal{O}_W) (V) = (\varphi \circ \varphi)_* (\mathcal{O}_W) (V) = \varphi_* \mathcal{O}_W (V \times \mathbb{C}^*) \\
\cong \mathcal{O}_{V \times \mathbb{C}^*} (V \times \mathbb{C}^*) \cong \mathcal{O}_C (V) \otimes_{\mathbb{C}} \mathcal{O}_C (\mathbb{C}^*) \\
\cong \mathcal{O}_C (V) \otimes_{\mathbb{C}} \mathbb{C}^m, \ t^{-m} \cong \bigoplus_{m \in \mathbb{Z}} \mathcal{O}_C (V) \cdot t^{-m}.
\]

This implies that any section of \( q_* (\mathcal{O}_W) \) can be locally uniquely represented as a finite sum of the form \( \sum f_m \cdot t^{-m} \) with \( f_m \in \mathcal{O}_C (V) \). Using the induced local trivialisations of \( W \) and \( \tilde{L}^{\otimes m} \) over \( V \) together with the identifications \( \mathbb{C}^{\otimes m} \cong \mathbb{C} \) and \( \mathbb{C}^{\otimes (-1)} \cong \text{Hom}(\mathbb{C}, \mathbb{C}) \cong \mathbb{C} \) we can construct a bijection between sections in \( \tilde{L}^m \) over \( V \) and functions in \( \mathcal{O}_C (V) \). We obtain a family of isomorphisms
\[
(q_* (\mathcal{O}_W) (V) \to \bigoplus_{m \in \mathbb{Z}} \mathcal{O}_C (V) \cdot t^{-m}),
\]

which does not depend on the chosen trivialisations, is compatible with the restriction maps and hence induces an isomorphism of \( \mathcal{O}_C \)-algebras \( q_* (\mathcal{O}_W) \) and \( \bigoplus_{m \in \mathbb{Z}} \mathcal{O}_C (\tilde{L}^m) \). \( \square \)

Now we prove lemma 5:

**Proof.** We have to prove that \( \Omega^2 (W) \cong q^* (\Omega^1_C) \). To this end we consider the sheaf of relative forms \( \Omega^1_{W|C} \). This sheaf is trivial and generated by a relative form given in local coordinates by \( \not\exists \). The following short exact sequence of locally free sheaves of ranks 1, 2, and 1
\[
0 \to q^* (\Omega^1_C) \to \Omega^1_W \to \Omega^1_{W|C} \to 0
\]
implies
\[
\Lambda^2 (\Omega^1_W) \cong \Lambda^1 (q^* (\Omega^1_C)) \otimes \Lambda^1 (\Omega^1_{W|C}) \cong q^* (\Omega^1_C). \ \square
\]

Finally we prove lemma 6:

**Proof.** We have to prove that if the bundle \( \tilde{L} \) is non-trivial and the sheaf \( \Omega^2, (W) \) is trivial, then the nowhere vanishing section in \( \Omega^2, (W) \) is unique up to complex multiples. Consider two nowhere vanishing sections in \( \Omega^2, (W) \). There quotient is a
nowhere vanishing regular function. It remains to prove that all nowhere vanishing regular functions on \( W \) are constant. Using lemma 4 we obtain

\[
\mathcal{O}_W(W) \cong q_* (\mathcal{O}_W)(C) \cong \bigoplus_{m \in \mathbb{Z}} \mathcal{O}_C (\mathring{L}^m)(C).
\]

Nowhere vanishing functions on \( W \) correspond to nowhere vanishing sections in \( \mathring{L}^m \). A non-homogeneous section in \( \mathring{L}^m \) can not be nowhere vanishing. A homogeneous nowhere vanishing section in \( \mathring{L}^m \) exists if and only if \( \mathring{L}^m \) is trivial. But the bundle \( \mathring{L} \) is negative, hence \( \mathring{L}^m \) is trivial if and only if \( m = 0 \). \( \square \)

5. Automorphy Factors of \( \mathbb{Q} \)-Gorenstein Quasi-Homogeneous Surface Singularities

In this section we use the results of section 4 to prove Theorem 3:

**Theorem.** A quasi-homogeneous surface singularity is \( \mathbb{Q} \)-Gorenstein of index \( r \) if and only if for the corresponding automorphy factor \( (U, \Gamma, L) \) there is an integer \( m \) (called the level or the exponent of the automorphy factor) without common divisors with \( r \) and a \( \Gamma \)-equivariant isomorphism \( \mathring{L}^m \cong T_U^r \).

**Proof.** We first assume that for some positive integer \( r \) and integer \( m \) there is a \( \Gamma \)-equivariant isomorphism \( \mathring{L}^m \cong T_U^r \). This isomorphism induces a \( (\Gamma / \Gamma') \)-equivariant automorphy factor \( (U, \Gamma', L') \). Then according to proposition 7 there exist global nowhere vanishing \( (\Gamma / \Gamma') \)-invariant sections in \( \Omega^{2,r}(W) \). Such a section induces a nowhere vanishing section \( \Omega^{2,r}(W/\Gamma / \Gamma') = \Omega^{2,r}(X^*) \), hence the corresponding singularity \( (X(U, \Gamma, L), o) \) is \( \mathbb{Q} \)-Gorenstein.

Now let us assume that singularity \( (X, x) \) with automorphy factor \( (U, \Gamma, L) \) is \( \mathbb{Q} \)-Gorenstein of index \( r \), i.e. there exist nowhere vanishing sections in \( \Omega^{2,r}(W/\Gamma / \Gamma') \). We consider the singularity \( (X, \hat{x}) \), which corresponds to the automorphy factor \( (U, \Gamma', L) \). For this singularity we have \( X \cong \hat{X} / (\Gamma / \Gamma') \). The pull-back of a nowhere vanishing section in \( \Omega^{2,r}(X^*) \) along the unramified covering \( \hat{X}^* \to X^* \) is a nowhere vanishing \( (\Gamma / \Gamma') \)-invariant section in \( \Omega^{2,r}(X^*) \), hence the singularity \( (\hat{X}, \hat{x}) \) is also \( \mathbb{Q} \)-Gorenstein of index \( r \).

A nowhere vanishing \( (\Gamma / \Gamma') \)-invariant section in \( \Omega^{2,r}(X^*) \) induces a nowhere vanishing \( (\Gamma / \Gamma') \)-invariant section in \( \Omega^{2,r}(W) = \Omega^{2,r}(\mathring{L}^m) \). Then proposition 7 implies the existence of an \( (\Gamma / \Gamma') \)-equivariant isomorphism \( \mathring{L}^m \cong T_U^r \) for some integer \( m \), i.e. the induced action of \( (\Gamma / \Gamma') \) on \( \mathring{L}^m \cong T_U^r \) coincides with the canonical action of \( (\Gamma / \Gamma') \) on \( T_U^r \). Hence the action of \( \Gamma \) on \( \mathring{L}^m \cong T_U^r \) also coincides with the canonical action of \( \Gamma \) on \( T_U^r \), i.e. there exists a \( \Gamma \)-equivariant isomorphism \( \mathring{L}^m \cong T_U^r \). \( \square \)

**Remark.** Theorem 3 also follows from the following result of K. Watanabe [Wat81], appearing in the context of the theory of commutative rings: Let \( R = R(X, D) \) be a normal graded ring, which is presented by the Pinkham-Demazure method. Then the canonical module \( K_R \) of \( R \) is \( \mathbb{Q} \)-Cartier of index \( r \), if and only if, there exists a rational function \( \phi \) on \( X \) such that \( r(K + D') = mD = \text{div}_X(\phi) \) for some integer \( m \) and \( r \) is the minimum of such \( r \). In our case, \( X = C = U / \Gamma' \) and \( \pi : U \to C \) is the Galois cover associated to the automorphy factor, and the result is translated to the relation on \( U \) as equivariant isomorphism \( T_U^{-r} \cong \mathring{L}^m \). Our proofs of Proposition 7 and Theorem 3 give a more direct description of the automorphy factors in question.
6. From Hyperbolic Automorphy Factors to Biquotients

In this section we prove Theorem 1:

Theorem. The link space of a hyperbolic \( \mathbb{Q} \)-Gorenstein quasi-homogeneous surface singularity of level \( m \) and index \( r \) is diffeomorphic to a quotient

\[
\tilde{\Gamma}_1 \backslash \tilde{G} / \tilde{\Gamma}_2,
\]

where \( \tilde{G} \) is the universal cover \( \tilde{\mathrm{PSL}}(2, \mathbb{R}) \) of the 3-dimensional Lie group \( \mathrm{PSL}(2, \mathbb{R}) \), while \( \tilde{\Gamma}_1 \) and \( \tilde{\Gamma}_2 \) are discrete subgroups of level \( m \) in \( \tilde{G} \), \( \tilde{\Gamma}_1 \) is cocompact, and the image of \( \tilde{\Gamma}_2 \) in \( \mathrm{PSL}(2, \mathbb{R}) \) is a cyclic subgroup of order \( r \). Conversely, any biquotient as above is diffeomorphic to the link space of a quasi-homogeneous hyperbolic \( \mathbb{Q} \)-Gorenstein singularity.

Before we explain the proof of this theorem, we give a description of the Lie group \( G = \mathrm{PSL}(2, \mathbb{R}) \) and its coverings.

As topological space \( \mathrm{PSL}(2, \mathbb{R}) \) is homeomorphic to the solid torus \( S^1 \times \mathbb{C} \). The fundamental group of the solid torus \( G \) is infinite cyclic. Therefore, for each natural number \( m \) there is a unique connected \( m \)-fold covering

\[
G_m = \tilde{G} / (m \cdot Z(\tilde{G}))
\]

of \( G \), where \( Z(\tilde{G}) \) is the central subgroup of \( \tilde{G} \). For \( m = 2 \) this is the group \( G_2 = \mathrm{SL}(2, \mathbb{R}) \).

We use the following description of the covering groups \( G_m \) of \( G = \mathrm{PSL}(2, \mathbb{R}) \), which fixes a group structure. Let \( \mathrm{Hol}(H, \mathbb{C}^*) \) be the set of all holomorphic functions \( H \to \mathbb{C}^* \).

Proposition 8. The \( m \)-fold covering group \( G_m \) of \( G \) can be described as

\[
\{(g, \delta) \in G \times \mathrm{Hol}(H, \mathbb{C}^*) \mid \delta^m(z) = g'(z) \text{ for all } z \in H \}
\]

with multiplication

\[
(g_2, \delta_2) \cdot (g_1, \delta_1) = (g_2 \cdot g_1, (\delta_2 \circ g_1) \cdot \delta_1).
\]

Remark. This description of \( G_m \) and the description of \( \tilde{G} \) that we give later are inspired by the notion of automorphic differential forms of fractional degree, introduced by J. Milnor in [Mil75]. For a more detailed discussion of this fact see [LV80], section 1.8.

We now explain the connection between automorphy factors in question and lifts of Fuchsian groups into the finite coverings of \( \tilde{G} \).

Definition. A lift of the Fuchsian group \( \Gamma \) into \( G_m \) is a subgroup \( \Gamma^* \) of \( G_m \) such that the restriction of the covering map \( G_m \to \Gamma \) to \( \Gamma^* \) is an isomorphism between \( \Gamma^* \) and \( \Gamma \).

Proposition 9. There is a 1-1-correspondence between hyperbolic \( \mathbb{Q} \)-Gorenstein automorphy factors \((H, \Gamma, H \times \mathbb{C})\) of level \( m \) and index \( r \) and the lifts of \( \Gamma \) into \( G_m \).

Proof. On the one hand using the description of the covering \( G_m \) from Proposition 10 we see that there is a 1-1-correspondence between lifts of \( \Gamma \) into \( G_m \) and families \( \{\delta_g\}_{g \in \Gamma} \) of holomorphic functions \( \delta_g : H \to \mathbb{C}^* \) such that for any \( g \in \Gamma \)

\[
\delta_g^m = g'
\]
and for any \( g_1, g_2 \in \Gamma \)

\[
\delta_{g_2 \cdot g_1} = (\delta_{g_2} \circ g_1) \cdot \delta_{g_1}.
\]

Let \( \mathcal{D} \) be the set of all such families \( \delta_g \).

On the other hand there is a 1-1-correspondence between hyperbolic \( \mathbb{Q} \)-Gorenstein automorphy factors \( (H, \Gamma, H \times \mathbb{C}) \) of level \( m \) and index \( r \) and and families \( \{e_g\}_{g \in \Gamma} \) of holomorphic functions \( e_g : H \to \mathbb{C}^* \) such that for any \( g \in \Gamma \)

\[
e_m^g = (g')^r
\]

and for any \( g_1, g_2 \in \Gamma \)

\[
e_{g_2 \cdot g_1} = (e_{g_2} \circ g_1) \cdot e_{g_1}.
\]

Let \( \mathcal{E} \) be the set of all such families \( \{e_g\}_{g \in \Gamma} \).

It remains to establish a 1-1-correspondence between the sets \( \mathcal{D} \) and \( \mathcal{E} \). This correspondence is defined as follows: Let us assign to a family \( \delta_g \in \mathcal{D} \) the family \( \{e_g\} \) given by \( e_g := \delta_g^r \). One checks easily that \( \{e_g\} \in \mathcal{E} \).

If the images \( (e_g), (\tilde{e}_g) \in \mathcal{E} \) of \( \delta_g \), \( \tilde{\delta}_g \) \( \in \mathcal{D} \) coincide then on the one hand

\[
\left( \frac{\tilde{\delta}_g}{\delta_g} \right)^r = \frac{\tilde{\delta}_g^r}{\delta_g^r} = \frac{\tilde{e}_g}{e_g} = 1,
\]

on the other hand

\[
\left( \frac{\tilde{\delta}_g}{\delta_g} \right)^m = \frac{\tilde{\delta}_g^m}{\delta_g^m} = \frac{\tilde{g}}{g} = 1.
\]

But the integers \( m \) and \( r \) are relatively prime, hence there exists only one complex number \( \xi \) with the property \( \xi^m = \xi^r = 1 \), namely \( \xi = 1 \). Hence for any \( g \in \Gamma \)

\[
\frac{\tilde{\delta}_g}{\delta_g} \equiv 1,
\]

i.e. the families \( \delta_g \) and \( \tilde{\delta}_g \) coincide. So we have shown that the mapping \( \mathcal{D} \to \mathcal{E} \) is injective.

Now let us consider a family \( \{e_g\} \in \mathcal{E} \). It holds \( g'(z) \notin \mathbb{R}_+ \cup \{0\} \) for all \( z \in H \), hence there exist functions \( \rho_g : H \to \mathbb{R}_+ \) and \( \varphi_g : H \to (0,1) \) such that \( g' = \rho_g \cdot \exp(2\pi i \varphi_g) \). The chain rule implies

\[
\rho_{g_2 \cdot g_1} = (\rho_{g_2} \circ g_1) \cdot \rho_g,
\]

and

\[
\varphi_{g_2 \cdot g_1} = \varphi_{g_2} \circ g_1 - \varphi_{g_2} \in \mathbb{Z}.
\]

The function \( e_g \) is then of the form

\[
e_g = \frac{\tilde{e}_g}{\rho_g} \cdot \exp \left( 2\pi i \cdot \frac{r \cdot \varphi_g + k_g}{m} \right)
\]

for some function \( k_g : H \to \mathbb{Z} \). The function \( k_g \) is continuous and hence constant.

The integers \( m \) and \( r \) are relatively prime, hence there is an integer \( n_g \) such that \( r \cdot n_g \equiv k_g \mod m \). Let us define a family \( \delta_g \) by setting

\[
\delta_g = \rho_g^{\frac{1}{r}} \cdot \exp \left( 2\pi i \cdot \frac{\varphi_g + n_g}{m} \right).
\]

We now prove that the family \( \delta_g \) is in \( \mathcal{D} \). The first property

\[
\delta_g^m = \rho_g \cdot \exp \left( 2\pi i \cdot (\varphi_g + n_g) \right) = \rho_g \cdot \exp (2\pi i \cdot \varphi_g) = g'
\]

is satisfied. The second property
\[ \delta_{g_2, g_1} = (\delta_{g_2} \circ g_1) \cdot \delta_{g_1} \]
is equivalent to
\[ \rho_{g_2, g_1} = (\rho_{g_2} \circ g_1) \cdot \rho_{g_1} \]
and
\[ (\varphi_{g_2, g_1} - \varphi_{g_2} \circ g_1 - \varphi_{g_1}) + (n_{g_2, g_1} - n_{g_2} \circ g_1 - n_{g_1}) \equiv 0 \mod m. \]
The first of these equations follows from
\[ \rho_{g_2, g_1} = (\rho_{g_2} \circ g_1) \cdot \rho_{g_1}. \]
To prove the second equations we observe that
\[ e_{g_2, g_1} = (e_{g_2} \circ g_1) \cdot e_{g_1} \]
implies that \( m \) is a divisor of the integer
\[ r \cdot (\varphi_{g_2, g_1} - \varphi_{g_2} \circ g_1 - \varphi_{g_1}) + (k_{g_2, g_1} - k_{g_2} \circ g_1 - k_{g_1}). \]
Because of \( r \cdot n_g \equiv k_g \mod m \) also the integer
\[ r \cdot \left( (\varphi_{g_2, g_1} - \varphi_{g_2} \circ g_1 - \varphi_{g_1}) + (n_{g_2, g_1} - n_{g_2} \circ g_1 - n_{g_1}) \right) \]
is divisible by \( m \). Since \( m \) and \( r \) are relatively prime, the number \( m \) must be a
divisor of the integer
\[ (\varphi_{g_2, g_1} - \varphi_{g_2} \circ g_1 - \varphi_{g_1}) + (n_{g_2, g_1} - n_{g_2} \circ g_1 - n_{g_1}). \]
So the family \( (\delta_g) \) is in \( \mathcal{D} \). The image of the family \( (\delta_g) \) under the map \( \mathcal{D} \to \mathcal{E} \) is
\[ \delta_g^e = \rho_{\mathcal{D}} \cdot \exp \left( 2\pi i \cdot \frac{r \cdot \varphi_g + r \cdot n_g}{m} \right) = \rho_{\mathcal{D}} \cdot \exp \left( 2\pi i \cdot \frac{r \cdot \varphi_g + k_g}{m} \right) = e_g. \]
So we have proved that the mapping \( \mathcal{D} \to \mathcal{E} \) is surjective. \( \square \)

Now we explain the connection between lifts of Fuchsian groups into the finite
coverings of \( \hat{G} \) and discrete subgroup of finite index in \( \hat{G} \).

We use the following description of the covering groups \( \hat{G} \) of \( G = \text{PSL}(2, \mathbb{R}) \),
which fixes a group structure. Let \( \text{Hol}(H, \mathbb{C}) \) be the set of all holomorphic functions
\( H \to \mathbb{C} \).

**Proposition 10.** The universal covering group \( \hat{G} \) of \( G \) can be described as
\[ \{(g, \delta) \in G \times \text{Hol}(H, \mathbb{C}) \mid e \circ \delta = \delta' \}, \]
where \( e(w) = \exp(2\pi i w) \). The multiplication is given by
\[ (g_2, \delta_2) \cdot (g_1, \delta_1) = (g_2 \cdot g_1, \delta_2 \circ g_1 + \delta_1). \]
The covering map \( \hat{G} \to G_m \) is given by
\[ (g, \delta) \mapsto (g, e(\delta/m)). \]

**Remark.** The center of the group \( \hat{G} \) is infinite cyclic and is equal to the preimage
of the unit element in \( \hat{G} \):
\[ Z(\hat{G}) = \{(g, \delta) \in \hat{G} \mid g = \text{Id}, \quad \delta \text{ is an integer constant}\}. \]

**Definition.** The level of a discrete subgroup \( \hat{\Gamma} \subset \hat{G} \) is the index of \( \hat{\Gamma} \cap Z(\hat{G}) \) as a
subgroup of \( Z(\hat{G}) \).
The following fact is well known (see for example section 4 in [KR85]):

**Proposition 11.** There is a one-to-one correspondence between discrete co-compact subgroups of level $m$ in $G$ and liftings of discrete co-compact subgroups in $\text{PSL}(2, \mathbb{R})$ into the $m$-fold covering of $\text{PSL}(2, \mathbb{R})$. The correspondence is given by mapping a subgroup in $\hat{G}$ into its image under the covering map $\hat{G} \to G_m$.

We now prove Theorem 1.

**Proof.** Let $(X, x)$ be a hyperbolic $\mathbb{Q}$-Gorenstein quasi-homogeneous surface singularity of level $m$ and index $r$ and let $(H, \Gamma_1, L)$ be the corresponding automorphism factor. Let us consider a trivialisation $L \cong H \times \mathbb{C}$ of the bundle $L$. Combining the results of Propositions 9 and 11 we see that there is a discrete co-compact subgroup $\hat{\Gamma}_1$ of level $m$ in $\hat{G}$ such that the action of the group $\Gamma_1$ can be described as

$$g \cdot (z, t) = (g(z), e(\delta(z)r/m) \cdot t),$$

where $\delta : H \to \mathbb{C}$ is a holomorphic function such that $(g, \delta)$ is an element of $\hat{\Gamma}_1$. This action of $\hat{\Gamma}_1$ can be obtained as a restriction of the action of the group $\hat{G}$ on $L$ via

$$(g, \delta) \cdot (z, t) = (g(z), e(\delta(z)r/m) \cdot t).$$

It is easy to check, that this is an action of $\hat{G}$. The unit subbundle of $L$ can be identified with the subbundle

$$S = \{(z, t) \in H \times \mathbb{C} \mid |t|^m = (\text{Im}(z))^r \}.$$

The bundle $S$ is invariant under $\hat{G}$: For $(z', t') = (g, \delta) \cdot (z, t) = (g(z), e(\delta(z)r/m) \cdot t)$ we have

$$|t'|^m = |g(z)|^m = |\text{Im}(z)|^r = \left( \frac{\text{Im}(z)}{\text{Im}(z')} \right)^r = \left( \frac{\text{Im}(z)}{\text{Im}(z')} \right)^r.$$

The stabiliser of a point $(z_0, t_0) \in S$ is

$$\hat{\Gamma}_2 := \text{Stab}_{\hat{G}}((z_0, t_0)) = \{ (g, \delta) \in \hat{G} \mid g(z_0) = z_0, \quad \delta(z) \cdot \frac{r}{m} \in \mathbb{Z} \}.$$

We now determine the level of the subgroup $\hat{\Gamma}_2$:

$$\hat{\Gamma}_2 \cap Z(\hat{G}) = \{(g, \delta) \in Z(\hat{G}) \mid \delta \text{ is an integer constant divisible by } m \}\bigm/ m \cdot Z(\hat{G}).$$

The map $(g, \delta) \mapsto (g, \delta) \cdot (i, 1)$ defines a $\hat{\Gamma}_1$-equivariant diffeomorphism $\hat{G}/\hat{\Gamma}_2 \to S$. Here $\hat{\Gamma}_1$ acts on $\hat{G}$ by left multiplication. We obtain the following commutative diagram

$$\begin{array}{ccc}
\hat{G}/\hat{\Gamma}_2 & \longrightarrow & L^* / \mathbb{R}_+ \cong S \\
\downarrow & & \downarrow \\
\hat{\Gamma}_1 \backslash \hat{G}/\hat{\Gamma}_2 & \longrightarrow & X^* / \mathbb{R}_+ \cong M 
\end{array}$$

Hence we have

$$M \cong \hat{\Gamma}_1 \backslash \hat{G}/\hat{\Gamma}_2.$$

Conversely, let $\hat{\Gamma}_1$ and $\hat{\Gamma}_2$ be discrete subgroups of level $m$ in $\hat{G}$, let $\hat{\Gamma}_1$ be co-compact, and let the image of $\hat{\Gamma}_2$ in $\text{PSL}(2, \mathbb{R})$ be a cyclic subgroup of order $r$. 

Then $\Gamma_1 = \hat{\Gamma}_1/(\hat{\Gamma}_1 \cap Z(\hat{G}))$ is a discrete co-compact subgroup of $\text{PSL}(2, \mathbb{R})$. We can define an automorphy factor $(H, \Gamma_1, H \times \mathbb{C})$ by setting

$$g \cdot (z, t) = (g(z), e(\delta(z)r/m) \cdot t),$$

where $\delta : H \to \mathbb{C}$ is a holomorphic function such that $(g, \delta)$ is an element of $\hat{\Gamma}_1$. From the first part of the proof we know that the link of the corresponding quasihomogeneous $\mathbb{Q}$-Gorenstein surface singularity is diffeomorphic to $\hat{\Gamma}_1 \backslash \hat{G}/\hat{\Gamma}_2$. \hfill \Box

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