ON THE LINK SPACE OF A $\mathbb{Q}$-GORENSTEIN QUASI-HOMOGENEOUS SURFACE SINGULARITY

ANNA PRATOUSSEVITCH

ABSTRACT. In this paper we prove the following theorem: Let M be the link space of a quasi-homogeneous hyperbolic $\mathbb{Q}$-Gorenstein surface singularity. Then M is diffeomorphic to a coset space $\tilde{\Gamma}_1\backslash\tilde{G}/\tilde{\Gamma}_2$, where $\tilde{G}$ is the 3-dimensional Lie group $\widetilde{\mathrm{PSL}}(2,\mathbb{R})$, while $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ are discrete subgroups of G, $\tilde{\Gamma}_1$ is co-compact and $\tilde{\Gamma}_2$ is cyclic. Conversely, if M is diffeomorphic to a coset space as above, then M is diffeomorphic to the link space of a quasi-homogeneous hyperbolic $\mathbb{Q}$-Gorenstein singularity. We also prove the following characterisation of quasi-homogeneous $\mathbb{Q}$-Gorenstein surface singularities: A quasi-homogeneous surface singularity is $\mathbb{Q}$-Gorenstein of index r if and only if for the corresponding automorphy factor (U,Γ,L) some tensor power of the complex line bundle L is Γ-equivariantly isomorphic to some tensor power of the tangent bundle of the Riemannian surface U.

1. INTRODUCTION

Graded affine coordinate rings of quasi-homogeneous surface singularities can be identified with graded rings of generalised automorphic forms. The description in terms of automorphy factors was found in 1975–77 by Dolgachev, Milnor, Neumann and Pinkham [Dol75, Dol77, Mil75, Neu77, Pin77].

For some special classes of quasi-homogeneous surface singularities as Gorenstein and $\mathbb{Q}$-Gorenstein singularities one can obtain more precise descriptions of the corresponding automorphy factors.

In Theorem 3 we obtain a characterisation of hyperbolic and spherical $\mathbb{Q}$-Gorenstein quasi-homogeneous surface singularities in terms of their automorphy factors. This characterisation leads to a description of their links as quotients of certain 3-dimensional Lie groups by discrete subgroups. More precisely, we prove the following statement

Theorem 1. *The link space of a hyperbolic $\mathbb{Q}$-Gorenstein quasi-homogeneous surface singularity of level m and index r is diffeomorphic to a biquotient*

$$\tilde{\Gamma}_1\backslash\tilde{G}/\tilde{\Gamma}_2,$$

where $\tilde{G}$ is the universal cover $\widetilde{\mathrm{PSL}}(2,\mathbb{R})$ of the 3-dimensional Lie group $\mathrm{PSL}(2,\mathbb{R})$, while $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ are discrete subgroups of level m in $\tilde{G}$, $\tilde{\Gamma}_1$ is co-compact, and the image of $\tilde{\Gamma}_2$ in $\mathrm{PSL}(2,\mathbb{R})$ is a cyclic subgroup of order r. Conversely, any biquotient as above is diffeomorphic to the link space of a quasi-homogeneous hyperbolic $\mathbb{Q}$-Gorenstein singularity.

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These statements are generalisations of the results of Dolgachev [Dol83] on Gorenstein quasi-homogeneous surface singularities. The Gorenstein quasi-homogeneous surface singularities correspond to the case of a trivial group $\tilde{\Gamma}_2$.

Similar statements are also true in the case of Euclidean automorphy factors and corresponding singularities. This case was already discussed by Dolgachev in [Dol83]. The Euclidean $\mathbb{Q}$-Gorenstein quasi-homogeneous surface singularities are rational singularities, which are quotients of Gorenstein Euclidean singularities by actions of finite groups.

The description of the link space of a hyperbolic Gorenstein quasi-homogeneous surface singularity as a quotient of the Lie group $\widetilde{\mathrm{PSL}}(2,\mathbb{R})$ by the action of a discrete subgroup was the motivation for the study in [Pra03], [BPR03] of a certain construction of fundamental domains for such actions. This construction leads to interesting results on the combinatorial geometry of the link spaces of Gorenstein quasi-homogeneous surface singularities.

We expect that our construction of fundamental domains can be generalised in order to study the combinatorial geometry of the link spaces of $\mathbb{Q}$-Gorenstein quasi-homogeneous surface singularities. We shall discuss the combinatorial geometry of the link spaces in the $\mathbb{Q}$-Gorenstein case in an ongoing paper.

The paper is organised as follows: In section 2 we explain the description of quasi-homogeneous surface singularities via automorphy factors. In section 3 we define $\mathbb{Q}$-Gorenstein quasi-homogeneous surface singularities and introduce our characterisation of the corresponding automorphy factors (Theorem 3). Then in section 4 we prove some technical results needed to prove this characterisation. After that we prove Theorem 3 in section 5. Finally we prove Theorem 1 in section 6.

Notation: In this paper we use $\mathbb{R}_+$ for $\{x \in \mathbb{R} \mid x > 0\}$. We denote by L^* the associated $\mathbb{C}^*$-bundle of a complex line bundle L, while $L^\vee$ is the dual bundle of L.

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2. Automorphy Factors

In this section we recall the results of Dolgachev, Milnor, Neumann and Pinkham [Dol75, Dol77, Mil75, Neu77, Pin77] on the graded affine coordinate rings, which correspond to quasi-homogeneous surface singularities.

Definition. *A (negative unramified) automorphy factor* (U, Γ, L) is a complex line bundle L over a simply connected Riemann surface U together with a discrete cocompact subgroup $\Gamma \subset \mathrm{Aut}(U)$ acting compatibly on U and on the line bundle L, such that the following two conditions are satisfied:

1) The action of Γ is free on L^*, the complement of the zero-section in L.
2) Let $\Gamma' \triangleleft \Gamma$ be a normal subgroup of finite index, which acts freely on U, and let $\bar{L} \to C$ be the complex line bundle $\bar{L} = L/\Gamma'$ over the compact Riemann surface $C = U/\Gamma'$. Then $\bar{L}$ is a negative line bundle.

A simply connected Riemann surface U can be $\mathbb{C}\mathrm{P}^1$, $\mathbb{C}$, or H, the real hyperbolic plane. We call the corresponding automorphy factor and the corresponding singularity *spherical*, *Euclidean*, resp. *hyperbolic*.

Remark. There always exists a normal freely acting subgroup of Γ of finite index. In the hyperbolic case the existence follows from the theorem of Fox-Bundgaard-Nielsen. If the second assumption in the last definition holds for some normal freely acting subgroup of finite index, then it holds for any such subgroup.

The simplest examples of such a complex line bundle with group action are the cotangent bundle of the complex projective line $U = \mathbb{C}\mathrm{P}^1$ and the tangent bundle of the hyperbolic plane $U = H$ equipped with the canonical action of a subgroup $\Gamma \subset \mathrm{Aut}(U)$.

Let (U, Γ, L) be a negative unramified automorphy factor. Since the bundle $\bar{L} = L/\Gamma'$ is negative, one can contract the zero section of $\bar{L}$ to get a complex surface with one isolated singularity corresponding to the zero section. There is a canonical action of the group Γ/Γ' on this surface. The quotient is a complex surface $X(U, \Gamma, L)$ with an isolated singular point o, which depends only on the automorphy factor (U, Γ, L).

The following theorem summarises the results of Dolgachev, Milnor, Neumann, and Pinkham:

Theorem 2. *The surface $X(U, \Gamma, L)$ associated to a negative unramified automorphy factor (U, Γ, L) is a quasi-homogeneous affine algebraic surface with a normal isolated singularity. Its affine coordinate ring is the graded $\mathbb{C}$-algebra of generalised Γ-invariant automorphic forms*

$$A = \bigoplus_{m \geqslant 0} H^0(U, L^{-m})^\Gamma .$$

All normal isolated quasi-homogeneous surface singularities (X, x) are obtained in this way, and the automorphy factors with $(X(U, \Gamma, L), o)$ isomorphic to (X, x) are uniquely determined by (X, x) up to isomorphism.

3. $\mathbb{Q}$-Gorenstein Quasi-Homogeneous Surface Singularities

In this section we recall the definition of $\mathbb{Q}$-Gorenstein singularities and the characterisation of the corresponding automorphy factors.

A normal isolated singularity of dimension n is Gorenstein if and only if there is a nowhere vanishing n-form on a punctured neighbourhood of the singular point. For example all isolated singularities of complete intersections are Gorenstein.

A natural generalisation of Gorenstein singularities are the $\mathbb{Q}$-Gorenstein singularities (compare [Rei87, Ish87, Ish00]). A normal isolated singularity of dimension at least 2 is *$\mathbb{Q}$-Gorenstein* if there is a natural number r such that the divisor $r \cdot \mathcal{K}_X$ is defined on a punctured neighbourhood of the singular point by a function. Here $\mathcal{K}_X$ is the canonical divisor of X. The smallest such number r is called the *index* of the singularity. A normal isolated surface singularity is Gorenstein if and only if it is $\mathbb{Q}$-Gorenstein of index 1.

In section 5 we prove the following characterisation of $\mathbb{Q}$-Gorenstein quasi-homogeneous surface singularities in terms of the corresponding automorphy factors:

Theorem 3. *A quasi-homogeneous surface singularity is $\mathbb{Q}$-Gorenstein of index r if and only if for the corresponding automorphy factor (U, Γ, L) there is an integer m (called the level or the exponent of the automorphy factor) without common divisors with r and a Γ-invariant isomorphism $L^m \cong T_U^r$.*

Let (U, Γ, L) be a negative unramified automorphy factor of level m and index r, which corresponds to a $\mathbb{Q}$-Gorenstein singularity. The isomorphism $L^m \cong T_U^r$ induces an isomorphism $\bar{L}^m \cong T_C^r$. The bundle $\bar{L}$ is negative. A simple computation with Chern numbers shows that the possible values of the exponent are $m = -1$ or $m = -2$ for $U = \mathbb{C}\mathrm{P}^1$, whereas $m = 0$ for $U = \mathbb{C}$ and $m \in \mathbb{N}$ for $U = H$.

4. The associated bundle of the quotient bundle

Let (U, Γ, L) be a spherical or hyperbolic negative unramified automorphy factor. As in the definition let $\Gamma' \triangleleft \Gamma$ be a normal subgroup of Γ acting freely on U, and let $p : \bar{L} \to C$ be the complex line bundle with total space $\bar{L} = L/\Gamma'$ and base $C = U/\Gamma'$. In this section we consider the associated $\mathbb{C}^*$-bundle of the bundle p, i.e. $p|_{\bar{L}^*} : \bar{L}^* \to C$. For ease of notation we set $W := \bar{L}^*$ and $q := p|_W$. We first present some technical lemmas, which will be used later to determine $\Omega^{2,r}(W) := (\Omega^2(W))^{\otimes r}$.

Lemma 4. *The following $\mathcal{O}_C$-algebras are isomorphic*

$$q_*(\mathcal{O}_W) \cong \bigoplus_{m \in \mathbb{Z}} \mathcal{O}_C(\bar{L}^m).$$

Lemma 5. *We have $\Omega^2(W) \cong q^*(\Omega^1_C)$.*

Lemma 6. *If the bundle $\bar{L}$ is non-trivial and the sheaf $\Omega^{2,r}(W)$ is trivial, then there exists up to complex multiples only one nowhere vanishing section in $\Omega^{2,r}(W)$.*

We postpone the proofs of these lemmas until the end of this section and discuss first the main result of the section, the description of (Γ/Γ')-invariant sections in $\Omega^{2,r}(W)$.

Proposition 7. *The sheaf $\Omega^{2,r}(W)$ is trivial if and only if there exists an integer m and an isomorphism $\bar{L}^m \cong T_C^r$.*

Assume that $\Omega^{2,r}(W)$ is trivial and let m be the integer such that $\bar{L}^m \cong T_C^r$. Then the global nowhere vanishing sections in $\Omega^{2,r}(W)$ are (Γ/Γ')-invariant if and only if the isomorphism $\bar{L}^m \cong T_C^r$ is (Γ/Γ')-equivariant.

Proof. 1) We first prove that $\Omega^{2,r}(W) \cong \mathcal{O}_W$ implies $\bar{L}^m \cong T_C^r$ for some $m \in \mathbb{Z}$.

Assume that $\Omega^{2,r}(W) \cong \mathcal{O}_W$. This implies on the one hand using lemma 4 that

$$q_*(\Omega^{2,r}(W)) \cong q_*(\mathcal{O}_W) \cong \bigoplus_{i \in \mathbb{Z}} \mathcal{O}_C(\bar{L}^i).$$

On the other hand we have using lemma 5

$$q_*(\Omega^{2,r}(W)) \cong q_*((q^*(\Omega^1_C))^{\otimes r}).$$

Now we obtain for $\Omega^{1,r}_C := (\Omega^1_C)^{\otimes r}$

$$q_*((q^*(\Omega^1_C))^{\otimes r}) \cong q_*(q^*(\Omega^{1,r}_C)),$$

because q^* is compatible with tensor products. The projection formula implies

$$q_*(q^*(\Omega^{1,r}_C)) \cong q_*(\mathcal{O}_W \otimes_{\mathcal{O}_W} q^*(\Omega^{1,r}_C)) \cong q_*(\mathcal{O}_W) \otimes_{\mathcal{O}_C} \Omega^{1,r}_C.$$

Finally using lemma 4 again we obtain

$$\begin{aligned} q_*(\Omega^{2,r}(W)) &\cong q_*(\mathcal{O}_W) \otimes_{\mathcal{O}_C} \Omega_C^{1,r} \\ &\cong \Big(\bigoplus_{m\in\mathbb{Z}} \mathcal{O}_C(\bar{L}^m)\Big) \otimes_{\mathcal{O}_C} \Omega_C^{1,r} \\ &\cong \bigoplus_{m\in\mathbb{Z}} \mathcal{O}_C(\bar{L}^m \otimes T_C^{-r}). \end{aligned}$$

Comparing both equations for $q_*(\Omega^{2,r}(W))$ we obtain

$$\bigoplus_{i\in\mathbb{Z}} \mathcal{O}_C(\bar{L}^i) \cong \bigoplus_{m\in\mathbb{Z}} \mathcal{O}_C(\bar{L}^m \otimes T_C^{-r}),$$

hence

$$\mathcal{O}_C(\bar{L}^m \otimes T_C^{-r}) \cong \mathcal{O}_C(\bar{L}^0) \cong \mathcal{O}_C$$

for some $m \in \mathbb{Z}$. This implies $\bar{L}^m \cong T_C^r$.

2) We now assume that $\bar{L} \cong T_C^r$, i.e. $m = 1$. The Riemann surface U is then the real hyperbolic plane H. We study the tangent bundle T_C of the hyperbolic surface $C = H/\Gamma'$. We define a 2-form on $(T_C)^*$ in local coordinates by $\eta = \frac{1}{t^2}\cdot(dz\wedge dt)$. Using the fact that any change of coordinates is of the form $(z,t) \mapsto (\varphi(z), \varphi'(z)\cdot t)$ we can verify that this local definition gives rise to a global nowhere vanishing 2-form on $(T_C)^*$, and that this 2-form is invariant under an action of $g \in \Gamma$ if and only if the action is given by $(z,t) \mapsto (g(z), g'(z)\cdot t)$, i.e. the action coincides with the canonical action of g on T_C. The 2-form on $(T_C)^*$ induces a nowhere vanishing section η in $\Omega^{2,r}((T_C^r)^*)$, which is invariant under an action of $g \in \Gamma$ if and only if the action is given in local coordinates by $(z,t) \mapsto (g(z), (g'(z))^r \cdot t)$, i.e. the action coincides with the canonical action of g on T_C^r. Hence if the isomorphism $\bar{L} \cong T_C^r$ is (Γ/Γ')-equivariant, there exists a (Γ/Γ')-invariant nowhere vanishing section in $\Omega^{2,r}(W)$.

3) We now assume that $\bar{L} \cong T_C^{-r} \cong (T_C^\vee)^r$, i.e. $m = -1$. The Riemann surface U is then the complex projective line $\mathbb{C}\mathrm{P}^1$. We study the cotangent bundle $T_C^\vee$ of the surface $C = \mathbb{C}\mathrm{P}^1/\Gamma'$. We define a 2-form on $(T_C^\vee)^*$ in local coordinates by $\eta = dz \wedge dt$. Using the fact that any change of coordinates is of the form $(z,t) \mapsto (\varphi(z), \frac{1}{\varphi'(z)}\cdot t)$ we can verify that this local definition gives rise to a global nowhere vanishing 2-form on $(T_C^\vee)^*$. We continue in the proof as for $m = 1$ and obtain a nowhere vanishing section η in $\Omega^{2,r}((T_C^{-r})^*)$, which is invariant under an action of $g \in \Gamma$ if and only if the action coincides with the canonical action of g on $(T_C^\vee)^r$. Hence if the isomorphism $\bar{L} \cong T_C^{-r}$ is (Γ/Γ')-equivariant, there exists a (Γ/Γ')-invariant nowhere vanishing section in $\Omega^{2,r}(W)$.

4) As the next step of the proof we consider the case $\bar{L}^m \cong T_C^r$ with $m \neq 0$. Let η be the nowhere vanishing section in $\Omega^{2,r}((T_C^{\pm r})^*)$, i.e. in $\Omega^{2,r}((\bar{L}^{|m|})^*)$, constructed in subsections 2 and 3. We consider the covering $\tau : \bar{L} \to \bar{L}^m$. The pull-back $\tau^*(\eta)$ of the section η under the covering τ is a nowhere vanishing section in $\Omega^{2,r}(\bar{L}^*) = \Omega^{2,r}(W)$. If the isomorphism $\bar{L}^m \cong T_C^r$ is (Γ/Γ')-equivariant, the induced section in $\Omega^{2,r}(W)$ is (Γ/Γ')-invariant.

5) We now assume that $\Omega^{2,r}(W)$ is trivial and that there exists a nowhere vanishing section ω in $\Omega^{2,r}(W)$, which is (Γ/Γ')-invariant. Then in particular there exists an integer m such that $\bar{L}^m \cong T_C^r$. Let η be the nowhere vanishing section in

$\Omega^{2,r}((T_C^{\pm r})^*)$ constructed before. By lemma 6 the sections ω and η are complex multiples of each other, hence the isomorphism $\bar{L}^m \cong T_C^r$ is (Γ/Γ')-equivariant. □

Now it remains to show the technical lemmas, which we have used in the proof of lemma 7. We first prove lemma 4:

Proof. We have to prove that the following $\mathcal{O}_C$-algebras are isomorphic

$$q_*(\mathcal{O}_W) \cong \bigoplus_{m\in\mathbb{Z}} \mathcal{O}_C(\bar{L}^m).$$

Let us consider a local trivialisation $\varphi : \bar{L}|_V \to V \times \mathbb{C}$ of the complex line bundle $\bar{L}$ over an open affine subset $V \subset C$. This trivialisation induces trivialisations $\varphi : W|_V \to V \times \mathbb{C}^*$ of the $\mathbb{C}^*$-bundle $W \to C$ and $\varphi^{\otimes m} : \bar{L}^m|_V \to V \times \mathbb{C}^{\otimes m}$ of the complex line bundle $\bar{L}^m = \bar{L}^{\otimes m} \to C$. Then we obtain

$$\begin{aligned} q_*(\mathcal{O}_W)(V) &= (\mathrm{pr} \circ \varphi)_*(\mathcal{O}_W)(V) = \varphi_*\mathcal{O}_W(V \times \mathbb{C}^*) \\ &\cong \mathcal{O}_{V\times\mathbb{C}^*}(V \times \mathbb{C}^*) \cong \mathcal{O}_C(V) \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^*}(\mathbb{C}^*) \\ &\cong \mathcal{O}_C(V) \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}] \cong \bigoplus_{m\in\mathbb{Z}} \mathcal{O}_C(V) \cdot t^{-m}. \end{aligned}$$

This implies that any section of $q_*(\mathcal{O}_W)$ can be locally uniquely represented as a finite sum of the form $\sum f_m \cdot t^{-m}$ with $f_m \in \mathcal{O}_C(V)$. Using the induced local trivialisations of W and $\bar{L}^{\otimes m}$ over V together with the identifications $\mathbb{C}^{\otimes m} \cong \mathbb{C}$ and $\mathbb{C}^{\otimes(-1)} \cong \mathrm{Hom}(\mathbb{C}, \mathbb{C}) \cong \mathbb{C}$ we can construct a bijection between sections in $\bar{L}^m$ over V and functions in $\mathcal{O}_C(V)$. We obtain a family of isomorphisms

$$(q_*(\mathcal{O}_W)(V) \to \oplus_{m\in\mathbb{Z}}\mathcal{O}_C(V) \cdot t^{-m})_V,$$

which does not depend on the chosen trivialisations, is compatible with the restriction maps and hence induces an isomorphism of $\mathcal{O}_C$-algebras $q_*(\mathcal{O}_W)$ and $\oplus_{m\in\mathbb{Z}}\mathcal{O}_C(\bar{L}^m)$. □

Now we prove lemma 5:

Proof. We have to prove that $\Omega^2(W) \cong q^*(\Omega^1_C)$. To this end we consider the sheaf of relative forms $\Omega^1_{W|C}$. This sheaf is trivial and generated by a relative form given in local coordinates by $\frac{dt}{t}$. The following short exact sequence of locally free sheaves of ranks 1, 2, and 1

$$0 \to q^*(\Omega^1_C) \to \Omega^1_W \to \Omega^1_{W|C} \to 0$$

implies

$$\Lambda^2(\Omega^1_W) \cong \Lambda^1(q^*(\Omega^1_C)) \otimes \Lambda^1(\Omega^1_{W|C}) \cong q^*(\Omega^1_C). \quad \square$$

Finally we prove lemma 6:

Proof. We have to prove that if the bundle $\bar{L}$ is non-trivial and the sheaf $\Omega^{2,r}(W)$ is trivial, then the nowhere vanishing section in $\Omega^{2,r}(W)$ is unique up to complex multiples. Consider two nowhere vanishing sections in $\Omega^{2,r}(W)$. There quotient is a nowhere vanishing regular function. It remains to prove that all nowhere vanishing regular functions on W are constant. Using lemma 4 we obtain

$$\mathcal{O}_W(W) \cong q_*(\mathcal{O}_W)(C) \cong \bigoplus_{m\in\mathbb{Z}} \mathcal{O}_C(\bar{L}^m)(C).$$

Nowhere vanishing functions on W correspond to nowhere vanishing sections in $\bar{L}^m$. A non-homogeneous section in $\bar{L}^m$ can not be nowhere vanishing. A homogeneous nowhere vanishing section in $\bar{L}^m$ exists if and only if $\bar{L}^m$ is trivial. But the bundle $\bar{L}$ is negative, hence $\bar{L}^m$ is trivial if and only if $m = 0$. □

5. Automorphy factors of $\mathbb{Q}$-Gorenstein quasi-homogeneous surface singularities

In this section we use the results of section 4 to prove Theorem 3:

Theorem. *A quasi-homogeneous surface singularity is $\mathbb{Q}$-Gorenstein of index r if and only if for the corresponding automorphy factor (U, Γ, L) there is an integer m (called the level or the exponent of the automorphy factor) without common divisors with r and a Γ-equivariant isomorphism $L^m \cong T_U^r$.*

Proof. We first assume that for some positive integer r and integer m there is a Γ-equivariant isomorphism $L^m \cong T_U^r$. This isomorphism induces a (Γ/Γ')-equivariant isomorphism $\bar{L}^m \cong T_C^r$. Then according to proposition 7 there exist global nowhere vanishing (Γ/Γ')-invariant sections in $\Omega^{2,r}(W)$. Such a section induces a nowhere vanishing section in $\Omega^{2,r}(W/(\Gamma/\Gamma')) = \Omega^{2,r}(X^*)$, hence the corresponding singularity $(X(U, \Gamma, L), o)$ is $\mathbb{Q}$-Gorenstein.

Now let us assume that singularity (X, x) with automorphy factor (U, Γ, L) is $\mathbb{Q}$-Gorenstein of index r, i.e. there exist nowhere vanishing sections in $\Omega^{2,r}(X^*) \cong \Omega^{2,r}(W/(\Gamma/\Gamma'))$. We consider the singularity $(\bar{X}, \bar{x})$, which corresponds to the automorphy factor (U, Γ', L). For this singularity we have $X \cong \bar{X}/(\Gamma/\Gamma')$. The pull-back of a nowhere vanishing section in $\Omega^{2,r}(X^*)$ along the unramified covering $\bar{X}^* \to X^*$ is a nowhere vanishing (Γ/Γ')-invariant section in $\Omega^{2,r}(\bar{X}^*)$, hence the singularity $(\bar{X}, \bar{x})$ is also $\mathbb{Q}$-Gorenstein of index r.

A nowhere vanishing (Γ/Γ')-invariant section in $\Omega^{2,r}(\bar{X}^*)$ induces a nowhere vanishing (Γ/Γ')-invariant section in $\Omega^{2,r}(W) = \Omega^{2,r}(\bar{L}^*)$. Then proposition 7 implies the existence of an (Γ/Γ')-equivariant isomorphism $\bar{L}^m \cong T_C^r$ for some integer m, i.e. the induced action of (Γ/Γ') on $\bar{L}^m \cong T_C^r$ coincides with the canonical action of (Γ/Γ') on T_C^r. Hence the action of Γ on $L^m \cong T_U^r$ also coincides with the canonical action of Γ on T_U^r, i.e. there exists a Γ-equivariant isomorphism $L^m \cong T_U^r$. □

Remark. Theorem 3 also follows from the following result of K. Watanabe [Wat81], appearing in the context of the theory of commutative rings: Let $R = R(X, D)$ be a normal graded ring, which is presented by the Pinkham-Demazure method. Then the canonical module K_R of R is Q-Cartier of index r, if and only if, there exists a rational function ϕ on X such that $r(K + D') - mD = \mathrm{div}_X(\phi)$ for some integer m and r is the minimum of such r. In our case, $X = C = U/\Gamma'$ and $\pi : U \to C$ is the Galois cover associated to the automorphy factor, and the result is translated to the relation on U as equivariant isomorphism $T_U^{-r} \cong L^m$. Our proofs of Proposition 7 and Theorem 3 give a more direct description of the automorphy factors in question.

6. From Hyperbolic Automorphy Factors to Biquotients

In this section we prove Theorem 1:

Theorem. *The link space of a hyperbolic $\mathbb{Q}$-Gorenstein quasi-homogeneous surface singularity of level m and index r is diffeomorphic to a quotient*

$$\tilde{\Gamma}_1\backslash\tilde{G}/\tilde{\Gamma}_2,$$

where $\tilde{G}$ is the universal cover $\widetilde{\mathrm{PSL}}(2,\mathbb{R})$ of the 3-dimensional Lie group $\mathrm{PSL}(2,\mathbb{R})$, while $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ are discrete subgroups of level m in $\tilde{G}$, $\tilde{\Gamma}_1$ is co-compact, and the image of $\tilde{\Gamma}_2$ in $\mathrm{PSL}(2,\mathbb{R})$ is a cyclic subgroup of order r. Conversely, any biquotient as above is diffeomorphic to the link space of a quasi-homogeneous hyperbolic $\mathbb{Q}$-Gorenstein singularity.

Before we explain the proof of this theorem, we give a description of the Lie group $G=\mathrm{PSL}(2,\mathbb{R})$ and its coverings.

As topological space $\mathrm{PSL}(2,\mathbb{R})$ is homeomorphic to the solid torus $\mathbb{S}^1\times\mathbb{C}$. The fundamental group of the solid torus G is infinite cyclic. Therefore, for each natural number m there is a unique connected m-fold covering

$$G_m=\tilde{G}/(m\cdot Z(\tilde{G}))$$

of G, where $Z(\tilde{G})$ is the central subgroup of $\tilde{G}$. For $m=2$ this is the group $G_2=\mathrm{SL}(2,\mathbb{R})$.

We use the following description of the covering groups G_m of $G=\mathrm{PSL}(2,\mathbb{R})$, which fixes a group structure. Let $\mathrm{Hol}(H,\mathbb{C}^*)$ be the set of all holomorphic functions $H\to\mathbb{C}^*$.

Proposition 8. *The m-fold covering group G_m of G can be described as*

$$\{(g,\delta)\in G\times\mathrm{Hol}(H,\mathbb{C}^*)\;\big|\;\delta^m(z)=g'(z)\text{ for all }z\in H\}$$

with multiplication

$$(g_2,\delta_2)\cdot(g_1,\delta_1)=(g_2\cdot g_1,(\delta_2\circ g_1)\cdot\delta_1).$$

Remark. This description of G_m and the description of $\tilde{G}$ that we give later are inspired by the notion of automorphic differential forms of fractional degree, introduced by J. Milnor in [Mil75]. For a more detailed discussion of this fact see [LV80], section 1.8.

We now explain the connection between automorphy factors in question and lifts of Fuchsian groups into the finite coverings of $\tilde{G}$.

Definition. A *lift* of the Fuchsian group Γ into G_m is a subgroup Γ^* of G_m such that the restriction of the covering map $G_m\to G$ to Γ^* is an isomorphism between Γ^* and Γ.

Proposition 9. *There is a 1-1-correspondence between hyperbolic $\mathbb{Q}$-Gorenstein automorphy factors $(H,\Gamma,H\times\mathbb{C})$ of level m and index r and the lifts of Γ into G_m.*

Proof. On the one hand using the description of the covering G_m from Proposition 10 we see that there is a 1-1-correspondence between lifts of Γ into G_m and families $\{\delta_g\}_{g\in\Gamma}$ of holomorphic functions $\delta_g:H\to\mathbb{C}^*$ such that for any $g\in\Gamma$

$$\delta_g^m=g'$$

and for any $g_1, g_2 \in \Gamma$

$$\delta_{g_2 \cdot g_1} = (\delta_{g_2} \circ g_1) \cdot \delta_{g_1}.$$

Let $\mathcal{D}$ be the set of all such families (δ_g).

On the other hand there is a 1-1-correspondence between hyperbolic $\mathbb{Q}$-Gorenstein automorphy factors $(H, \Gamma, H \times \mathbb{C})$ of level m and index r and and families $\{e_g\}_{g\in\Gamma}$ of holomorphic functions $e_g : H \to \mathbb{C}^*$ such that for any $g \in \Gamma$

$$e_g^m = (g')^r$$

and for any $g_1, g_2 \in \Gamma$

$$e_{g_2 \cdot g_1} = (e_{g_2} \circ g_1) \cdot e_{g_1}.$$

Let $\mathcal{E}$ be the set of all such families $(e_g)_{g\in\Gamma}$.

It remains to establish a 1-1-correspondence between the sets $\mathcal{D}$ and $\mathcal{E}$. This correspondence is defined as follows: Let us assign to a family $(\delta_g) \in \mathcal{D}$ the family (e_g) given by $e_g := \delta_g^r$. One checks easily that $(e_g) \in \mathcal{E}$.

If the images $(e_g), (\tilde{e}_g) \in \mathcal{E}$ of $(\delta_g), (\tilde{\delta}_g) \in \mathcal{D}$ coincide then on the one hand

$$\left(\frac{\tilde{\delta}_g}{\delta_g}\right)^r = \frac{\tilde{\delta}_g^r}{\delta_g^r} = \frac{\tilde{e}_g}{e_g} = 1,$$

on the other hand

$$\left(\frac{\tilde{\delta}_g}{\delta_g}\right)^m = \frac{\tilde{\delta}_g^m}{\delta_g^m} = \frac{g'}{g'} = 1.$$

But the integers m and r are relatively prime, hence there exists only one complex number ξ with the property $\xi^m = \xi^r = 1$, namely $\xi = 1$. Hence for any $g \in \Gamma$

$$\frac{\tilde{\delta}_g}{\delta_g} \equiv 1,$$

i.e. the families (δ_g) and $(\tilde{\delta}_g)$ coincide. So we have shown that the mapping $\mathcal{D} \to \mathcal{E}$ is injective.

Now let us consider a family $(e_g) \in \mathcal{E}$. It holds $g'(z) \notin \mathbb{R}_+ \cup \{0\}$ for all $z \in H$, hence there exist functions $\rho_g : H \to \mathbb{R}_*$ and $\varphi_g : H \to (0,1)$ such that $g' = \rho_g \cdot \exp(2\pi i \varphi_g)$. The chain rule implies

$$\rho_{g_2 \cdot g_1} = (\rho_{g_2} \circ g_1) \cdot \rho_{g_1}$$

and

$$\varphi_{g_2 \cdot g_1} - \varphi_{g_2} \circ g_1 - \varphi_{g_2} \in \mathbb{Z}.$$

The function e_g is then of the form

$$e_g = \rho_g^{\frac{r}{m}} \cdot \exp\left(2\pi i \cdot \frac{r \cdot \varphi_g + k_g}{m}\right)$$

for some function $k_g : H \to \mathbb{Z}$. The function k_g is continuous and hence constant. The integers m and r are relatively prime, hence there is an integer n_g such that $r \cdot n_g \equiv k_g \mod m$. Let us define a family (δ_g) by setting

$$\delta_g = \rho_g^{\frac{1}{m}} \cdot \exp\left(2\pi i \cdot \frac{\varphi_g + n_g}{m}\right).$$

We now prove that the family (δ_g) is in $\mathcal{D}$. The first property

$$\delta_g^m = \rho_g \cdot \exp\left(2\pi i \cdot (\varphi_g + n_g)\right) = \rho_g \cdot \exp\left(2\pi i \cdot \varphi_g\right) = g'$$

is satisfied. The second property

$$\delta_{g_2\cdot g_1} = (\delta_{g_2}\circ g_1)\cdot \delta_{g_1}$$

is equivalent to

$$\rho_{g_2\cdot g_1}^{\frac{1}{m}} = (\rho_{g_2}\circ g_1)^{\frac{1}{m}}\cdot \rho_{g_1}^{\frac{1}{m}}$$

and

$$(\varphi_{g_2\cdot g_1}-\varphi_{g_2}\circ g_1-\varphi_{g_1})+(n_{g_2\cdot g_1}-n_{g_2}\circ g_1-n_{g_1})\equiv\ 0 \mod m.$$

The first of these equations follows from

$$\rho_{g_2\cdot g_1} = (\rho_{g_2}\circ g_1)\cdot \rho_{g_1}.$$

To prove the second equations we observe that

$$e_{g_2\cdot g_1} = (e_{g_2}\circ g_1)\cdot e_{g_1}$$

implies that m is a divisor of the integer

$$r\cdot(\varphi_{g_2\cdot g_1}-\varphi_{g_2}\circ g_1-\varphi_{g_1})+(k_{g_2\cdot g_1}-k_{g_2}\circ g_1-k_{g_1}).$$

Because of $r\cdot n_g\equiv k_g \mod m$ also the integer

$$r\cdot\Big((\varphi_{g_2\cdot g_1}-\varphi_{g_2}\circ g_1-\varphi_{g_1})+(n_{g_2\cdot g_1}-n_{g_2}\circ g_1-n_{g_1})\Big)$$

is divisible by m. Since m and r are relatively prime, the number m must be a divisor of the integer

$$(\varphi_{g_2\cdot g_1}-\varphi_{g_2}\circ g_1-\varphi_{g_1})+(n_{g_2\cdot g_1}-n_{g_2}\circ g_1-n_{g_1}).$$

So the family (δ_g) is in $\mathcal{D}$. The image of the family (δ_g) under the map $\mathcal{D}\to\mathcal{E}$ is

$$\delta_g^r=\rho_g^{\frac{r}{m}}\cdot\exp\left(2\pi i\cdot\frac{r\cdot\varphi_g+r\cdot n_g}{m}\right)=\rho_g^{\frac{r}{m}}\cdot\exp\left(2\pi i\cdot\frac{r\cdot\varphi_g+k_g}{m}\right)=e_g.$$

So we have proved that the mapping $\mathcal{D}\to\mathcal{E}$ is surjective. □

Now we explain the connection between lifts of Fuchsian groups into the finite coverings of $\tilde{G}$ and discrete subgroup of finite index in $\tilde{G}$.

We use the following description of the covering groups $\tilde{G}$ of $G=\mathrm{PSL}(2,\mathbb{R})$, which fixes a group structure. Let $\mathrm{Hol}(H,\mathbb{C})$ be the set of all holomorphic functions $H\to\mathbb{C}$.

Proposition 10. *The universal covering group $\tilde{G}$ of G can be described as*

$$\{(g,\delta)\in G\times \mathrm{Hol}(H,\mathbb{C})\ \big|\ e\circ\delta=g'\},$$

where $e(w)=\exp(2\pi i w)$. The multiplication is given by

$$(g_2,\delta_2)\cdot(g_1,\delta_1)=(g_2\cdot g_1,\delta_2\circ g_1+\delta_1).$$

The covering map $\tilde{G}\to G_m$ is given by

$$(g,\delta)\mapsto(g,e(\delta/m)).$$

Remark. The center of the group $\tilde{G}$ is infinite cyclic and is equal to the preimage of the unit element in G:

$$Z(\tilde{G})=\{(g,\delta)\in\tilde{G}\ \big|\ g=\mathrm{Id},\quad \delta \text{ is an integer constant}\}.$$

Definition. The level of a discrete subgroup $\tilde{\Gamma}\subset\tilde{G}$ is the index of $\tilde{\Gamma}\cap Z(\tilde{G})$ as a subgroup of $Z(\tilde{G})$.

The following fact is well known (see for example section 4 in [KR85]):

Proposition 11. *There is a one-to-one correspondence between discrete co-compact subgroups of level m in $\tilde{G}$ and liftings of discrete co-compact subgroups in $\mathrm{PSL}(2,\mathbb{R})$ into the m-fold covering of $\mathrm{PSL}(2,\mathbb{R})$. The correspondence is given by mapping a subgroup in $\tilde{G}$ into its image under the covering map $\tilde{G} \to G_m$.*

We now prove Theorem 1.

Proof. Let (X,x) be a hyperbolic $\mathbb{Q}$-Gorenstein quasi-homogeneous surface singularity of level m and index r and let (H,Γ_1,L) be the corresponding automorphy factor. Let us consider a trivialisation $L \simeq H \times \mathbb{C}$ of the bundle L. Combining the results of Propositions 9 and 11 we see that there is a discrete co-compact subgroup $\tilde{\Gamma}_1$ of level m in $\tilde{G}$ such that the action of the group Γ_1 can be described as

$$g \cdot (z,t) = (g(z), e(\delta(z)r/m) \cdot t),$$

where $\delta : H \to \mathbb{C}$ is a holomorphic function such that (g,δ) is an element of $\tilde{\Gamma}_1$. This action of $\tilde{\Gamma}_1$ can be obtained as a restriction of the action of the group $\tilde{G}$ on L via

$$(g,\delta) \cdot (z,t) = (g(z), e(\delta(z)r/m) \cdot t).$$

It is easy to check, that this is an action of $\tilde{G}$. The unit subbundle of L can be identified with the subbundle

$$S = \{(z,t) \in H \times \mathbb{C} \mid |t|^m = (\mathrm{Im}(z))^r\}.$$

The bundle S is invariant under $\tilde{G}$: For $(z',t') = (g,\delta)\cdot(z,t) = (g(z), e(\delta(z)r/m)\cdot t)$ we have

$$\frac{|t'|^m}{|t|^m} = \left|e\left(\delta(z) \cdot \frac{r}{m}\right)\right|^m = |e(\delta(z))|^r = |g'(z)|^r = \left(\frac{\mathrm{Im}\, g(z)}{\mathrm{Im}\, z}\right)^r = \frac{(\mathrm{Im}(z'))^r}{(\mathrm{Im}(z))^r}.$$

The stabiliser of a point $(z_0,t_0) \in S$ is

$$\tilde{\Gamma}_2 := \mathrm{Stab}_{\tilde{G}}((z_0,t_0)) = \left\{(g,\delta) \in \tilde{G} \;\middle|\; g(z_0) = z_0, \quad \delta(z) \cdot \frac{r}{m} \in \mathbb{Z}\right\}.$$

We now determine the level of the subgroup $\tilde{\Gamma}_2$:

$$\begin{aligned} \tilde{\Gamma}_2 \cap Z(\tilde{G}) &= \{(g,\delta) \in Z(\tilde{G}) \mid \delta \text{ is an integer constant divisible by } m\} \\ &= m \cdot Z(\tilde{G}). \end{aligned}$$

The map $(g,\delta) \mapsto (g,\delta)\cdot(i,1)$ defines a $\tilde{\Gamma}_1$-equivariant diffeomorphism $\tilde{G}/\tilde{\Gamma}_2 \to S$. Here $\tilde{\Gamma}_1$ acts on $\tilde{G}$ by left multiplication. We obtain the following commutative diagram

$$\begin{array}{ccc} \tilde{G}/\tilde{\Gamma}_2 & \longrightarrow & L^*/\mathbb{R}_+ \cong S \\ \big\downarrow & & \big\downarrow \\ \tilde{\Gamma}_1\backslash\tilde{G}/\tilde{\Gamma}_2 & \longrightarrow & X^*/\mathbb{R}_+ \cong M \end{array}$$

Hence we have

$$M \cong \tilde{\Gamma}_1\backslash\tilde{G}/\tilde{\Gamma}_2.$$

Conversely, let $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ be discrete subgroups of level m in $\tilde{G}$, let $\tilde{\Gamma}_1$ be co-compact, and let the image of $\tilde{\Gamma}_2$ in $\mathrm{PSL}(2,\mathbb{R})$ be a cyclic subgroup of order r.

Then $\Gamma_1 = \tilde{\Gamma}_1/(\tilde{\Gamma}_1 \cap Z(\tilde{G}))$ is a discrete co-compact subgroup of $\mathrm{PSL}(2,\mathbb{R})$. We can define an automorphy factor $(H, \Gamma_1, H \times \mathbb{C})$ by setting

$$g \cdot (z,t) = (g(z), e(\delta(z)r/m) \cdot t),$$

where $\delta : H \to \mathbb{C}$ is a holomorphic function such that (g,δ) is an element of $\tilde{\Gamma}_1$. From the first part of the proof we know that the link of the corresponding quasi-homogeneous $\mathbb{Q}$-Gorenstein surface singularity is diffeomorphic to $\tilde{\Gamma}_1 \backslash \tilde{G}/\tilde{\Gamma}_2$. $\square$

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Mathematisches Institut, Universität Bonn, Beringstrasse 1, 53115 Bonn, Germany
E-mail address: anna@math.uni-bonn.de