THE COMBINATORIAL GEOMETRY OF $\mathbb{Q}$-GORENSTEIN QUASI-HOMOGENEOUS SURFACE SINGULARITIES

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Abstract. The main result of this paper is a construction of fundamental domains for certain group actions on Lorentz manifolds of constant curvature. We consider the simply connected Lie group $\tilde{G}=\widetilde{\mathrm{SU}}(1,1)$. The Killing form on the Lie group $\tilde{G}$ gives rise to a biinvariant Lorentz metric of constant curvature. We consider a discrete subgroup Γ_1 and a cyclic discrete subgroup Γ_2 in $\tilde{G}$ which satisfy certain conditions. We describe the Lorentz space form $\Gamma_1\backslash\tilde{G}/\Gamma_2$ by constructing a fundamental domain for the action of $\Gamma_1\times\Gamma_2$ on $\tilde{G}$ by $(g,h)\cdot x=gxh^{-1}$. This fundamental domain is a polyhedron in the Lorentz manifold $\tilde{G}$ with totally geodesic faces. For a co-compact subgroup the corresponding fundamental domain is compact. The class of subgroups for which we construct fundamental domains corresponds to an interesting class of singularities. In particular the bi-quotients of the form $\Gamma_1\backslash\tilde{G}/\Gamma_2$ are diffeomorphic to the links of quasi-homogeneous $\mathbb{Q}$-Gorenstein surface singularities.

1. Introduction

In the context of Riemannian manifolds, there are standard constructions for fundamental domains, for example Dirichlet regions. However, in the context of semi-Riemannian manifolds, such constractions are rare. The main result of this paper is a construction of fundamental domains for certain group actions on Lorentz manifolds of constant curvature.

We consider the universal cover $\tilde{G}$ of the group G of orientation-preserving isometries of the hyperbolic plane. The Killing form on the Lie group $\tilde{G}$ gives rise to a biinvariant Lorentz metric of constant curvature. We consider a discrete subgroup Γ_1 and a discrete cyclic subgroup Γ_2 in $\tilde{G}$ which satisfy the conditions $(*)$ specified below. In this paper we describe a construction of fundamental domains for the action of $\Gamma_1\times\Gamma_2$ on $\widetilde{\mathrm{SU}}(1,1)$ by $(g,h)\cdot x=gxh^{-1}$. The resulting fundamental domain is a polyhedron in the Lorentz manifold $\tilde{G}$ with totally geodesic faces. For a co-compact subgroup the corresponding fundamental domain is compact. The precise formulation of these results is contained in Theorems A and B.

The study of discrete subgroups of finite level is motivated by some deep connections between these subgroups and quasi-homogeneous isolated singularities of complex surfaces studied by J. Milnor, I. Dolgachev, and W. Neumann [Mil75, Dol83, Neu77, Neu83]. The class of subgroups for which we construct fundamental domains corresponds to an interesting class of singularities. There is a 1-1-correspondence

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between the subgroups from this class and quasi-homogeneous $\mathbb{Q}$-Gorenstein surface singularities. In particular the bi-quotients of the form $\Gamma_1\backslash\tilde{G}/\Gamma_2$ are diffeomorphic to the links of quasi-homogeneous $\mathbb{Q}$-Gorenstein surface singularities. For a more detailed treatment of this connection see [Pra06] and [BPR03], §1–2.

The construction described in [Pra01], [BPR03], [Pra07] can be understood as a special case of the construction described in this paper when the subgroup Γ_2 is trivial.

A bi-quotient of the form $\Gamma_1\backslash\tilde{G}/\Gamma_2$ is a standard Lorentz space form. The standard Lorentz space forms were studied by R.S. Kulkarni and F. Raymond [KR85]. Examples of non-standard Lorentz space forms were found by W. Goldman [Gol85], É. Ghys [Ghy87], and recently by F. Salein [Sal00]. The survey [BZ04] of Th. Barbot and A. Zeghib and the paper [Fra05] of Ch. Frances are good references for the reader interested in group actions on Lorentz manifolds. The results of this paper suggest that the description of Lorentz space forms by means of fundamental domains could be extended to include non-standard Lorentz space forms.

Let us specify the conditions that we want to impose on the subgroups Γ_1 and Γ_2. We consider the universal cover of the group $G = \mathrm{PSU}(1,1)$ of orientation-preserving isometries of the hyperbolic plane. Here our model of the hyperbolic plane is the unit disc $\mathbb{D}$ in $\mathbb{C}$. The kernel of the universal covering map $\widetilde{\mathrm{SU}}(1,1) \to \mathrm{PSU}(1,1)$ is the centre Z of the group $\widetilde{\mathrm{SU}}(1,1)$, an infinite cyclic group. Therefore, for each natural number k there is a unique connected k-fold covering of $\mathrm{PSU}(1,1)$. For $k = 2$ this is the group

$$\mathrm{SU}(1,1) = \left\{ \begin{pmatrix} w & z \\ \bar{z} & \bar{w} \end{pmatrix} \;\middle|\; (w,z) \in \mathbb{C}^2,\ |w|^2 - |z|^2 = 1 \right\}.$$

The *level* of a discrete subgroup $\Gamma \subset \widetilde{\mathrm{SU}}(1,1)$ is the index of $\Gamma \cap Z$ as a subgroup of Z.

Condition (*): We consider a discrete subgroup Γ_1 and a discrete cyclic subgroup Γ_2 in $\widetilde{\mathrm{SU}}(1,1)$ of finite level k. We suppose that the images $\overline{\Gamma}_1$ and $\overline{\Gamma}_2$ of Γ_1 resp. Γ_2 in $\mathrm{PSU}(1,1)$ have a joint fixed point in $\mathbb{D}$, i.e. there is a point u in $\mathbb{D}$ which is fixed by a nontrivial element of $\overline{\Gamma}_1$ and by a nontrivial element of $\overline{\Gamma}_2$. For $i = 1, 2$, let p_i be the smallest order of a non-trivial element in $\overline{\Gamma}_i$ that has u as a fixed point. Let $p = \mathrm{lcm}(p_1, p_2)$ be the least common multiple of p_1 and p_2. Furthermore we assume that $p > k$. (Our construction depends on the choice of the fixed point $u \in \mathbb{D}$.)

The paper is organized as follows: We start in Section 2 with some general remarks on the Lie groups $\mathrm{SU}(1,1)$ and $\widetilde{\mathrm{SU}}(1,1)$ and their embeddings in the 4-dimensional pseudo-Euclidean space resp. in a certain $\mathbb{R}_+$-bundle, the universal cover of a positive cone in that pseudo-Riemannian space. We describe in section 3 some elements of the construction, such as affine half-spaces and their substitutes in the $\mathbb{R}_+$-bundle. We also define prismatic sets Q_x, certain finite intersections of half-spaces, and study their properties. After that we are prepared to state in section 4 our main results, Theorems A and B, and to prove them. In section 5 we describe our explicit computations of fundamental domains for particular pairs of discrete subgroups and give pictures of these fundamental domains.

I would like to thank Egbert Brieskorn and Ludwig Balke for useful conversations related to this work.

2. Preliminaries

We consider the 4-dimensional pseudo-Euclidean space $E^{2,2}$ of signature $(2,2)$. We think of $E^{2,2}$ as the real vector space $\mathbb{C}^2 \cong \mathbb{R}^4$ with the symmetric bilinear form

$$\langle (z_1, w_1), (z_2, w_2) \rangle = \mathrm{Re}(z_1 \bar{z}_2 - w_1 \bar{w}_2).$$

In the pseudo-Euclidean space $E^{2,2}$ we consider the quadric

$$\begin{aligned} G &= \left\{ a \in E^{2,2} \;\middle|\; \langle a, a \rangle = -1 \right\} \\ &= \left\{ (z, w) \in E^{2,2} \;\middle|\; |z|^2 - |w|^2 = -1 \right\}. \end{aligned}$$

For a fixed $z \in \mathbb{C}$ the intersection

$$\{ w \in \mathbb{C} \mid (z, w) \in G \} = \{ w \in \mathbb{C} \mid |w|^2 = |z|^2 + 1 \}$$

is the circle of radius $\sqrt{|z|^2 + 1} \geqslant 1$. It holds $|w| \geqslant 1$ for any $(z, w) \in G$. The bilinear form on $E^{2,2}$ induces a Lorentz metric of signature $(2,1)$ on G. The quadric G is a model of the pseudo-hyperbolic space.

Furthermore we consider the cone over G

$$L = \mathbb{R}_+ \cdot G = \{ \lambda \cdot a \mid \lambda > 0,\ a \in G \}.$$

The cone L can be described as

$$\begin{aligned} L &= \left\{ a \in E^{2,2} \;\middle|\; \langle a, a \rangle < 0 \right\} \\ &= \left\{ (z, w) \in E^{2,2} \;\middle|\; |z| < |w| \right\}. \end{aligned}$$

For a fixed $z \in \mathbb{C}$ the intersection

$$\{ w \in \mathbb{C} \mid (z, w) \in L \} = \{ w \in \mathbb{C} \mid |w| > |z| \}$$

is the complement of the disc of radius $|z|$. It holds $w \neq 0$ for any $(z, w) \in L$. The bilinear form on $E^{2,2}$ induces a pseudo-Riemannian metric of signature $(2,2)$ on L.

We may think of L as a $\mathbb{R}_+$-bundle over G with radial projection $\theta : L \to G$ as bundle map. The map $L \to \mathbb{D}$ defined by $(z, w) \mapsto z/w$ is a principal $\mathbb{C}^*$-bundle, where the action of $\lambda \in \mathbb{C}^*$ is defined by $\lambda \cdot (z, w) = (\lambda^{-1} z, \lambda^{-1} w)$. Let $\pi : \tilde{G} \to G$ be the universal covering. Henceforth we identify the Lie group $\mathrm{SU}(1,1)$ with G via

$$\begin{pmatrix} w & z \\ \bar{z} & \bar{w} \end{pmatrix} \mapsto (z, \bar{w}),$$

and $\widetilde{\mathrm{SU}}(1,1)$ with $\tilde{G}$. The biinvariant metrics on G and $\tilde{G}$ are proportional to the Killing forms. We denote the pull-back $\tilde{L} \to \tilde{G}$ of the $\mathbb{R}_+$-bundle $\theta : L \to G$ under the covering map $\pi : \tilde{G} \to G$ also by θ. The following diagram commutes

$$\begin{array}{ccc} \tilde{L} & \xrightarrow{\pi} & L \\ {\scriptstyle\theta}\big\downarrow & & \big\downarrow{\scriptstyle\theta} \\ \tilde{G} & \xrightarrow{\pi} & G \end{array}$$

G resp. $\tilde{G}$ is canonically embedded in L resp. $\tilde{L}$ and therefore there exist canonical trivializations $L \cong G \times \mathbb{R}_+$ resp. $\tilde{L} \cong \tilde{G} \times \mathbb{R}_+$. The covering $\tilde{L}$ inherits canonically a pseudo-Riemannian metric from L.

We now give a brief description of the full isometry group of $\tilde{G}$ (compare sections 2.1–2.3 in [KR85]). The product $\tilde{G} \times \tilde{G}$ acts on $\tilde{G}$ via

$$(g, h) \cdot x = gxh^{-1}$$

by Lorentz isometries since the metric is biinvariant. The identity component $\mathrm{Isom}_0(\tilde{G})$ of the isometry group is isomorphic to $(\tilde{G} \times \tilde{G})/\Delta_Z$, where

$$\Delta_Z = \{(z, z) \bigm| z \in Z\}$$

and Z is the centre of $\tilde{G}$. The full isometry group of $\tilde{G}$ has four components corresponding to time- and/or space-reversals. Let ε be the geodesic symmetry at the identity given by $g \mapsto g^{-1}$ and η the lift of the conjugation by the matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ in G fixing the identity. Then ε preserves the space-orientation and reverses the time-orientation, while η reverses both the space- and time-orientation. Moreover, the group $\mathrm{Isom}^+(\tilde{G}) = \langle \mathrm{Isom}_0(\tilde{G}), \eta \rangle$ is the full group of orientation-preserving isometries and

$$\mathrm{Isom}(\tilde{G}) = \langle \mathrm{Isom}_0(\tilde{G}), \eta, \varepsilon \rangle \cong \mathrm{Isom}_0(\tilde{G}) \rtimes (\langle \eta \rangle \times \langle \varepsilon \rangle)$$

is the full isometry group of $\tilde{G}$.

The universal covering $\pi : \tilde{L} \to L$ of

$$L = \left\{(z, w) \in E^{2,2} \bigm| |z| < |w|\right\}$$

can also be described as

$$\begin{aligned} &\tilde{L} = \{(z, \alpha, r) \in \mathbb{C} \times \mathbb{R} \times \mathbb{R}_+ \bigm| |z| < r\}, \\ &\pi(z, \alpha, r) = (z, re^{i\alpha}). \end{aligned}$$

We call the number $\alpha \in \mathbb{R}$ the argument of the element $(z, \alpha, r) \in \tilde{L}$.

The restriction of the covering map $\pi : \tilde{L} \to L$ gives the description of the universal covering $\pi : \tilde{G} \to G$ of

$$G = \left\{(z, w) \in E^{2,2} \bigm| |z|^2 - |w|^2 = -1\right\}$$

as

$$\begin{aligned} &\tilde{G} = \{(z, \alpha, r) \in \mathbb{C} \times \mathbb{R} \times \mathbb{R}_+ \bigm| |z|^2 = r^2 - 1\}, \\ &\pi(z, \alpha, r) = (z, re^{i\alpha}). \end{aligned}$$

For $(z, \alpha, r) \in \tilde{G}$ the positive real number r can be computed from z and α, hence we can also identify $\tilde{G}$ with $\mathbb{C} \times \mathbb{R}$ via $(z, \alpha, r) \mapsto (z, \alpha)$.

The map $\theta : \tilde{L} \to \tilde{G}$ can be described as

$$\theta(z, \alpha, r) = \left(\lambda^{-1} z, \alpha, \lambda^{-1} r\right) \quad \text{with} \quad \lambda = \sqrt{r^2 - |z|^2}.$$

3. The Elements of the Construction

For $g \in \tilde{G}$ let E_g resp. I_g be the connected component of $\pi^{-1}(\bar{E}_{\bar{g}})$ resp. $\pi^{-1}(\bar{I}_{\bar{g}})$ containing g, where $\bar{g} := \pi(g)$ is the image of g in G,

$$\bar{E}_{\bar{g}} := \{a \in L \mid \langle g, a\rangle = -1\} = T_{\bar{g}}G \cap L$$

is the intersection of the affine tangent space $T_{\bar{g}}G$ on G in the point $\bar{g}$ with L and

$$\bar{I}_{\bar{g}} := \{a \in L \mid \langle g, a\rangle \leqslant -1\} = T^-_{\bar{g}}G \cap L$$

is the intersection the half-space $T^-_{\bar{g}}G$ of $\mathbb{C}^2$ bounded by $\bar{E}_{\bar{g}}$ and not containing 0 with L. $\bar{E}_{\bar{g}}$ and $\bar{I}_{\bar{g}}$ are simply connected and even contractible, hence their pre-images under the covering map π consist of infinitely many connected components, one of them containing g.

The three-dimensional submanifold E_g subdivides $\tilde{L}$ in two connected components, the closure of one of them is I_g, and we denote the closure of the other by H_g. The boundary of I_g, resp. H_g, is equal to E_g.

As an example, for the unit elements $e = (0, 0, 1)$ in $\tilde{G}$ and $\bar{e} = \pi(e) = (0, 1)$ in G, we have

$$\bar{I}_{\bar{e}} = \{(z, w) \in \mathbb{C}^2 \mid \operatorname{Re}(w) \geqslant 1,\ |z| < |w|\},$$

the boundary $\bar{E}_{\bar{e}}$ of $\bar{I}_{\bar{e}}$ is a one-sheeted hyperboloid of revolution. The pre-image of $\bar{I}_{\bar{e}}$ is

$$\pi^{-1}(\bar{I}_{\bar{e}}) = \{(z, \alpha, r) \in \mathbb{C} \times \mathbb{R} \times \mathbb{R}_+ \mid r \cdot \cos\alpha \geqslant 1,\ |z| < r\}.$$

The connected components of $\pi^{-1}(\bar{I}_{\bar{e}})$ resp. $\pi^{-1}(\bar{E}_{\bar{e}})$ containing e are

$$I_e = \left\{(z, \alpha, r) \in \mathbb{C} \times \mathbb{R} \times \mathbb{R}_+ \;\middle|\; |\alpha| < \frac{\pi}{2},\ r \geqslant \frac{1}{\cos\alpha},\ |z| < r\right\}$$

and

$$E_e = \left\{(z, \alpha, r) \in \mathbb{C} \times \mathbb{R} \times \mathbb{R}_+ \;\middle|\; |\alpha| < \frac{\pi}{2},\ r = \frac{1}{\cos\alpha},\ |z| < r\right\}.$$

The subsets E_g resp. I_g have the analogous properties because $E_g = g \cdot E_e$ and $I_g = g \cdot I_e$.

We make use of the following construction (compare [Mil75]). Given a base-point $x \in \mathbb{D}$ and a real number t, let $\rho_x(t) \in \mathrm{PSU}(1, 1)$ denote the rotation through angle t about the point x. Thus we obtain a homomorphism $\rho_x : \mathbb{R} \to \mathrm{PSU}(1, 1)$, which clearly lifts to the unique homomorphism $r_x : \mathbb{R} \to \widetilde{\mathrm{SU}}(1, 1)$ into the universal covering group. Since $\rho_x(2\pi) = \mathrm{Id}_{\mathbb{D}}$, it follows that the lifted element $r_x(2\pi)$ belongs to the central subgroup Z of $\widetilde{\mathrm{SU}}(1, 1)$. Note that this element $r_x(2\pi) \in Z$ depends continuously on x, and therefore is independent of the choice of x. We easily compute $r_0(2t) = (0, -t, 1)$ and hence $r_x(2\pi) = r_0(2\pi) = (0, -\pi, 1)$ for all $x \in \mathbb{D}$. Moreover we obtain

$$\begin{aligned}
r_0(2t) \cdot (z, \alpha, r) &= (ze^{it}, \alpha - t, r),\\
(z, \alpha, r) \cdot r_0(2t) &= (ze^{-it}, \alpha - t, r),\\
(z, \alpha, r) \cdot r_0(-2t) &= (ze^{it}, \alpha + t, r).
\end{aligned}$$

Let Γ_1 and Γ_2 be discrete subgroups of finite level k in $\widetilde{\mathrm{SU}}(1,1)$. For $i=1,2$, let $\overline{\Gamma}_i$ be the image of Γ_i in $\mathrm{PSU}(1,1)$. We assume the existence of a joint fixed point $u\in\mathbb{D}$ of $\overline{\Gamma}_1$ and $\overline{\Gamma}_2$.

For $i=1,2$, the isotropy group $(\overline{\Gamma}_i)_u$ of u in $\overline{\Gamma}_i$ is a finite cyclic group generated by $\rho_u(2\pi/p_i)$, where $p_i=|(\overline{\Gamma}_i)_u|$. The isotropy group $(\Gamma_i)_u$ of u in Γ_i is an infinite cyclic group generated by $d_i:=r_u(2\vartheta_i)$, where $\vartheta_i=\frac{\pi k}{p_i}$. We can assume without loss of generality that $u=0\in\mathbb{D}$. Under this assumption it follows

$$d_i=r_0(2\vartheta_i)=(0,-\vartheta_i,1)\quad\text{ and }\quad d_i\cdot(z,\alpha,r)=\left(ze^{i\vartheta_i},\alpha-\vartheta_i,r\right).$$

Now let us start with the construction of fundamental domains for the action of $\Gamma_1\times\Gamma_2$ on $\tilde{G}$. For a point x in the orbit $\Gamma_1(u)$ let $T(x)$ be

$$T(x)=\{(g_1,g_2)\in\Gamma_1\times\Gamma_2\ \big|\ g_1(u)=x\}.$$

Let

$$Q_x=\bigcap_{(g_1,g_2)\in T(x)}H_{g_1g_2}.$$

As an example, for $x=u$ we have that

$$T(u)=(\Gamma_1)_u\times\Gamma_2=\{(d_1^{m_1},d_2^{m_2})\ \big|\ m_1,m_2\in\mathbb{Z}\}=\langle(d_1,e),(e,d_2)\rangle.$$

The generator (d_1,e) acts on $\tilde{G}$ by left multiplication

$$d_1\cdot(z,\alpha,r)=(ze^{i\vartheta_1},\alpha-\vartheta_1,r).$$

The generator (e,d_2) acts on $\tilde{G}$ by right multiplication

$$(z,\alpha,r)\cdot d_2^{-1}=(ze^{i\vartheta_2},\alpha+\vartheta_2,r).$$

Let $p=\mathrm{lcm}(p_1,p_2)$ be the least common multiple of p_1 and p_2. Let

$$d=r_u(2\pi k/p)=r_u(2\vartheta),\quad\text{ where }\vartheta=\frac{\pi k}{p}.$$

The element d acts on $\tilde{G}$ by left multiplication

$$d\cdot(z,\alpha,r)=(ze^{i\vartheta},\alpha-\vartheta,r)$$

and it acts on the (α,r)-half-plane by the translation mapping

$$\tau(\alpha,r)=(\alpha-\vartheta,r).$$

An important assumption for the following construction is

$$p>k.$$

In terms of the element d the assumption $p>k$ means that the argument ϑ of d is less then π.

We have

$$Q_u=\bigcap_{(g_1,g_2)\in T(u)}H_{g_1g_2}=\bigcap_{m_1,m_2\in\mathbb{Z}}H_{d_1^{m_1}d_2^{m_2}}=\bigcap_{m\in\mathbb{Z}}H_{d^m},$$

since $\langle d_1,d_2\rangle=\langle d\rangle$.

What does the set

$$Q_u=\bigcap_{m\in\mathbb{Z}}H_{d^m}$$

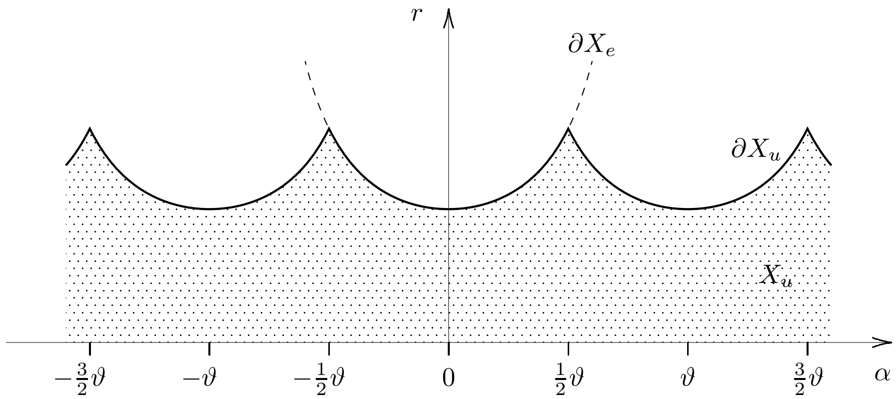

Figure 1: The image X_u of Q_u in the (α, r)-half-plane

look like? The image of the set H_e under the projection $(z, \alpha, r) \mapsto (\alpha, r)$ is

$$X_e = \{(\alpha, r) \in \mathbb{R} \times \mathbb{R}_+ \bigm| r \cdot \cos\alpha \leqslant 1 \text{ or } |\alpha| \geqslant \pi/2\}.$$

The images of the sets $H_{d^m} = d^m \cdot H_e$ under the projection $(z, \alpha, r) \mapsto (\alpha, r)$ are the translates $\tau^m(X_e)$ of the set X_e. The manifold Q_u is a disc bundle over its image $X_u = \bigcap_{m \in \mathbb{Z}} \tau^m(X_e)$ in the (α, r)-plane. The shaded area in figure 1 is X_u. (The real line is not part of X_u.) The subsets Q_x are images of the subset Q_u under the action of the group $\Gamma_1 \times \Gamma_2$. For any $x \in \Gamma_1(u)$ there is an element $g \in \Gamma_1$ such that $g(x) = u$. Then $Q_x = g \cdot Q_u$.

The manifolds gQ_u play a central role in our construction. We want to explain the geometric nature of these objects. We have described Q_u as a disc bundle over the set X_u in the (α, r)-half-plane $\mathbb{R} \times \mathbb{R}_+$. We may describe $Q_u \subset \tilde{L} \subset \mathbb{C} \times \mathbb{R} \times \mathbb{R}_+$ as

$$Q_u = (\mathbb{C} \times X_u) \cap \tilde{L}.$$

We think of X_u as a universal covering of a punctured plane polygon. Consider the following diagram of covering maps

$$\begin{tikzcd} \mathbb{R} \times \mathbb{R}_+ \arrow[r, "\pi'"] \arrow[dr, "\pi"'] & \mathbb{C}^* \arrow[d, "\pi''"] \\ & \mathbb{C}^* \end{tikzcd}$$

where $\pi(\alpha, r) = re^{i\alpha}$, $\pi'(\alpha, r) = r^{1/k} e^{i\alpha/k}$ and $\pi''(z) = z^k$. We now consider the curve $\pi(\partial X_u)$. It is easy to see that this is a regular star polygon $\left\{\frac{2p}{k}\right\}$ when k is odd and a regular star polygon $\left\{\frac{p}{k}\right\}$ when k is even, whereby a star polygon $\left\{\frac{n}{m}\right\}$, with n and m positive integers, is a figure formed by connecting with straight lines every m-th point out of n regularly spaced points lying on a circle (see H.S.M. Coxeter [Cox69], §2.8, pp. 36–38).

Remark: $k = 2$, $p_1 = 5$, $p_2 = 3$, $p = 15$: $\left\{\frac{15}{2}\right\}$ star polygon.

Therefore the curve $\pi'(\partial X_u)$ is a curvilinear $2p$-gon covering the star polygon once or twice. Let $P' \subset \mathbb{C}$ and $P = P_u \subset \mathbb{C}$ be the plane areas bounded by the curvilinear polygon $\pi'(\partial X_u)$ and by the star polygon $\pi(X_u)$. The images of X_u

are the punctured plane polygons $\pi'(X_u) = P' \setminus \{0\}$ and $\pi(X_u) = P \setminus \{0\}$. We think of the product $\mathbb{C} \times P'$ as a 4-dimensional $2p$-gonal *prism*. $\mathbb{C} \times X_u$ is the universal covering of the pierced prism $\mathbb{C} \times (P' \setminus \{0\})$. The product $\mathbb{C} \times P \subset \mathbb{C}^2$ might be considered as a 4-dimensional *"star prism"*. Its axis $\mathbb{C} \times \{0\}$ does not meet $L \subset \mathbb{C} \times \mathbb{C}^*$. Therefore the universal covering $\pi : \tilde{L} \to L$ maps Q_u to the intersection of L with the star prism:

$$\pi(Q_u) = L \cap (\mathbb{C} \times P_u).$$

In the following lemma we prove some properties of the sets Q_x. We first give some definitions. Let $s : \tilde{G} \to \mathbb{R}_+$ be a section in the bundle $\tilde{L} \cong \tilde{G} \times \mathbb{R}_+$. We call the set

$$\{(a, \lambda) \in \tilde{G} \times \mathbb{R}_+ \mid \lambda = s(a)\}$$

the *graph* of s and the set

$$\{(a, \lambda) \in \tilde{G} \times \mathbb{R}_+ \mid \lambda \leqslant s(a)\}$$

the *subgraph* of s.

Lemma 1. *For a point $x \in \mathbb{D}$ in the orbit $\Gamma_1(u)$ of the point u under the action of the group Γ_1 the following holds:*

(i) For any point $(z, w) \in \pi(Q_x)$

$$|w| - |z| \leqslant |w - \bar{x}z| \leqslant f(|x|),$$

where

$$f(t) := \frac{\sqrt{1 - t^2}}{\cos \frac{\vartheta}{2}}.$$

(ii) The set Q_x is a subgraph of a section in the bundle $\tilde{L} \cong \tilde{G} \times \mathbb{R}_+$, while its boundary is the graph of this section.

Proof. Our proof is in two steps. We first check the properties of Q_x in the case $x = u$. In this case the properties follow from the explicit description of the set Q_u. Then we use the fact that for any $x \in \Gamma_1(u)$ there is an element $g \in \Gamma_1$ such that $Q_u = g \cdot Q_x$ to prove the properties of Q_x for $x \neq u$.

Let us first describe explicitly the image X_u of the set Q_u in the (α, r)-plane $\mathbb{R} \times \mathbb{R}_+$. The set X_u is the shaded area in figure 1. It is a subgraph of a function $\mathbb{R} \to \mathbb{R}_+$. Let us denote this function by φ. We now describe the function φ explicitly. The function φ is periodic with period ϑ, hence it is sufficient to describe φ on $[-\vartheta/2, \vartheta/2]$. For $\alpha \in [-\vartheta/2, \vartheta/2]$ it holds

$$\varphi(\alpha) = \frac{1}{\cos \alpha}.$$

For any $\alpha \in \mathbb{R}$ it holds

$$\varphi(\alpha) \leqslant \frac{1}{\cos \frac{\vartheta}{2}}$$

(with equality for $\alpha = (2k + 1)\vartheta/2$, $k \in \mathbb{Z}$).

Now let us verify the first assertion of the lemma. The inequality

$$|w| - |z| \leqslant |w - \bar{x}z|$$

follows from $|z| < |w|$ and $|x| < 1$. It remains to prove the second inequality.

Let us verify the first assertion of the lemma in the case $x = u$. (Recall that we assumed $u = 0$.) For $x = u = 0$ the second inequality in the first part of the lemma reduces to

$$|w| \leqslant \frac{1}{\cos\frac{\vartheta}{2}}$$

for any point $(z, w) \in \pi(Q_u)$. Let us consider a point $(z, w) \in \pi(Q_u)$ and its preimage $(z, \alpha, r) \in Q_u$. By definition of the map π it holds $w = re^{i\alpha}$. For the point $(z, \alpha, r) \in Q_u$ it holds $(\alpha, r) \in X_u$. The set X_u is the subgraph of the function φ, hence

$$r \leqslant \varphi(\alpha) \leqslant \frac{1}{\cos\frac{\vartheta}{2}}$$

for any point $(\alpha, r) \in X_u$. Hence

$$|w| = r \leqslant \frac{1}{\cos\frac{\vartheta}{2}}.$$

Let us verify the first assertion of the lemma for any x. Let us consider a point $x \in \Gamma_1(u)$ and an element $g \in \Gamma_1$ such that $g(x) = u$. Let $(a, b) \in G$ be the image of the element g under π. The element $(a, b) \in G$ corresponds to the matrix

$$\begin{pmatrix} \bar{b} & a \\ \bar{a} & b \end{pmatrix} \in \mathrm{SU}(1, 1)$$

and acts on $\mathbb{D}$ by

$$(a, b) \cdot x = \frac{\bar{b}x + a}{\bar{a}x + b}.$$

The property $(a, b) \cdot x = u = 0$ implies $a = -\bar{b}x$. From $(a, b) \in G$ we conclude

$$-1 = |a|^2 - |b|^2 = |-\bar{b}x|^2 - |b|^2 = -|b|^2 \cdot (1 - |x|^2)$$

and hence

$$|b| = \frac{1}{\sqrt{1 - |x|^2}}.$$

Let us consider $(z, w) \in \pi(Q_x)$ and $(z', w') = g \cdot (z, w) \in \pi(Q_u)$. On the one hand $(z', w') \in \pi(Q_u)$ implies

$$|w'| \leqslant \frac{1}{\cos\frac{\vartheta}{2}}.$$

On the other hand

$$|w'| = |\bar{a}z + bw| = |-b\bar{x}z + bw| = \frac{1}{\sqrt{1 - |x|^2}} \cdot |w - \bar{x}z|.$$

Hence

$$|w - \bar{x}z| \leqslant \frac{\sqrt{1 - |x|^2}}{\cos\frac{\vartheta}{2}}.$$

Let us verify the second assertion of the lemma in the case $x = u$. For the set Q_u we can describe the corresponding section $s_u : \tilde{G} \to \mathbb{R}_+$ explicitly as

$$s_u(z, \alpha, r) = \frac{\varphi(\alpha)}{r}.$$

Let us verify the second assertion of the lemma for any x. Let us consider a point $x \in \Gamma(u)$ and an element $g \in \Gamma$ such that $Q_u = g \cdot Q_x$. Then the section $s_x : \tilde{G} \to \mathbb{R}_+$ is given by

$$s_x(a) = s_u(g \cdot a). \quad \square$$

Lemma 2. *The family $(Q_x)_{x \in \Gamma_1(u)}$ is locally finite in the sense that any point of $\tilde{L}$ has a neighbourhood intersecting only finitely many prisms Q_x.*

Proof. We prove that the family $(\pi(Q_x))_{x \in \Gamma_1(u)}$ is locally finite (in L). This fact implies the local finiteness of the family $(Q_x)_{x \in \Gamma(u)}$, since if a subset U of L has an empty intersection with $\pi(Q_x)$ then the intersection of the pre-image $\pi^{-1}(U)$ with Q_x is empty too. By lemma 1(i) for any point $x \in \Gamma_1(u)$ and any point $(z, w) \in \pi(Q_x)$ the difference $|w| - |z|$ is bounded from above by $f(|x|)$. The values $f(t)$ tend to zero as t tends to 1. Choosing a point $(z_0, w_0) \in L$ and a positive number $\varepsilon < |w_0| - |z_0|$, the neighbourhood $U := \{(w, z) \in L \mid |w| - |z| > \varepsilon\}$ of the point (z_0, w_0) can intersect $\pi(Q_x)$ only for $|x|$ sufficiently small (so that $f(|x|) > \varepsilon$). But the group Γ_1 is discrete, so there are only finitely many points x in $\Gamma(u)$ with norm $|x|$ under a given bound. This finishes the proof. $\square$

Remark. This property of Q_x allows us to deal with $P = \cup Q_x$ in a similar way as with a finite union of polytopes.

Lemma 3. *The family $(E_g \cap Q_{g(u)})_{g \in \Gamma_1}$ is locally finite.*

Proof. This is immediate from the local finiteness of the family $(Q_x)_{x \in \Gamma_1(u)}$ plus the easy observation that the family $(E_g \cap Q_{g(u)})_{g \in (\Gamma_1)_u}$ is locally finite. $\square$

We consider in $\tilde{L}$ the four-dimensional polytope

$$P := \bigcup_{x \in \Gamma_1(u)} Q_x = \bigcup_{x \in \Gamma_1(u)} \bigcap_{g \in T(x)} H_g.$$

Lemma 4. *The projection $\partial P \to \tilde{G}$ is a $\Gamma_1 \times \Gamma_2$-equivariant homeomorphism.*

Proof. From lemma 1(ii) we know that the set Q_x is a subgraph of a section in the bundle $\tilde{L} \cong \tilde{G} \times \mathbb{R}_+$. A union of a locally finite family of subgraphs of sections in $\tilde{L}$ is again a subgraph of a section in $\tilde{L}$. To see this, let us first consider the following toy version of this statement: A union of subgraphs of functions $f_1, \ldots, f_k : \mathbb{R} \to \mathbb{R}_+$ is again a subgraph of a function $f : \mathbb{R} \to \mathbb{R}_+$, where $f = \max(f_1, \ldots, f_k)$. This is clear in the toy case and generalizes to the case of a locally finite family of subgraphs of sections in $\tilde{L}$. Thus the polyhedron $P = \cup Q_x$ is a subgraph of a section in the bundle $\tilde{L} \cong \tilde{G} \times \mathbb{R}_+$ as a union of a locally finite family of subgraphs. But for a subgraph of a section in the bundle $\tilde{L}$ it is clear that the bundle map $\tilde{L} \to \tilde{G}$ induces a homeomorphism from its boundary (equal to the graph of the section) onto $\tilde{G}$. This homeomorphism is $\Gamma_1 \times \Gamma_2$-equivariant since the projection $\tilde{L} \to \tilde{G}$ is $\Gamma_1 \times \Gamma_2$-equivariant. $\square$

4. The Main Results

Now we can state the main result

Theorem A. *The boundary of P is invariant with respect to the action of $\Gamma_1 \times \Gamma_2$. The subset*

$$F_g = \mathrm{Cl}_{\partial P}(\mathrm{Int}(\partial H_g \cap \partial P))$$

is a fundamental domain for the action of $\Gamma_1 \times \Gamma_2$ *on* ∂P. *The family*

$$(F_{g_1 g_2})_{g_1 \in \Gamma_1, g_2 \in \Gamma_2}$$

is locally finite in ∂P. *The projection* $\tilde{L} \to \tilde{G}$ *induces a* $\Gamma_1 \times \Gamma_2$*-equivariant homeomorphism*

$$\partial P \to \tilde{G}.$$

The image $\mathcal{F}_g$ *of* F_g *under the projection is a fundamental domain for the action of* $\Gamma_1 \times \Gamma_2$ *on* $\tilde{G}$. *The family* $(\mathcal{F}_{g_1 g_2})_{g_1 \in \Gamma_1, g_2 \in \Gamma_2}$ *is locally finite. For every elements* $g_1, h_1 \in \Gamma_1$, $g_2, h_2 \in \Gamma_2$ *with* $g_1 g_2 \neq h_1 h_2$ *the intersection* $\mathcal{F}_{g_1 g_2} \cap \mathcal{F}_{h_1 h_2}$ *lies in a totally geodesic submanifold of* $\tilde{G}$.

Remark. In this section all closures are taken in ∂P. We use the shorthand Cl instead of $\mathrm{Cl}_{\partial P}$.

Lemma 5. *Let* X *be a topological space. Let* A *and* B *be closed subsets of* X. *Then*

(i) $\operatorname{Int} \operatorname{Cl} \operatorname{Int} A = \operatorname{Int} A$,
(ii) $\operatorname{Int} A \cap \operatorname{Cl} \operatorname{Int} B \neq \varnothing \Rightarrow \operatorname{Int}(A \cap B) \neq \varnothing$.

Lemma 6.

$$\operatorname{Int} F_g = \operatorname{Int}(E_g \cap \partial P) \quad \textit{and} \quad \operatorname{Cl} \operatorname{Int} F_g = F_g.$$

Proof. The assertions follow from Lemma 5(i) with $A = E_g \cap \partial P$. □

Proof. To prove that F_g is a fundamental domain we have to prove two properties. The first property is that the images of F_g have no common inner points, i.e. the intersection $\operatorname{Int}(F_g) \cap F_h$ is empty if $g \neq h$. The second property is that $\operatorname{Cl}(\cup_{g \in \Gamma} \operatorname{Int} F_g) = \partial P$, i.e. roughly speaking the images of F_g cover the whole space ∂P.

Let us first prove that the intersection $\operatorname{Int}(F_g) \cap F_h$ is empty if $g \neq h$. Suppose on the contrary that there are elements $g, h \in \Gamma$ such that $g \neq h$ and $\operatorname{Int}(F_g) \cap F_h \neq \varnothing$. Let us consider the closed subsets $A = E_g \cap \partial P$ and $B = E_h \cap \partial P$. By Lemma 6 it holds $\operatorname{Int}(F_g) = \operatorname{Int} A$, hence the assumption $\operatorname{Int}(F_g) \cap F_h \neq \varnothing$ can be rewritten as $\operatorname{Int} A \cap \operatorname{Cl} \operatorname{Int} B \neq \varnothing$. From Lemma 5(ii) it follows that $\operatorname{Int}(A \cap B) \neq \varnothing$. This means that the set $\operatorname{Int}(E_g \cap E_h \cap \partial P)$ is not empty. But since the totally geodesic submanifolds E_g and E_h intersect transversally, the intersection $E_g \cap E_h$ has no inner points in ∂P.

Since $F_g \subset E_g \cap Q_{g(u)}$ lemma 3 implies that the family $(F_g)_{g \in \Gamma}$ is locally finite in ∂P. Lemma 4 says that the projection $\partial P \to \tilde{G}$ is a Γ-equivariant homeomorphism.

Now let us prove the property $\operatorname{Cl}(\cup_{g \in \Gamma} \operatorname{Int} F_g) = \partial P$. Since

$$\operatorname{Cl}\Big(\bigcup_{g \in \Gamma} \operatorname{Int} F_g\Big) \supset \bigcup_{g \in \Gamma} \operatorname{Cl} \operatorname{Int} F_g = \bigcup_{g \in \Gamma} F_g$$

(where the last equality holds by Lemma 6), it suffices to prove that $\cup_{g \in \Gamma} F_g = \partial P$. Consider $a \in \partial P$. From the definition of P and local finiteness (according to Lemma 3) of the family $(E_g \cap Q_{g(u)})_{g \in \Gamma}$ it follows that in some neighbourhood of the point a only finitely many elements of Γ are relevant, i.e. there exists a neighbourhood U of the point a in $\tilde{L}$ and elements $g_1,\dots,g_n\in\Gamma$ such that

$$\partial P\cap U=\bigcup_{i=1}^{n}(E_{g_i}\cap\partial P\cap U).$$

We may assume without loss of generality that the map $\pi|_U : U\to\pi(U)$ is a homeomorphism. The image of $P\cap U$ under this homeomorphism is an intersection of an open subset of L with a finite union of finite intersections of half-spaces H_g with the property $a\in\partial H_g$. Suppose that $a\notin \operatorname{Cl}\operatorname{Int}(E_{g_i}\cap\partial P)=F_{g_i}$ for all $i\in\{1,\dots,n\}$. This is only possible if for each $i\in\{1,\dots,n\}$ the set $E_{g_i}\cap\partial P\cap U$ is contained in a 2-dimensional submanifold of $\tilde{L}$. Thus $\partial P\cap U$ is contained in the union of finitely many 2-dimensional submanifolds. On the other hand it follows from lemma 4 that ∂P is homeomorphic to a 3-dimensional manifold $\tilde{G}$. This contradiction implies that $a\in F_g$ for some $g\in\Gamma$. □

Lemma 7. *The boundary ∂P of $P=\cup_{x\in\Gamma(u)}Q_x$ can be described as follows*

$$\partial P=\partial\Big(\bigcup_{x\in\Gamma(u)}Q_x\Big)=\Big(\bigcup_{x\in\Gamma(u)}\partial Q_x\Big)\backslash\Big(\bigcup_{x\in\Gamma(u)}\operatorname{Int}Q_x\Big).$$

This means that a point p is in the boundary of P if and only if p is not an interior point of any Q_x with $x\in\Gamma(u)$ and p is a boundary point of Q_x for some $x\in\Gamma(u)$.

Proof. From lemma 1(ii) we know that the set Q_x is a subgraph of a section s_x in the bundle $\tilde{L}\cong\tilde{G}\times\mathbb{R}_+$

$$Q_x=\{(a,\lambda)\in\tilde{G}\times\mathbb{R}_+ \mid \lambda\leqslant s_x(a)\}.$$

The set $P=\cup Q_x$ is the subgraph of the section $s_P=\max s_x$. (In this proof max means $\max_{x\in\Gamma(u)}$, $\cup$ means $\cup_{x\in\Gamma(u)}$, $\exists x$ means $\exists x\in\Gamma(u)$ and so on.) This property would be obvious for a finite union of subgraphs. Using local finiteness (according to Lemma 2) we prove that this property also holds for P. But for a subgraph

$$X=\{(a,\lambda)\in\tilde{G}\times\mathbb{R}_+ \mid \lambda\leqslant s(a)\}$$

of a section s in the bundle $\tilde{L}$ it is clear that $(a,\lambda)\in\partial X$ if and only if $\lambda=s(a)$. Hence

$$(a,\lambda)\in\partial P \iff \lambda=s_P(a).$$

By definition of s_P

$$\lambda=s_P(a)\iff(\exists x\quad\lambda=s_x(a))\quad\text{and}\quad(\forall x\quad\lambda\geqslant s_x(a)).$$

On the other hand

$$\begin{aligned}(a,\lambda)\in\cup\,\partial Q_x &\iff \exists x\quad\lambda=s_x(a),\\ (a,\lambda)\notin\cup\operatorname{Int}Q_x &\iff \forall x\quad\lambda\geqslant s_x(a).\end{aligned}$$

□

Lemma 8. $\operatorname{Int}F_e\subset\partial Q_u$.

Proof. By Lemma 6 it holds $\operatorname{Int}F_e=\operatorname{Int}(E_e\cap\partial P)$. Suppose that there is a point $a\in\operatorname{Int}F_e=\operatorname{Int}(E_e\cap\partial P)$ such that $a\notin\partial Q_u$. Since $a\in\partial P$ and $a\notin\partial Q_u$ there exists $x\in\Gamma(u)\backslash\{u\}$ such that $a\in\partial Q_x$. Then any neighbourhood of a intersects $E_e\cap\operatorname{Int}Q_x\subset E_e\backslash\partial P$. The projection $\theta:\tilde{L}\to\tilde{G}$ is continuous and the restriction $\theta|_{\partial P} : \partial P \to \tilde{G}$ is a homeomorphism, therefore any neighbourhood of a intersects $((\theta|_{\partial P})^{-1} \circ \theta)(E_e \backslash \partial P)) \subset \partial P \backslash E_e$. This implies $a \notin \operatorname{Int}(E_e \cap \partial P) = \operatorname{Int} F_e$. Contradiction. □

Proposition 9.

$$F_e = \operatorname{Cl} \operatorname{Int} \left((E_e \cap \partial Q_u) - (\bigcup_{x \in \Gamma(u) \backslash \{u\}} \operatorname{Int} Q_x) \right).$$

Proof. Let $\hat{F} := (E_e \cap \partial Q_u) - (\cup_{x \in \Gamma(u) \backslash \{u\}} \operatorname{Int} Q_x)$. We claim that F_e and $\hat{F}$ coincide up to the boundary, i.e. $\operatorname{Int} F_e = \operatorname{Int} \hat{F}$. To prove this we show the inclusions $\operatorname{Int} F_e \subset \operatorname{Int} \hat{F}$ and $\operatorname{Int} \hat{F} \subset \operatorname{Int} F_e$. We first prove that $\operatorname{Int} F_e \subset \operatorname{Int} \hat{F}$. To that end we show that $\operatorname{Int} F_e \subset \hat{F}$. Then $\operatorname{Int} \operatorname{Int} F_e \subset \operatorname{Int} \hat{F}$ and $\operatorname{Int} F_e = \operatorname{Int} \operatorname{Int} F_e$ imply $\operatorname{Int} F_e \subset \operatorname{Int} \hat{F}$. To see that $\operatorname{Int} F_e$ is contained in $\hat{F}$ we have to show (by definition of $\hat{F}$) that $\operatorname{Int} F_e$ is contained in E_e, in ∂Q_u, and does not intersect $\operatorname{Int} Q_x$ for all $x \in \Gamma(u) \backslash \{u\}$. By definition of F_e it holds $\operatorname{Int} F_e \subset E_e$. By Lemma 8 it holds $\operatorname{Int} F_e \subset \partial Q_u$. Finally for any $x \in \Gamma(u) \backslash \{u\}$ it holds $F_e \cap \operatorname{Int} Q_x = \varnothing$ because of the fact that F_e is contained in ∂P, and $\partial P \cap \operatorname{Int} Q_x = \varnothing$ by Lemma 7. This implies $\operatorname{Int} F_e \subset \hat{F}$ and therefore $\operatorname{Int} F_e \subset \operatorname{Int} \hat{F}$. We now have to prove the inclusion $\operatorname{Int} \hat{F} \subset \operatorname{Int} F_e$. From the definition of $\hat{F}$ it follows that $\hat{F} \subset E_e$. Moreover $\hat{F} \subset \partial Q_u \subset (\cup_{x \in \Gamma(u)} \partial Q_x)$ and $\hat{F} \cap (\cup_{x \in \Gamma(u) \backslash \{u\}} \operatorname{Int} Q_x) = \varnothing$ imply by Lemma 7 that $\hat{F} \subset \partial P$. Now from $\hat{F} \subset E_e \cap \partial P$ it follows that $\operatorname{Int} \hat{F} \subset \operatorname{Int}(E_e \cap \partial P) = \operatorname{Int} F_e$, where the last equality holds by Lemma 6. We now have proved both inclusions, i.e. we know that $\operatorname{Int} \hat{F} = \operatorname{Int} F_e$. From this it follows that $\operatorname{Cl} \operatorname{Int} \hat{F} = \operatorname{Cl} \operatorname{Int} F_e = F_e$. □

Lemma 10. *If Γ is co-compact, then F_g is compact.*

Proof. Consider a sequence a_k in $\operatorname{Int} F_g$. Let φ be the composition of the projection maps $\partial P \to \tilde{G}$ and $\tilde{G} \to \tilde{G}/\Gamma$. Since the quotient $\tilde{G}/\Gamma$ is compact we may assume without loss of generality that the sequence $\varphi(a_k)$ tends to a limit $\bar{a} \in \tilde{G}/\Gamma$. Since φ is surjective there exists a pre-image $a \in \partial P$ of $\bar{a}$ under φ. Hence there is a sequence h_k in Γ such that the sequence $h_k a_k$ tends to a. Since the family $(F_g)_{g \in \Gamma}$ is locally finite there exists a neighbourhood U of a that intersects only finitely many fundamental domains F_g. Therefore the set $\{h_k | k \in \mathbb{N}\}$ is finite. After choosing a subsequence we may assume that the sequence h_k is constant, say $h_k = h$. Then the sequence $h a_k$ tends to a, hence the sequence a_k tends to $h^{-1} a$. This implies $h^{-1} a \in F_g$. □

Theorem B. *If Γ is co-compact then F_g is a compact polyhedron, i.e. a finite union of finite compact intersections of half-spaces I_a.*

Proof. The family $(Q_x)_{x \in \Gamma(u)}$ is locally finite and the fundamental domain F_e is compact by lemma 10. From this it follows that there is a finite subset $E \subset \Gamma(u)$ such that $F_e \cap Q_x = \varnothing$ for all $x \in \Gamma(u) \backslash E$. By proposition 8 this implies the assertion. □

5. Examples

We have computed the fundamental domains explicitly for those infinite series of pairs of discrete subgroups which correspond via the construction described in [Pra06] to certain series of $\mathbb{Q}$-Gorenstein quasi-homogeneous surface singularities. In particular the quotient of $\widetilde{\mathrm{SU}}(1,1)$ by one of the corresponding group action is diffeomorphic to the link of the corresponding quasi-homogeneous singularity.

A discrete co-compact subgroup Γ of level k in $\widetilde{\mathrm{SU}}(1,1)$ such that the image in $\mathrm{PSU}(1,1)$ is a triangle group with signature $(\alpha_1, \alpha_2, \alpha_3)$ will be denoted by $\Gamma(\alpha_1, \alpha_2, \alpha_3)^k$.

The following figures show some of the explicitly computed fundamental domains.

Some explanations are required to make the figures of fundamental domains comprehensible. The image $\pi(F_e)$ of the fundamental domain F_e is a compact polyhedron in $\mathfrak{su}(1,1)$ with flat faces. The Lie algebra $\mathfrak{su}(1,1)$ is a 3-dimensional flat Lorentz space of signature $(n_+, n_-) = (2,1)$. Such a polyhedron has a distinguished rotational axis of symmetry. The direction of this axis is negative definite, and the orthogonal complement is positive definite. Changing the sign of the pseudo-metric in the direction of the rotational axis transforms Lorentz space into a well-defined Euclidean space. The image $\pi(F_e)$ of the fundamental domain is then transformed into a polyhedron in Euclidean space with dihedral symmetry. Figures 2, 4 and 6 show the Euclidean polyhedra obtained in this way in the cases $\Gamma(5,3,3)^2 \times (C_3)^2$, $\Gamma(7,3,3)^2 \times (C_3)^2$ and $\Gamma(9,3,3)^2 \times (C_3)^2$. The direction of the rotational axis is vertical. The top and bottom faces are removed.

The polyhedra in figures 2, 4 and 6 are all scaled by the same factor to illustrate the proportions between different fundamental domains.

Figures 3, 5, 7, 8 illustrate the identification schemes for the cases $\Gamma(5,3,3)^2 \times (C_3)^2$, $\Gamma(7,3,3)^2 \times (C_3)^2$ and $\Gamma(9,3,3)^2 \times (C_3)^2$. The face identification is equivariant with respect to the dihedral symmetry of the polyhedron. The faces labeled with the same letter and shaded in the same way are identified. Numbers on the edges of shaded faces indicate the identified flags (face, edge, vertex).

References

[BPR03] Egbert Brieskorn, Anna Pratoussevitch, and Frank Rothenhäusler, *The Combinatorial Geometry of Singularities and Arnold's Series E, Z, Q*, Moscow Mathematical Journal **3** (2003), no. 2, 273–333, the special issue dedicated to Vladimir I. Arnold on the occasion of his 65th birthday.

[BZ04] Thierry Barbot and Abdelghani Zeghib, *Group actions on Lorentz spaces, mathematical aspects: a survey*, The Einstein equations and the large scale behavior of gravitational fields, Birkhäuser, Basel, 2004, pp. 401–439.

[Cox69] H. S. M. Coxeter, *Introduction to geometry*, second ed., John Wiley & Sons Inc., New York, 1969.

[Dol83] Igor V. Dolgachev, *On the Link Space of a Gorenstein Quasihomogeneous Surface Singularity*, Math. Ann. **265** (1983), 529–540.

[Fra05] Charles Frances, *Lorentzian kleinian groups*, Comment. Math. Helv. **80** (2005), 883–910.

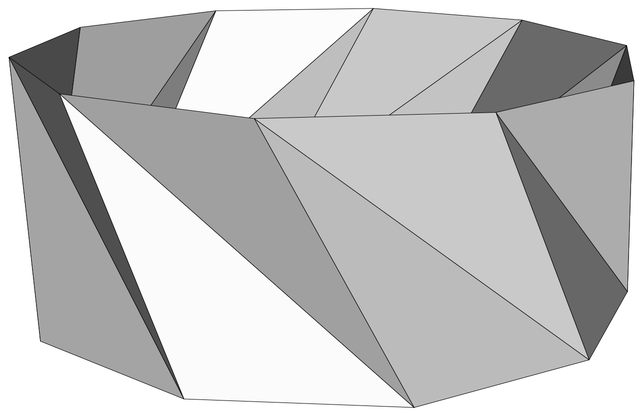

Figure 2: Fundamental domain for $\Gamma(5,3,3)^2 \times (C_3)^2$

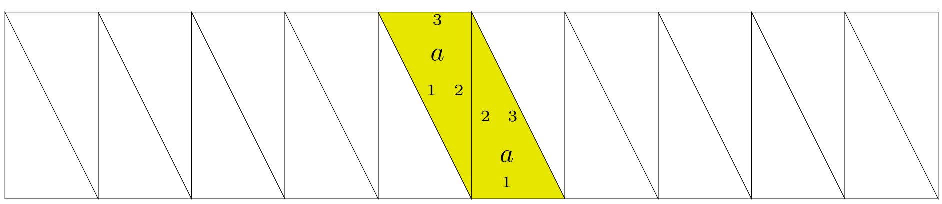

Figure 3: Identification scheme for $\Gamma(5,3,3)^2 \times (C_3)^2$

[Ghy87] Étienne Ghys, *Flots d'Anosov dont les feuilletages stables sont différentiables*, Ann. Sci. École Norm. Sup. (4) **20** (1987), no. 2, 251–270.

[Gol85] William M. Goldman, *Nonstandard Lorentz space forms*, J. Differential Geom. **21** (1985), no. 2, 301–308.

[KR85] Ravi S. Kulkarni and Frank Raymond, *3-dimensional Lorentz space forms and Seifert fiber spaces*, Journal of Differential Geometry **21** (1985), 231–268.

[Mil75] John Milnor, *On the 3-dimensional Brieskorn manifolds $M(p,q,r)$*, Knots, groups and 3-manifolds (L. P. Neuwirth, ed.), Annals of Math. Studies, vol. 84, Princeton University Press, Princeton, 1975, pp. 175–225.

[Neu77] Walter D. Neumann, *Brieskorn complete intersections and automorphic forms*, Invent. Math. **42** (1977), 285–293.

[Neu83] ———, *Geometry of quasihomogeneous surface singularities*, Proceedings of the summer institute on singularities held at Humboldt State University, Arcata, Calif., 1981 (P. Orlik, ed.), Proceedings of Symposia in Pure Mathematics, vol. 40, 2, American Math. Soc., Providence, 1983, pp. 245–258.

[Pra01] Anna Pratoussevitch, *Polyedrische Fundamentalbereiche diskreter Untergruppen von* $\widetilde{\mathrm{SU}}(1,1)$, Bonner Mathematische Schriften [Bonn Mathematical Publications], 346, Universität Bonn, Mathematisches Institut, Bonn, 2001, Dissertation, Rheinische Friedrich-Wilhelms-Universität Bonn, Bonn, 2001.

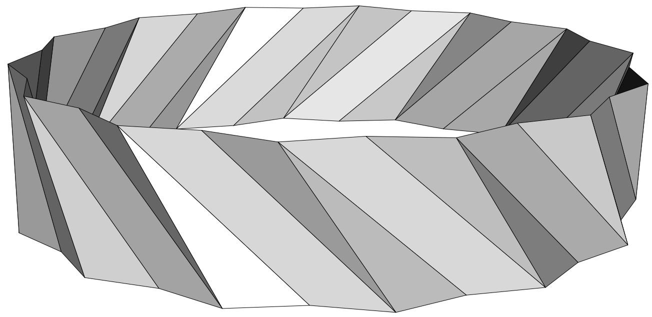

Figure 4: Fundamental domain for $\Gamma(7,3,3)^2 \times (C_3)^2$

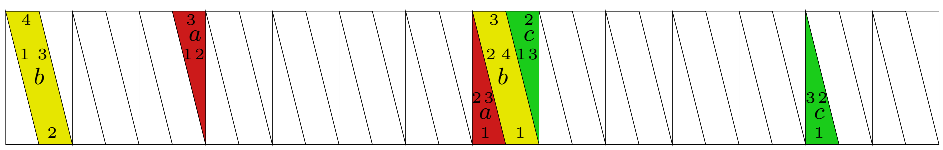

Figure 5: Identification scheme for $\Gamma(7,3,3)^2 \times (C_3)^2$

[Pra06] ______, *On the Link Space of a Q-Gorenstein Quasi-Homogeneous Surface Singularity*, Proceedings of the VIII Workshop on Real and Complex Singularities at CIRM (Luminy), Birkhäuser, 2006, pp. 311–325.

[Pra07] ______, *Fundamental Domains in Lorentzian Geometry*, Geometriae Dedicata **126** (2007), 155–175.

[Sal00] Francois Salein, *Variétés anti-de Sitter de dimension 3 exotiques*, Ann. Inst. Fourier (Grenoble) **50** (2000), no. 1, 257–284.

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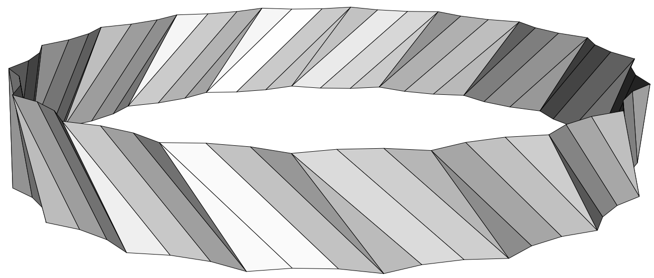

Figure 6: Fundamental domain for $\Gamma(9,3,3)^2 \times (C_3)^2$

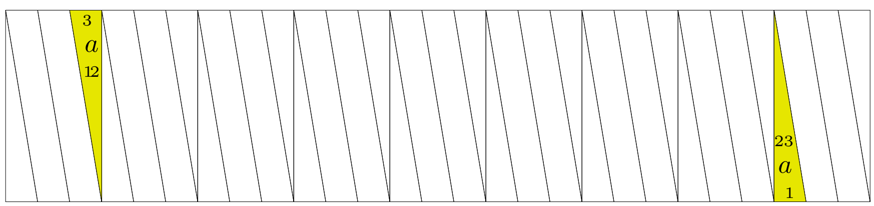

Figure 7: Identification scheme for $\Gamma(9,3,3)^2 \times (C_3)^2$

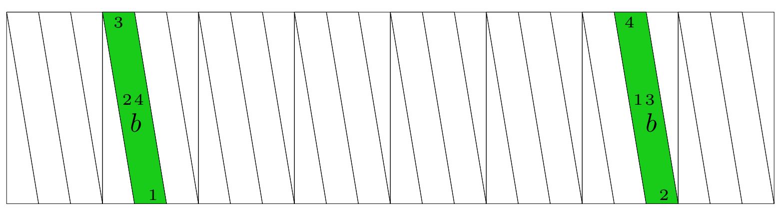

Figure 8: Identification scheme for $\Gamma(9,3,3)^2 \times (C_3)^2$