FUNDAMENTAL DOMAINS IN LORENTZIAN GEOMETRY

ANNA PRATOUSSEVITCH

Abstract. We consider discrete subgroups Γ of the simply connected Lie group $\widetilde{\mathrm{SU}}(1,1)$ of finite level, i.e. the subgroup intersects the centre of $\widetilde{\mathrm{SU}}(1,1)$ in a subgroup of finite index, this index is called the level of the group. The Killing form induces a Lorentzian metric of constant curvature on the Lie group $\widetilde{\mathrm{SU}}(1,1)$. The discrete subgroup Γ acts on $\widetilde{\mathrm{SU}}(1,1)$ by left translations. We describe the Lorentz space form $\widetilde{\mathrm{SU}}(1,1)/\Gamma$ by constructing a fundamental domain F for Γ. We want F to be a polyhedron with totally geodesic faces. We construct such F for all Γ satisfying the following condition: The image $\bar{\Gamma}$ of Γ in $\mathrm{PSU}(1,1)$ has a fixed point u in the unit disk of order larger than the index of Γ. The construction depends on the group Γ and on the orbit $\Gamma(u)$ of the fixed point u.

1. Introduction

We consider the universal cover of $\mathrm{PSU}(1,1) \cong \mathrm{PSL}(2,\mathbb{R})$, the group of orientation-preserving isometries of the hyperbolic plane. Here our model of the hyperbolic plane is the unit disc $\mathbb{D}$ in $\mathbb{C}$.

The kernel of the universal covering map $\widetilde{\mathrm{SU}}(1,1) \to \mathrm{PSU}(1,1)$ is the centre Z of the group $\widetilde{\mathrm{SU}}(1,1)$, an infinite cyclic group. Therefore, for each natural number k there is a unique connected k-fold covering of $\mathrm{PSU}(1,1)$. For $k=2$ this is the group

$$\mathrm{SU}(1,1) = \left\{ \begin{pmatrix} w & z \\ \bar{z} & \bar{w} \end{pmatrix} \;\middle|\; (w,z) \in \mathbb{C}^2,\ |w|^2 - |z|^2 = 1 \right\}.$$

The level of a discrete subgroup $\Gamma \subset \widetilde{\mathrm{SU}}(1,1)$ is the index of $\Gamma \cap Z$ as a subgroup of Z. There is a one-to-one correspondence between discrete subgroups of level k in $\widetilde{\mathrm{SU}}(1,1)$ and liftings of discrete subgroups in $\mathrm{PSU}(1,1)$ into the k-fold covering of $\mathrm{PSU}(1,1)$.

We consider a discrete subgroup Γ in $\widetilde{\mathrm{SU}}(1,1)$ of finite level k. We suppose that the image $\overline{\Gamma}$ of Γ in $\mathrm{PSU}(1,1)$ has at least one fixed point in $\mathbb{D}$ of order p, i.e. a point in $\mathbb{D}$, which is fixed by a nontrivial element of $\overline{\Gamma}$ of order p. Furthermore we assume that the order p of the fixed point is larger then the level k of the subgroup Γ. Our construction depends on the choice of the fixed point $u \in \mathbb{D}$ of $\overline{\Gamma}$, or actually on its orbit $\overline{\Gamma}(u)$.

Date: January 26, 2006.

2000 *Mathematics Subject Classification.* Primary 53C50; Secondary 14J17, 32S25, 51M20, 52B10.

Key words and phrases. Lorentz space form, polyhedral fundamental domain, quasihomogeneous singularity, Arnold singularity series.

Research partially supported by the Graduiertenkolleg Mathematik in Bonn, financed by DFG.

The Killing form on the Lie group $\widetilde{\mathrm{SU}}(1,1)$ gives rise to a Lorentz biinvariant metric of constant curvature. The quotient of $\widetilde{\mathrm{SU}}(1,1)$ by the discrete subgroup Γ is a Lorentz space form with respect to this metric, i.e. a complete Lorentz manifold of constant curvature (compare R.S. Kulkarni and F. Raymond [KR85]).

The main result of this paper is the construction of fundamental domains for the action of Γ on $\widetilde{\mathrm{SU}}(1,1)$ by left translations, applicable to any discrete subgroup Γ in $\widetilde{\mathrm{SU}}(1,1)$ as above. This fundamental domain is a polyhedron in the Lorentz manifold $\widetilde{\mathrm{SU}}(1,1)$ with totally geodesic faces. For a co-compact subgroup the corresponding fundamental domain is compact. The precise formulation of this result is contained in Theorems A and B.

The construction of fundamental domain was a part of the author's Ph.D. thesis [Pra01]. It was outlined in the survey article [BPR03]. The object of this paper is to provide complete proofs for this result together with a concise and self-contained description of the construction.

Our results generalize a construction by Th. Fischer [Fis92]. He suggested how to construct a fundamental domain for the action of a discrete subgroup of $\mathrm{PSU}(1,1)$ by left multiplication. His construction can be interpreted in our terms as a construction for discrete subgroup of $\widetilde{\mathrm{SU}}(1,1)$ of level 1. A less technical proof of Th. Fischer's result is given in [BKN+98].

The study of discrete subgroups of finite level was originally motivated by some deep connections between these subgroups and quasi-homogeneous isolated singularities of complex surfaces studied by J. Milnor, I. Dolgachev, and W. Neumann [Mil75, Dol83, Neu77, Neu83]. In particular the quotient $\widetilde{\mathrm{SU}}(1,1)/\Gamma$ is diffeomorphic to the link of some quasi-homogeneous Gorenstein singularity. For a more detailed treatment of this connection see [BPR03], §1–2.

The paper is organized as follows: We start in section 2 by discussion of low-dimensional analogues of our problem and use these examples to indicate some of the main ingredients of our construction. Section 3 contains some general remarks on the Lie groups $\mathrm{SU}(1,1)$ and $\widetilde{\mathrm{SU}}(1,1)$ and their embeddings in some 4-dimensional pseudo-Euclidean space resp. in a certain $\mathbb{R}_+$-bundle, the universal cover of a positive cone in that pseudo-Riemannian space. We describe in section 4 some elements of the construction, such as affine half-spaces and their substitutes in the $\mathbb{R}_+$-bundle. We also define prismatic sets Q_u, certain finite intersections of half-spaces, and study their properties.

After that we are prepared to state in section 5 our main results, Theorems A and B, and to prove them. In section 6 we report on our explicit computations of fundamental domains for certain infinite series of discrete subgroups. The choice of this series is motivated by the connections between them and some series of quasi-homogeneous surface singularities. We also give some pictures of fundamental domains. In section 7 we discuss some relations to and similarities with other fundamental domain constructions and give an outlook on possible generalizations. Finally, section 8 contains some facts from general topology, which are used in section 5.

The results described in section 6 have been announced in [BPR03] together with computations of fundamental domains for other singularities by E. Brieskorn and

F. Rothenhäusler. The synopsis of these results points at a very regular pattern for these series of singularities and their fundamental domains.

I am indebted to Egbert Brieskorn and Thomas Fischer who opened up this field of research. I am very grateful to Egbert Brieskorn for his guidance and helpful discussions. I would like to thank Ludwig Balke, Werner Ballmann, Ilya Dogolazky, Pierre Pansu, and Frank Rothenhäuser for useful conversations related to this work. I would like to thank the referee for his valuable remarks and suggestions. I thank Ilya Dogolazky for his help in producing the figures.

2. Low-Dimensional Analogues

Before we start to describe the construction for $\widetilde{\mathrm{SU}}(1,1)$, we discuss in this section the corresponding problem of finding fundamental domains for an action of a discrete subgroup for two toy cases, for the one-dimensional Lie groups $\mathrm{SO}(2)$ and $\mathrm{SO}(1,1)$. Some of the main ideas of the construction for $\widetilde{\mathrm{SU}}(1,1)$ can be seen already in the discussion of these low-dimensional examples.

We first consider the Lie group $\mathrm{SO}(2)$. We identify $\mathrm{SO}(2)$ with the circle

$$G = \{z(t) = (\cos t, \sin t) \bigm| t \in \mathbb{R}\}$$

in the Euclidean plane E^2. We consider a discrete subgroup Γ_m, the finite cyclic subgroup of order m generated by $z(2\pi/m)$, and its action on $\mathrm{SO}(2)$ by left multiplication, which extends to an action on E^2 by isometries.

Clearly, the segment of length $2\pi/m$ with midpoint $z(0)$ is a fundamental domain for this action, this is the Dirichlet domain with respect to the point $z(0)$. However, for the description of Dirichlet domains we need the distance. The following description of the same fundamental domain as projection of an affine construction with tangent half-planes is more appropriate for generalizations in pseudo-Riemannian setting.

For $g \in G$ let $H_g = \{a \in E^2 \bigm| \langle a, g\rangle \leqslant 1\}$ be the half-plane with boundary tangent to the circle G in the point g. Then the intersection of the tangent half-planes

$$P = \bigcap_{g\in\Gamma_m} H_g$$

is a regular Γ_m-invariant m-gon, its faces are fundamental domains for the action of Γ_m on the boundary ∂P, and the projection of the faces under the contraction $a \mapsto a/|a|$ yields to a tiling of the circle by (Dirichlet) fundamental domains with respect to Γ_m. Figure 1 illustrates the construction for $m = 6$.

A more involved example, where the construction of Dirichlet fundamental domains can be described in the same way, is the case of the action of discrete subgroups on $\mathrm{SU}(2)$, in particular the construction of the 4-dimensional regular polyhedron bounded by 120 dodecahedra, the tiling of the 3-dimensional sphere by 120 spherical dodecahedra and the resulting construction of the Poincaré homology sphere using the binary icosahedral group. Here we identify $\mathrm{SU}(2)$ with the 3-dimensional sphere in the Euclidean space E^4.

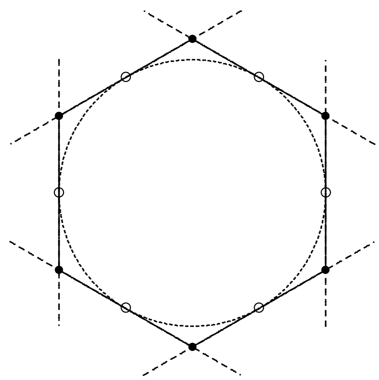

Figure 1: Fundamental domain construction in SO(2)

Our second description of the fundamental domain in SO(2) does not use any distance. It only uses the embedding of SO(2) in the (pseudo-)Euclidean space. However, the simple-minded attempt to generalize this affine construction to pseudo-Riemannian quadrics fails. Our second one-dimensional example, the hyperbola in the Minkowski plane, shows, why the naive approach fails and what can be done about this.

We now consider the Lie group $\mathrm{SO}(1,1)$ and identify it with the hyperbola

$$G = \{z(t,\varepsilon) = \varepsilon \cdot (\sinh t, \cosh t) \mid t \in \mathbb{R},\ \varepsilon = \pm 1\}$$

in the Minkowski plane $E^{1,1}$ with metric induced by $\langle a, b\rangle = a_1 b_1 - a_2 b_2$. Let us fix $d > 0$ and consider a discrete subgroup Γ_d, the subgroup generated by the elements $z(d, 1)$ and $z(d, -1)$ and isomorphic to $\mathbb{Z} \times \{\pm 1\}$. Moreover we consider the action of Γ_d on $\mathrm{SO}(1,1)$ by left multiplication, which extends to an action on $E^{1,1}$ by isometries. Clearly, the segment of length d with midpoint $z(0,+1)$ is a fundamental domain for this action. This is the Dirichlet domain with respect to the point $z(0,+1)$.

Let L be the cone over G, i.e. $L = \mathbb{R}_+ \cdot G = \{a \in E^{1,1} \mid \langle a, a\rangle < 0\}$. For $g \in G$ let H_g be the half-plane $H_g = \{a \in E^{1,1} \mid \langle a, g\rangle \geqslant -1\}$ with boundary tangent to the hyperbola G in the point g. The polytope $P = \cap_{g\in\Gamma_d} H_g$ contains in this case only one point $(0, 0)$, so we can not get any information about the tiling from this set. Instead we consider the set

$$P = \bigcup_{k\in\mathbb{Z}} \bigcap_{\varepsilon=\pm 1} H_{z(k\cdot d,\varepsilon)}$$

and the part of its boundary contained in L.

Figure 2 shows the polyhedron P and illustrates our construction of fundamental domains for Γ_d on the part of the boundary of P lying over G, i.e. in the positive cone L over G. The following assertions can be easily verified: The set $\partial P \cap L$ is Γ_d-invariant, and its faces are fundamental domains for the action of Γ_d on $\partial P \cap L$. The projection $\partial P \cap L \to G$ given by the contraction $a \mapsto a/\sqrt{-(a,a)}$ is a Γ_d-equivariant homeomorphism, and the projection of the faces of $\partial P \cap L$ yields to a tiling of the hyperbola by fundamental domains with respect to Γ_d.

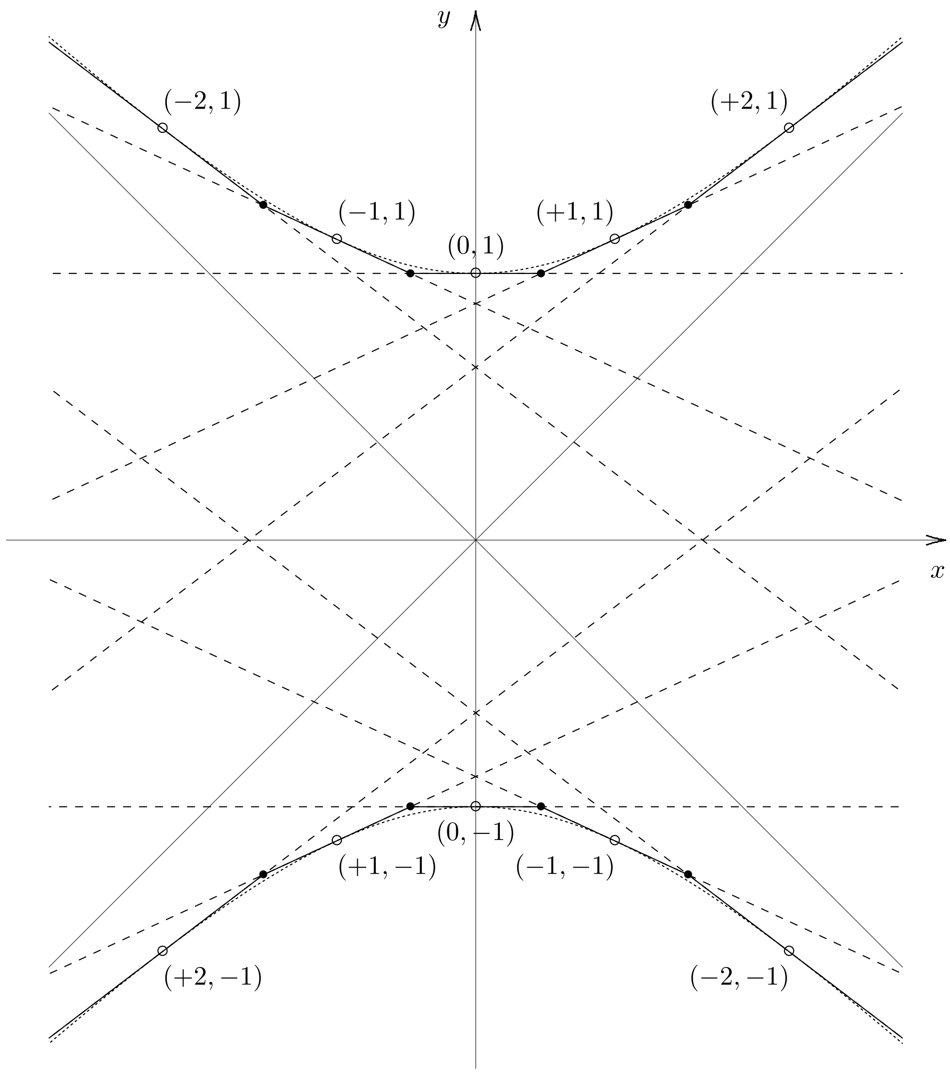

Figure 2: Fundamental domain construction in $\mathrm{SO}(1, 1)$

This construction suggests some important ingredients of the construction for $\mathrm{SU}(1, 1)$, namely the embedding of $\mathrm{SU}(1, 1)$ as a quadric in a 4-dimensional pseudo-Euclidean space, appropriate decomposition of the discrete subgroup in countable many finite subsets $T(x)$, $x \in X$, and finally the study of the 4-dimensional polyhedron

$$P = \bigcup_{x \in X} \bigcap_{g \in T(x)} H_g.$$

Some new ideas come in when we generalize the fundamental domain construction for $\widetilde{\mathrm{SU}}(1, 1)$. We consider an embedding of $\widetilde{\mathrm{SU}}(1, 1)$ as an image of a section in a (trivial) $\mathbb{R}_+$-bundle over $\widetilde{\mathrm{SU}}(1, 1)$, namely in the universal cover of the positive cone $\mathbb{R}_+ \cdot \widetilde{\mathrm{SU}}(1,1)$, and we define appropriate substitutes for tangent spaces and half-spaces there.

We also want to point out that these two one-dimensional examples are not only examples, we meet them again in the construction for $\mathrm{SU}(1,1)$. We identify $\mathrm{SU}(1,1)$ with a quadric in a 4-dimensional pseudo-Euclidean space. The subgroups $\mathrm{SO}(2) \cong \mathrm{U}(1)$ and $\mathrm{SO}(1,1)$ of $\mathrm{SU}(1,1)$ can be identified with certain plane sections of this quadric, and the corresponding constructions of fundamental domains are then sections of the construction for $\mathrm{SU}(1,1)$.

3. Preliminaries

We consider the 4-dimensional pseudo-Euclidean space $E^{2,2}$ of signature $(2,2)$. We think of $E^{2,2}$ as real vector space $\mathbb{C}^2 \cong \mathbb{R}^4$ with the symmetric bilinear form

$$\langle (z_1,w_1),(z_2,w_2)\rangle = \mathrm{Re}(z_1\bar{z}_2 - w_1\bar{w}_2).$$

In the pseudo-Euclidean space $E^{2,2}$ we consider the quadric

$$\begin{aligned} G &= \left\{ a \in E^{2,2} \;\middle|\; \langle a,a\rangle = -1 \right\} \\ &= \left\{ (z,w) \in E^{2,2} \;\middle|\; |z|^2 - |w|^2 = -1 \right\}. \end{aligned}$$

For a fixed $z \in \mathbb{C}$ the intersection

$$\{w \in \mathbb{C} \mid (z,w) \in G\} = \{w \in \mathbb{C} \mid |w|^2 = |z|^2 + 1\}$$

is the circle of radius $\sqrt{|z|^2+1} \geqslant 1$. It holds $|w| \geqslant 1$ for any $(z,w) \in G$. The bilinear form on $E^{2,2}$ induces a Lorentz metric of signature $(2,1)$ on G. The quadric G is a model of the pseudo-hyperbolic space.

Furthermore we consider the cone over G

$$L = \mathbb{R}_+ \cdot G = \{\lambda \cdot a \mid \lambda > 0,\ a \in G\}.$$

The cone L can be described as

$$\begin{aligned} L &= \left\{ a \in E^{2,2} \;\middle|\; \langle a,a\rangle < 0 \right\} \\ &= \left\{ (z,w) \in E^{2,2} \;\middle|\; |z| < |w| \right\}. \end{aligned}$$

For a fixed $z \in \mathbb{C}$ the intersection

$$\{w \in \mathbb{C} \mid (z,w) \in L\} = \{w \in \mathbb{C} \mid |w| > |z|\}$$

is the complement of the disc of radius $|z|$. It holds $w \neq 0$ for any $(z,w) \in L$. The bilinear form on $E^{2,2}$ induces a pseudo-Riemannian metric of signature $(2,2)$ on L.

We may think of L as a $\mathbb{R}_+$-bundle over G with radial projection $\theta : L \to G$ as bundle map. The map $L \to \mathbb{D}$ defined by $(z,w) \mapsto z/w$ is principal $\mathbb{C}^*$-bundle, where the action of $\lambda \in \mathbb{C}^*$ is defined by $\lambda \cdot (z,w) = (\lambda^{-1}z, \lambda^{-1}w)$. Let $\pi : \tilde{G} \to G$ be the universal covering. Henceforth we identify the Lie group $\mathrm{SU}(1,1)$ with G via

$$\begin{pmatrix} w & z \\ \bar{z} & \bar{w} \end{pmatrix} \mapsto (z,\bar{w}),$$

and $\widetilde{\mathrm{SU}}(1,1)$ with $\tilde{G}$. The biinvariant metrics on G and $\tilde{G}$ are proportional to the Killing forms. We denote the pull-back $\tilde{L} \to \tilde{G}$ of the $\mathbb{R}_+$-bundle $\theta : L \to G$ under the covering map $\pi : \tilde{G} \to G$ also by θ. The following diagram commutes

$$\begin{array}{ccc} \tilde{L} & \xrightarrow{\pi} & L \\ {\scriptstyle\theta}\big\downarrow & & \big\downarrow{\scriptstyle\theta} \\ \tilde{G} & \xrightarrow{\pi} & G \end{array}$$

G resp. $\tilde{G}$ is canonically embedded in L resp. $\tilde{L}$ and therefore there exist canonical trivializations $L \cong G \times \mathbb{R}_+$ resp. $\tilde{L} \cong \tilde{G} \times \mathbb{R}_+$. The covering $\tilde{L}$ inherits canonically a pseudo-Riemannian metric from L.

We now give a brief description of the full isometry group of $\tilde{G}$ (compare sections 2.1–2.3 in [KR85]). The product $\tilde{G} \times \tilde{G}$ acts on $\tilde{G}$ via

$$(g,h) \cdot x = gxh^{-1}$$

by Lorentz isometries since the metric is biinvariant. The identity component $\mathrm{Isom}_0(\tilde{G})$ of the isometry group is isomorphic to $(\tilde{G} \times \tilde{G})/\Delta_Z$, where

$$\Delta_Z = \{(z,z) \mid z \in Z\}$$

and Z is the centre of $\tilde{G}$. The full isometry group of $\tilde{G}$ has four components corresponding to time- and/or space-reversals. Let ε be the geodesic symmetry at the identity given by $g \mapsto g^{-1}$ and η the lift of the conjugation by the matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ in G fixing the identity. Then ε preserves the space-orientation and reverses the time-orientation, while η reverses both the space- and time-orientation. Moreover, the group $\mathrm{Isom}^+(\tilde{G}) = \langle \mathrm{Isom}_0(\tilde{G}), \eta \rangle$ is the full group of orientation-preserving isometries and

$$\mathrm{Isom}(\tilde{G}) = \langle \mathrm{Isom}_0(\tilde{G}), \eta, \varepsilon \rangle \cong \mathrm{Isom}_0(\tilde{G}) \rtimes (\langle \eta \rangle \times \langle \varepsilon \rangle)$$

is the full isometry group of $\tilde{G}$.

The universal covering $\pi : \tilde{L} \to L$ of

$$L = \left\{(z,w) \in E^{2,2} \mid |z| < |w|\right\}$$

can also be described as

$$\begin{aligned} &\tilde{L} = \{(z,\alpha,r) \in \mathbb{C} \times \mathbb{R} \times \mathbb{R}_+ \mid |z| < r\}, \\ &\pi(z,\alpha,r) = (z, re^{i\alpha}). \end{aligned}$$

We call the number $\alpha \in \mathbb{R}$ the argument of the element $(z,\alpha,r) \in \tilde{L}$.

The restriction of the covering map $\pi : \tilde{L} \to L$ gives the description of the universal covering $\pi : \tilde{G} \to G$ of

$$G = \left\{(z,w) \in E^{2,2} \mid |z|^2 - |w|^2 = -1\right\}$$

as

$$\begin{aligned} &\tilde{G} = \{(z,\alpha,r) \in \mathbb{C} \times \mathbb{R} \times \mathbb{R}_+ \mid |z|^2 = r^2 - 1\}, \\ &\pi(z,\alpha,r) = (z, re^{i\alpha}). \end{aligned}$$

For $(z,\alpha,r) \in \tilde{G}$ the positive real number r can be computed from z and α, hence we can also identify $\tilde{G}$ with $\mathbb{C} \times \mathbb{R}$ via $(z,\alpha,r) \mapsto (z,\alpha)$.

The map $\theta : \tilde{L} \to \tilde{G}$ can be described as

$$\theta(z, \alpha, r) = \left(\lambda^{-1} z, \alpha, \lambda^{-1} r\right) \quad \text{with} \quad \lambda = \sqrt{r^2 - |z|^2}.$$

4. The Elements of the Construction

For $g \in \tilde{G}$ let E_g resp. I_g be the connected component of $\pi^{-1}(\bar{E}_{\bar{g}})$ resp. $\pi^{-1}(\bar{I}_{\bar{g}})$ containing g, where $\bar{g} := \pi(g)$ is the image of g in G,

$$\bar{E}_{\bar{g}} := \{a \in L \mid \langle g, a\rangle = -1\}$$

is the intersection of L with the affine tangent space on G in the point $\bar{g}$ and

$$\bar{I}_{\bar{g}} := \{a \in L \mid \langle g, a\rangle \leqslant -1\}$$

is the intersection of L with the half-space of $\mathbb{C}^2$ bounded by $\bar{E}_{\bar{g}}$ and not containing 0. $\bar{E}_{\bar{g}}$ and $\bar{I}_{\bar{g}}$ are simply connected and even contractible, hence their pre-images under the covering map π consist of infinitely many connected components, one of them containing g.

The three-dimensional submanifold E_g subdivides $\tilde{L}$ in two connected components, the closure of one of them is I_g, and we denote the closure of the other by H_g. The boundary of I_g, resp. H_g, is equal to E_g.

As an example, for the unit elements $e = (0, 0, 1)$ in $\tilde{G}$ and $\bar{e} = \pi(e) = (0, 1)$ in G, we have

$$\bar{I}_{\bar{e}} = \{(z, w) \in \mathbb{C}^2 \mid \operatorname{Re}(w) \geqslant 1,\ |z| < |w|\},$$

the boundary $\bar{E}_{\bar{e}}$ of $\bar{I}_{\bar{e}}$ is a one-sheeted hyperboloid of revolution. The pre-image of $\bar{I}_{\bar{e}}$ is

$$\pi^{-1}(\bar{I}_{\bar{e}}) = \{(z, \alpha, r) \in \mathbb{C} \times \mathbb{R} \times \mathbb{R}_+ \mid r \cdot \cos\alpha \geqslant 1,\ |z| < r\}.$$

The connected components of $\pi^{-1}(\bar{I}_{\bar{e}})$ resp. $\pi^{-1}(\bar{E}_{\bar{e}})$ containing e are

$$I_e = \left\{(z, \alpha, r) \in \mathbb{C} \times \mathbb{R} \times \mathbb{R}_+ \mid |\alpha| < \frac{\pi}{2},\ r \geqslant \frac{1}{\cos\alpha},\ |z| < r\right\}$$

and

$$E_e = \left\{(z, \alpha, r) \in \mathbb{C} \times \mathbb{R} \times \mathbb{R}_+ \mid |\alpha| < \frac{\pi}{2},\ r = \frac{1}{\cos\alpha},\ |z| < r\right\}.$$

The subsets E_g resp. I_g have the analogous properties because $E_g = g \cdot E_e$ and $I_g = g \cdot I_e$.

We make use of the following construction (compare [Mil75]). Given a base-point $x \in \mathbb{D}$ and a real number t, let $\rho_x(t) \in \mathrm{PSU}(1,1)$ denote the rotation through angle t about the point x. Thus we obtain a homomorphism $\rho_x : \mathbb{R} \to \mathrm{PSU}(1,1)$, which clearly lifts to the unique homomorphism $r_x : \mathbb{R} \to \widetilde{\mathrm{SU}}(1,1)$ into the universal covering group. Since $\rho_x(2\pi) = \mathrm{Id}_{\mathbb{D}}$, it follows that the lifted element $r_x(2\pi)$ belongs to the central subgroup Z of $\widetilde{\mathrm{SU}}(1,1)$. Note that this element $r_x(2\pi) \in Z$ depends continuously on x, and therefore is independent of the choice of x. We easily compute $r_0(2t) = (0, -t, 1)$ and hence $r_x(2\pi) = r_0(2\pi) = (0, -\pi, 1)$ for all $x \in \mathbb{D}$. Moreover we obtain

$$r_0(2t) \cdot (z, \alpha, r) = (ze^{it}, \alpha - t, r).$$

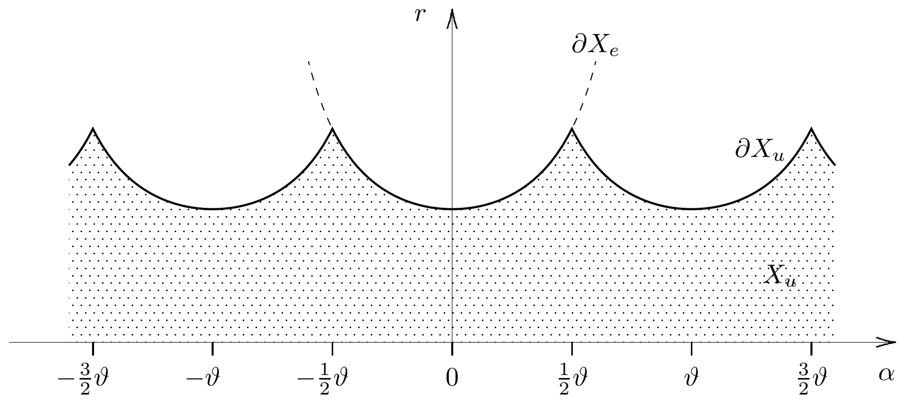

Figure 3: The image X_u of Q_u in the (α, r)-half-plane

Let Γ be a discrete subgroup of finite level k in $\widetilde{\mathrm{SU}}(1,1)$ and let $\overline{\Gamma}$ be the image of Γ in $\mathrm{PSU}(1,1)$. We assume the existence of a fixed point $u \in \mathbb{D}$ of $\overline{\Gamma}$. The isotropy group $\overline{\Gamma}_u$ of u in $\overline{\Gamma}$ is a finite cyclic group generated by $\rho_u(2\pi/p)$, where $p := |\overline{\Gamma}_u|$. The isotropy group Γ_u of u in Γ is a infinite cyclic group generated by $r_d := r_u(2\vartheta)$, where $\vartheta := \frac{\pi k}{p}$. We can assume without loss of generality that $u = 0 \in \mathbb{D}$. Under this assumption it follows

$$r_d = (0, -\vartheta, 1) \quad \text{and} \quad r_d \cdot (z, \alpha, r) = \left(ze^{i\vartheta}, \alpha - \vartheta, r\right).$$

An important assumption for the following construction is

$$p > k,$$

the order of u as a fixed point of $\overline{\Gamma}$ is greater than the level of the group. In terms of r_d this means that the argument ϑ of r_d is less then π.

Now let us start with the construction of fundamental domains for the action of Γ on $\tilde{G}$. For a point x in the orbit $\Gamma(u)$ let $T(x)$ be the left coset

$$T(x) = \{g \in \Gamma \mid g(u) = x\}$$

of the isotropy group Γ_u and let

$$Q_x = \bigcap_{g \in T(x)} H_g.$$

As an example, for $x = u$ we have $T(u) = \Gamma_u$, the infinite cyclic subgroup generated by the element $r_d = (0, -\vartheta, 1)$. The generator r_d acts on $\tilde{G}$ by left multiplication

$$r_d \cdot (z, \alpha, r) = (ze^{i\vartheta}, \alpha - \vartheta, r)$$

and it acts on the (α, r)-half-plane by the translation τ mapping (α, r) to $(\alpha - \vartheta, r)$. What does the set $Q_u = \cap_{m \in \mathbb{Z}} H_{r_d^m}$ look like? The images of the sets $H_{r_d^m} = r_d^m \cdot H_e$ under the projection $(z, \alpha, r) \mapsto (\alpha, r)$ are the translates $\tau^m(X_e)$ of the image

$$X_e = \{(\alpha, r) \in \mathbb{R} \times \mathbb{R}_+ \mid r \cdot \cos\alpha \leqslant 1 \ \text{ or } \ |\alpha| \geqslant \pi/2\}$$

of H_e. The manifold Q_u is a disc bundle over its image $X_u = \bigcap_{m \in \mathbb{Z}} \tau^m(X_e)$ in the (α, r)-plane. The shaded area in figure 3 is X_u.

The manifolds gQ_u play a central role in our construction. We want to explain the geometric nature of these objects. We have described Q_u as a disc bundle over the set X_u in the (α, r)-half-plane $\mathbb{R} \times \mathbb{R}_+$. We may describe $Q_u \subset \tilde{L} \subset \mathbb{C} \times \mathbb{R} \times \mathbb{R}_+$ as

$$Q_u = (\mathbb{C} \times X_u) \cap \tilde{L}.$$

We think of X_u as a universal covering of a punctured plane polygon. Consider the following diagram of covering maps

$$\begin{tikzcd} \mathbb{R} \times \mathbb{R}_+ \arrow[r, "\pi'"] \arrow[dr, "\pi"'] & \mathbb{C}^* \arrow[d, "\pi''"] \\ & \mathbb{C}^* \end{tikzcd}$$

where $\pi(\alpha, r) = re^{i\alpha}$ and $\pi'(\alpha, r) = r^{1/k}e^{i\alpha/k}$ and $\pi''(z) = z^k$. Consider the curve $\pi(\partial X_u)$. It is easy to see that this is a regular star polygon $\left\{\frac{2p}{k}\right\}$ when k is odd and a regular star polygon $\left\{\frac{p}{k}\right\}$ when k is even. (For the definition of a star polygon see for example H.S.M. Coxeter [Cox69], §2.8, pp. 36–38.) Therefore the curve $\pi'(\partial X_u)$ is a curvilinear $2p$-gon covering the star polygon once or twice. Let $P' \subset \mathbb{C}$ and $P = P_u \subset \mathbb{C}$ be the plane areas bounded by the curvilinear polygon $\pi'(\partial X_u)$ and by the star polygon $\pi(X_u)$. The images of X_u are the punctured plane polygons $\pi'(X_u) = P' \setminus \{0\}$ and $\pi(X_u) = P \setminus \{0\}$. We think of the product $\mathbb{C} \times P'$ as a 4-dimensional $2p$-gonal *prism.* $\mathbb{C} \times X_u$ is the universal covering of the pierced prism $\mathbb{C} \times (P' \setminus \{0\})$. The product $\mathbb{C} \times P \subset \mathbb{C}^2$ might be considered as a 4-dimensional *"star prism"*. Its axis $\mathbb{C} \times \{0\}$ does not meet $L \subset \mathbb{C} \times \mathbb{C}^*$. Therefore the universal covering $\pi : \tilde{L} \to L$ maps Q_u to the intersection of L with the star prism:

$$\pi(Q_u) = L \cap (\mathbb{C} \times P_u).$$

In the following lemma we prove some properties of the sets Q_x:

Lemma 1. *For a point $x \in \Gamma(u)$ the following holds:*

(i) For any point $(z, w) \in \pi(Q_x)$

$$|w| - |z| \leqslant |w - \bar{x}z| \leqslant f(|x|),$$

where

$$f(t) := \frac{\sqrt{1-t^2}}{\cos\frac{\vartheta}{2}}.$$

(ii) The set Q_x is a subgraph of a section in the bundle $\tilde{L} \cong \tilde{G} \times \mathbb{R}_+$, while its boundary is the graph of this section. This means that for some section $s : \tilde{G} \to \mathbb{R}_+$ in the bundle $\tilde{L}$

$$Q_x = \{(a, \lambda) \in \tilde{G} \times \mathbb{R}_+ \mid \lambda \leqslant s(a)\},$$
$$\partial Q_x = \{(a, \lambda) \in \tilde{G} \times \mathbb{R}_+ \mid \lambda = s(a)\}.$$

Proof. Our proof is in two steps. We first check the properties of Q_x in the case $x = u$. In this case the properties follow from the explicit description of the set Q_u. Then we use the fact that for any $x \in \Gamma(u)$ there is an element $g \in \Gamma$ such that $Q_u = g \cdot Q_x$ to prove the properties of Q_x for $x \neq u$.

Let us first describe explicitly the image X_u of the set Q_u in the (α, r)-plane $\mathbb{R}\times\mathbb{R}_+$. The set X_u is the shaded area in figure 3. It is a subgraph of a function $\mathbb{R} \to \mathbb{R}_+$. Let us denote this function by φ. We now describe the function φ explicitly. The function φ is periodic with period ϑ, hence it is sufficient to describe φ on $[-\vartheta/2, \vartheta/2]$. For $\alpha \in [-\vartheta/2, \vartheta/2]$ it holds

$$\varphi(\alpha) = \frac{1}{\cos\alpha}.$$

For any $\alpha \in \mathbb{R}$ it holds

$$\varphi(\alpha) \leqslant \frac{1}{\cos\frac{\vartheta}{2}}$$

(with equality for $\alpha = (2k+1)\vartheta/2$, $k \in \mathbb{Z}$).

Now let us verify the first assertion of the lemma. The inequality

$$|w| - |z| \leqslant |w - \bar{x}z|$$

follows from $|z| < |w|$ and $|x| < 1$. It remains to prove the second inequality.

Let us verify the first assertion of the lemma in the case $x = u$. (Recall that we assumed $u = 0$.) For $x = u = 0$ the second inequality in the first part of the lemma reduces to

$$|w| \leqslant \frac{1}{\cos\frac{\vartheta}{2}}$$

for any point $(z, w) \in \pi(Q_u)$. Let us consider a point $(z, w) \in \pi(Q_u)$ and its preimage $(z, \alpha, r) \in Q_u$. By definition of the map π it holds $w = re^{i\alpha}$. For the point $(z, \alpha, r) \in Q_u$ it holds $(\alpha, r) \in X_u$. The set X_u is the subgraph of the function φ, hence

$$r \leqslant \varphi(\alpha) \leqslant \frac{1}{\cos\frac{\vartheta}{2}}$$

for any point $(\alpha, r) \in X_u$. Hence

$$|w| = r \leqslant \frac{1}{\cos\frac{\vartheta}{2}}.$$

Let us verify the first assertion of the lemma for any x. Let us consider a point $x \in \Gamma(u)$ and an element $g \in \Gamma$ such that $g(x) = u$. Let $(a, b) \in G$ be the image of the element g under π. The element $(a, b) \in G$ corresponds to the matrix

$$\begin{pmatrix} \bar{b} & a \\ \bar{a} & b \end{pmatrix} \in \mathrm{SU}(1,1)$$

and acts on $\mathbb{D}$ by

$$(a, b) \cdot x = \frac{\bar{b}x + a}{\bar{a}x + b}.$$

The property $(a, b) \cdot x = u = 0$ implies $a = -\bar{b}x$. From $(a, b) \in G$ we conclude

$$-1 = |a|^2 - |b|^2 = |-\bar{b}x|^2 - |b|^2 = -|b|^2 \cdot (1 - |x|^2)$$

and hence

$$|b| = \frac{1}{\sqrt{1 - |x|^2}}.$$

Let us consider $(z,w)\in\pi(Q_x)$ and $(z',w')=g\cdot(z,w)\in\pi(Q_u)$. On the one hand $(z',w')\in\pi(Q_u)$ implies

$$|w'|\leqslant\frac{1}{\cos\frac{\vartheta}{2}}.$$

On the other hand

$$|w'|=|\bar{a}z+bw|=|-b\bar{x}z+bw|=\frac{1}{\sqrt{1-|x|^2}}\cdot|w-\bar{x}z|.$$

Hence

$$|w-\bar{x}z|\leqslant\frac{\sqrt{1-|x|^2}}{\cos\frac{\vartheta}{2}}.$$

Let us verify the second assertion of the lemma in the case $x=u$. For the set Q_u we can describe the corresponding section $s_u:\tilde{G}\to\mathbb{R}_+$ explicitly as

$$s_u(z,\alpha,r)=\frac{\varphi(\alpha)}{r}.$$

Let us verify the second assertion of the lemma for any x. Let us consider a point $x\in\Gamma(u)$ and an element $g\in\Gamma$ such that $Q_u=g\cdot Q_x$. Then the section $s_x:\tilde{G}\to\mathbb{R}_+$ is given by

$$s_x(a)=s_u(g\cdot a).\quad\square$$

Lemma 2. *The family $(Q_x)_{x\in\Gamma(u)}$ is locally finite in the sense that any point of $\tilde{L}$ has a neighbourhood intersecting only finitely many prisms Q_x.*

Proof. We prove that the family $(\pi(Q_x))_{x\in\Gamma(u)}$ is locally finite (in L). This fact implies the local finiteness of the family $(Q_x)_{x\in\Gamma(u)}$, since if a subset U of L has an empty intersection with $\pi(Q_x)$ then the intersection of the pre-image $\pi^{-1}(U)$ with Q_x is empty too. By lemma 1(i) for any point $x\in\Gamma(u)$ and any point $(z,w)\in\pi(Q_x)$ the difference $|w|-|z|$ is bounded from above by $f(|x|)$. The values $f(t)$ tend to zero as t tends to 1. Choosing a point $(z_0,w_0)\in L$ and a positive number $\varepsilon<|w_0|-|z_0|$, the neighbourhood $U:=\{(w,z)\in L\mid |w|-|z|>\varepsilon\}$ of the point (z_0,w_0) can intersect $\pi(Q_x)$ only for $|x|$ sufficiently small (so that $f(|x|)>\varepsilon$). But the group Γ is discrete, so there are only finitely many points x in $\Gamma(u)$ with norm $|x|$ under a given bound. This finishes the proof. $\square$

Remark. This property of Q_x allows us to deal with $P=\cup Q_x$ in a similar way as with a finite union of polytopes.

Lemma 3. *The family $(E_g\cap Q_{g(u)})_{g\in\Gamma}$ is locally finite.*

Proof. This is immediate from the local finiteness of the family $(Q_x)_{x\in\Gamma(u)}$ plus the easy observation that the family $(E_g\cap Q_{g(u)})_{g\in\Gamma_u}$ is locally finite. $\square$

We consider in $\tilde{L}$ the four-dimensional polytope

$$P:=\bigcup_{x\in\Gamma(u)}Q_x=\bigcup_{x\in\Gamma(u)}\bigcap_{g\in T(x)}H_g.$$

Lemma 4. *The projection $\partial P\to\tilde{G}$ is a Γ-equivariant homeomorphism.*

Proof. From lemma 1(ii) we know that the set Q_x is a subgraph of a section in the bundle $\tilde{L} \cong \tilde{G} \times \mathbb{R}_+$. The polyhedron $P = \cup Q_x$ inherits this property of the prisms Q_x as a union of a locally finite family of subgraphs. But for a subgraph of a section in the bundle $\tilde{L}$ it is clear that the bundle map $\tilde{L} \to \tilde{G}$ induces a homeomorphism from its boundary (equal to the graph of the section) onto $\tilde{G}$. This homeomorphism is Γ-equivariant since the projection $\tilde{L} \to \tilde{G}$ is Γ-equivariant. □

5. The Main Results

Now we can state the main result

Theorem A. *The boundary of P is invariant with respect to the action of Γ. The subset*

$$F_g = \mathrm{Cl}_{\partial P}(\mathrm{Int}(\partial H_g \cap \partial P))$$

is a fundamental domain for the action of Γ on ∂P. The family $(F_g)_{g\in\Gamma}$ is locally finite in ∂P. The projection $\tilde{L} \to \tilde{G}$ induces a Γ-equivariant homeomorphism

$$\partial P \to \tilde{G}.$$

The image $\mathcal{F}_g$ of F_g under the projection is a fundamental domain for the action of Γ on $\tilde{G}$. The family $(\mathcal{F}_g)_{g\in\Gamma}$ is locally finite. For every pair of elements $g, h \in \Gamma$ with $g \neq h$ the intersection $\mathcal{F}_g \cap \mathcal{F}_h$ lies in a totally geodesic submanifold of $\tilde{G}$.

Remark. In this section all closures are taken in ∂P. We use the shorthand Cl instead of $\mathrm{Cl}_{\partial P}$.

Lemma 5.

$$\mathrm{Int}\, F_g = \mathrm{Int}(E_g \cap \partial P) \quad \textit{and} \quad \mathrm{Cl}\,\mathrm{Int}\, F_g = F_g.$$

Proof. The assertions follow from Lemma 10(i) with $A = E_g \cap \partial P$. □

Proof. To prove that F_g is a fundamental domain we have to prove two properties. The first property is that the images of F_g have no common inner points, i.e. the intersection $\mathrm{Int}(F_g) \cap F_h$ is empty if $g \neq h$. The second property is that $\mathrm{Cl}(\cup_{g\in\Gamma} \mathrm{Int}\, F_g) = \partial P$, i.e. roughly speaking the images of F_g cover the whole space ∂P.

Let us first prove that the intersection $\mathrm{Int}(F_g) \cap F_h$ is empty if $g \neq h$. Suppose on the contrary that there are elements $g, h \in \Gamma$ such that $g \neq h$ and $\mathrm{Int}(F_g) \cap F_h \neq \varnothing$. Let us consider the closed subsets $A = E_g \cap \partial P$ and $B = E_h \cap \partial P$. By Lemma 5 it holds $\mathrm{Int}(F_g) = \mathrm{Int}\, A$, hence the assumption $\mathrm{Int}(F_g) \cap F_h \neq \varnothing$ can be rewritten as $\mathrm{Int}\, A \cap \mathrm{Cl}\,\mathrm{Int}\, B \neq \varnothing$. From Lemma 10(ii) it follows that $\mathrm{Int}(A \cap B) \neq \varnothing$. This means that the set $\mathrm{Int}(E_g \cap E_h \cap \partial P)$ is not empty. But since the totally geodesic submanifolds E_g and E_h intersect transversally, the intersection $E_g \cap E_h$ has no inner points in ∂P.

Since $F_g \subset E_g \cap Q_{g(u)}$ lemma 3 implies that the family $(F_g)_{g\in\Gamma}$ is locally finite in ∂P. Lemma 4 says that the projection $\partial P \to \tilde{G}$ is a Γ-equivariant homeomorphism.

Now let us prove the property $\mathrm{Cl}(\cup_{g\in\Gamma} \mathrm{Int}\, F_g) = \partial P$. Since

$$\mathrm{Cl}\,\Big(\bigcup_{g\in\Gamma} \mathrm{Int}\, F_g\Big) \supset \bigcup_{g\in\Gamma} \mathrm{Cl}\,\mathrm{Int}\, F_g = \bigcup_{g\in\Gamma} F_g$$

(where the last equality holds by Lemma 5), it suffices to prove that $\cup_{g\in\Gamma}F_g=\partial P$. Consider $a\in\partial P$. From the definition of P and local finiteness (according to Lemma 3) of the family $(E_g\cap Q_{g(u)})_{g\in\Gamma}$ it follows that in some neighbourhood of the point a only finitely many elements of Γ are relevant, i.e. there exists a neighbourhood U of the point a in $\tilde{L}$ and elements $g_1,\dots,g_n\in\Gamma$ such that

$$\partial P\cap U=\bigcup_{i=1}^{n}(E_{g_i}\cap\partial P\cap U).$$

We may assume without loss of generality that the map $\pi|_U : U\to\pi(U)$ is a homeomorphism. The image of $P\cap U$ under this homeomorphism is an intersection of an open subset of L with a finite union of finite intersections of half-spaces H_g with the property $a\in\partial H_g$. Suppose that $a\notin \operatorname{Cl}\operatorname{Int}(E_{g_i}\cap\partial P)=F_{g_i}$ for all $i\in\{1,\dots,n\}$. This is only possible if for each $i\in\{1,\dots,n\}$ the set $E_{g_i}\cap\partial P\cap U$ is contained in a 2-dimensional submanifold of $\tilde{L}$. Thus $\partial P\cap U$ is contained in the union of finitely many 2-dimensional submanifolds. On the other hand it follows from lemma 4 that ∂P is homeomorphic to a 3-dimensional manifold $\tilde{G}$. This contradiction implies that $a\in F_g$ for some $g\in\Gamma$. $\square$

Lemma 6. *The boundary ∂P of $P=\cup_{x\in\Gamma(u)}Q_x$ can be described as follows*

$$\partial P=\partial(\bigcup_{x\in\Gamma(u)}Q_x)=(\bigcup_{x\in\Gamma(u)}\partial Q_x)\backslash(\bigcup_{x\in\Gamma(u)}\operatorname{Int}Q_x).$$

This means that a point p is in the boundary of P if and only if p is not an interior point of any Q_x with $x\in\Gamma(u)$ and p is a boundary point of Q_x for some $x\in\Gamma(u)$.

Proof. From lemma 1(ii) we know that the set Q_x is a subgraph of a section s_x in the bundle $\tilde{L}\cong\tilde{G}\times\mathbb{R}_+$

$$Q_x=\{(a,\lambda)\in\tilde{G}\times\mathbb{R}_+\ \big|\ \lambda\leqslant s_x(a)\}.$$

The set $P=\cup Q_x$ is the subgraph of the section $s_P=\max s_x$. (In this proof max means $\max_{x\in\Gamma(u)}$, $\cup$ means $\cup_{x\in\Gamma(u)}$, $\exists\,x$ means $\exists\,x\in\Gamma(u)$ and so on.) This property would be obvious for a finite union of subgraphs. Using local finiteness (according to Lemma 2) we prove that this property also holds for P. But for a subgraph

$$X=\{(a,\lambda)\in\tilde{G}\times\mathbb{R}_+\ \big|\ \lambda\leqslant s(a)\}$$

of a section s in the bundle $\tilde{L}$ it is clear that $(a,\lambda)\in\partial X$ if and only if $\lambda=s(a)$. Hence

$$(a,\lambda)\in\partial P\iff\lambda=s_P(a).$$

By definition of s_P

$$\lambda=s_P(a)\iff(\exists\,x\quad\lambda=s_x(a))\quad\text{and}\quad(\forall\,x\quad\lambda\geqslant s_x(a)).$$

On the other hand

$$\begin{aligned}(a,\lambda)\in\cup\,\partial Q_x&\iff\exists\,x\quad\lambda=s_x(a),\\(a,\lambda)\notin\cup\operatorname{Int}Q_x&\iff\forall\,x\quad\lambda\geqslant s_x(a).\end{aligned}$$

$\square$

Lemma 7. $\operatorname{Int}F_e\subset\partial Q_u$.

Proof. By Lemma 5 it holds $\operatorname{Int} F_e = \operatorname{Int}(E_e \cap \partial P)$. Suppose that there is a point $a \in \operatorname{Int} F_e = \operatorname{Int}(E_e \cap \partial P)$ such that $a \notin \partial Q_u$. Since $a \in \partial P$ and $a \notin \partial Q_u$ there exists $x \in \Gamma(u)\backslash\{u\}$ such that $a \in \partial Q_x$. Then any neighbourhood of a intersects $E_e \cap \operatorname{Int} Q_x \subset E_e \backslash \partial P$. The projection $\theta : \tilde{L} \to \tilde{G}$ is continuous and the restriction $\theta|_{\partial P} : \partial P \to \tilde{G}$ is a homeomorphism, therefore any neighbourhood of a intersects $((\theta|_{\partial P})^{-1} \circ \theta)(E_e \backslash \partial P)) \subset \partial P \backslash E_e$. This implies $a \notin \operatorname{Int}(E_e \cap \partial P) = \operatorname{Int} F_e$. Contradiction. $\square$

Proposition 8.

$$F_e = \operatorname{Cl}\operatorname{Int}\left((E_e \cap \partial Q_u) - (\bigcup_{x \in \Gamma(u)\backslash\{u\}} \operatorname{Int} Q_x)\right).$$

Proof. Let $\hat{F} := (E_e \cap \partial Q_u) - (\cup_{x \in \Gamma(u)\backslash\{u\}} \operatorname{Int} Q_x)$. We claim that F_e and $\hat{F}$ coincide up to the boundary, i.e. $\operatorname{Int} F_e = \operatorname{Int} \hat{F}$. To prove this we show the inclusions $\operatorname{Int} F_e \subset \operatorname{Int} \hat{F}$ and $\operatorname{Int} \hat{F} \subset \operatorname{Int} F_e$. We first prove that $\operatorname{Int} F_e \subset \operatorname{Int} \hat{F}$. To that end we show that $\operatorname{Int} F_e \subset \hat{F}$. Then $\operatorname{Int}\operatorname{Int} F_e \subset \operatorname{Int} \hat{F}$ and $\operatorname{Int} F_e = \operatorname{Int}\operatorname{Int} F_e$ imply $\operatorname{Int} F_e \subset \operatorname{Int} \hat{F}$. To see that $\operatorname{Int} F_e$ is contained in $\hat{F}$ we have to show (by definition of $\hat{F}$) that $\operatorname{Int} F_e$ is contained in E_e, in ∂Q_u, and does not intersect $\operatorname{Int} Q_x$ for all $x \in \Gamma(u)\backslash\{u\}$. By definition of F_e it holds $\operatorname{Int} F_e \subset E_e$. By Lemma 7 it holds $\operatorname{Int} F_e \subset \partial Q_u$. Finally for any $x \in \Gamma(u)\backslash\{u\}$ it holds $F_e \cap \operatorname{Int} Q_x = \varnothing$ because of the fact that F_e is contained in ∂P, and $\partial P \cap \operatorname{Int} Q_x = \varnothing$ by Lemma 6. This implies $\operatorname{Int} F_e \subset \hat{F}$ and therefore $\operatorname{Int} F_e \subset \operatorname{Int} \hat{F}$. We now have to prove the inclusion $\operatorname{Int} \hat{F} \subset \operatorname{Int} F_e$. From the definition of $\hat{F}$ it follows that $\hat{F} \subset E_e$. Moreover $\hat{F} \subset \partial Q_u \subset (\cup_{x \in \Gamma(u)} \partial Q_x)$ and $\hat{F} \cap (\cup_{x \in \Gamma(u)\backslash\{u\}} \operatorname{Int} Q_x) = \varnothing$ imply by Lemma 6 that $\hat{F} \subset \partial P$. Now from $\hat{F} \subset E_e \cap \partial P$ it follows that $\operatorname{Int} \hat{F} \subset \operatorname{Int}(E_e \cap \partial P) = \operatorname{Int} F_e$, where the last equality holds by Lemma 5. We now have proved both inclusions, i.e. we know that $\operatorname{Int} \hat{F} = \operatorname{Int} F_e$. From this it follows that $\operatorname{Cl}\operatorname{Int} \hat{F} = \operatorname{Cl}\operatorname{Int} F_e = F_e$. $\square$

Lemma 9. *If Γ is co-compact, then F_g is compact.*

Proof. Consider a sequence a_k in $\operatorname{Int} F_g$. Let φ be the composition of the projection maps $\partial P \to \tilde{G}$ and $\tilde{G} \to \tilde{G}/\Gamma$. Since the quotient $\tilde{G}/\Gamma$ is compact we may assume without loss of generality that the sequence $\varphi(a_k)$ tends to a limit $\bar{a} \in \tilde{G}/\Gamma$. Since φ is surjective there exists a pre-image $a \in \partial P$ of $\bar{a}$ under φ. Hence there is a sequence h_k in Γ such that the sequence $h_k a_k$ tends to a. Since the family $(F_g)_{g \in \Gamma}$ is locally finite there exists a neighbourhood U of a that intersects only finitely many fundamental domains F_g. Therefore the set $\{h_k | k \in \mathbb{N}\}$ is finite. After choosing a subsequence we may assume that the sequence h_k is constant, say $h_k = h$. Then the sequence $h a_k$ tends to a, hence the sequence a_k tends to $h^{-1}a$. This implies $h^{-1}a \in F_g$. $\square$

Theorem B. *If Γ is co-compact then F_g is a compact polyhedron, i.e. a finite union of finite compact intersections of half-spaces I_a.*

Proof. The family $(Q_x)_{x \in \Gamma(u)}$ is locally finite and the fundamental domain F_e is compact by lemma 9. From this it follows that there is a finite subset $E \subset \Gamma(u)$ such that $F_e \cap Q_x = \varnothing$ for all $x \in \Gamma(u)\backslash E$. By proposition 7 this implies the assertion. $\square$

6. Examples

We have computed the fundamental domains explicitly for those infinite series of discrete subgroups, which correspond via the construction of I. Dolgachev [Dol83] to the Arnold series E, Z and Q of quasi-homogeneous surface singularities. In particular the quotient of $\widetilde{\mathrm{SU}}(1,1)$ by one of this groups is diffeomorphic to the link of the corresponding quasi-homogeneous singularity. Whenever it is convenient, we shall denote these subgroups also by the symbols E_n, Z_n, Q_n.

A discrete co-compact subgroup Γ of level k in $\widetilde{\mathrm{SU}}(1,1)$ such that the image in $\mathrm{PSU}(1,1)$ is a triangle group with signature $(\alpha_1,\alpha_2,\alpha_3)$ will be denoted by $\Gamma(\alpha_1,\alpha_2,\alpha_3)^k$. The subgroups E_n, Z_n, Q_n are of this type. The level k and the signature $(\alpha_1,\alpha_2,\alpha_3)$ are given in the following table (compare K. Möhring [Möh], table 19).

Type	$n \bmod 4$	k	$(\alpha_1,\alpha_2,\alpha_3)$
E_n	0	$(n-10)/2$	$(2,3,k+6)$
	2	$(n-10)/4$	$(3,3,k+3)$
	$1,3$	$(n-10)/3$	$(2,4,k+4)$
Z_n	3	$(n-9)/2$	$(2,3,2k+6)$
	1	$(n-9)/4$	$(3,3,2k+3)$
	$0,2$	$(n-9)/3$	$(2,4,2k+4)$
Q_n	2	$(n-8)/2$	$(2,3,3k+6)$
	0	$(n-8)/4$	$(3,3,3k+3)$
	$1,3$	$(n-8)/3$	$(2,4,3k+4)$

The following figures show some of the explicitly computed fundamental domains. We restrict ourselves here to the series E. For further figures of fundamental domains, for instance for the series Z and Q, for computations of fundamental domains for quadrangle rather than triangle subgroup, and for deeper discussion of connections with quasi-homogeneous surface singularities we refer to [BPR03], and also to [Rot01] and [Pra01].

Some explanations are required to make the figures of fundamental domains comprehensible. The image $\pi(F_e)$ of the fundamental domain F_e for a discrete co-compact group $\Gamma \subset \widetilde{\mathrm{SU}}(1,1)$ of finite level is a compact polyhedron in $\mathfrak{su}(1,1)$ with flat faces. The Lie algebra $\mathfrak{su}(1,1)$ is a 3-dimensional flat Lorentz space of signature $(n_+,n_-)=(2,1)$. Such a polyhedron has a distinguished rotational axis of symmetry. The direction of this axis is negative definite, and the orthogonal complement is positive definite. Changing the sign of the pseudo-metric in the direction of the rotational axis transforms Lorentz space into a well-defined Euclidean space. The image $\pi(F_e)$ of the fundamental domain is then transformed into a polyhedron in Euclidean space with dihedral symmetry. Figure 4 shows the Euclidean polyhedra obtained in this way. The direction of the rotational axis is vertical. The top and bottom faces are removed.

The polyhedra in figure 4 are all scaled by the same factor to illustrate the proportions between different fundamental domains.

Figure 5 illustrates the identification scheme for the cases E_{10+2n}. The face identification is equivariant with respect to the dihedral symmetry of the polyhedron.

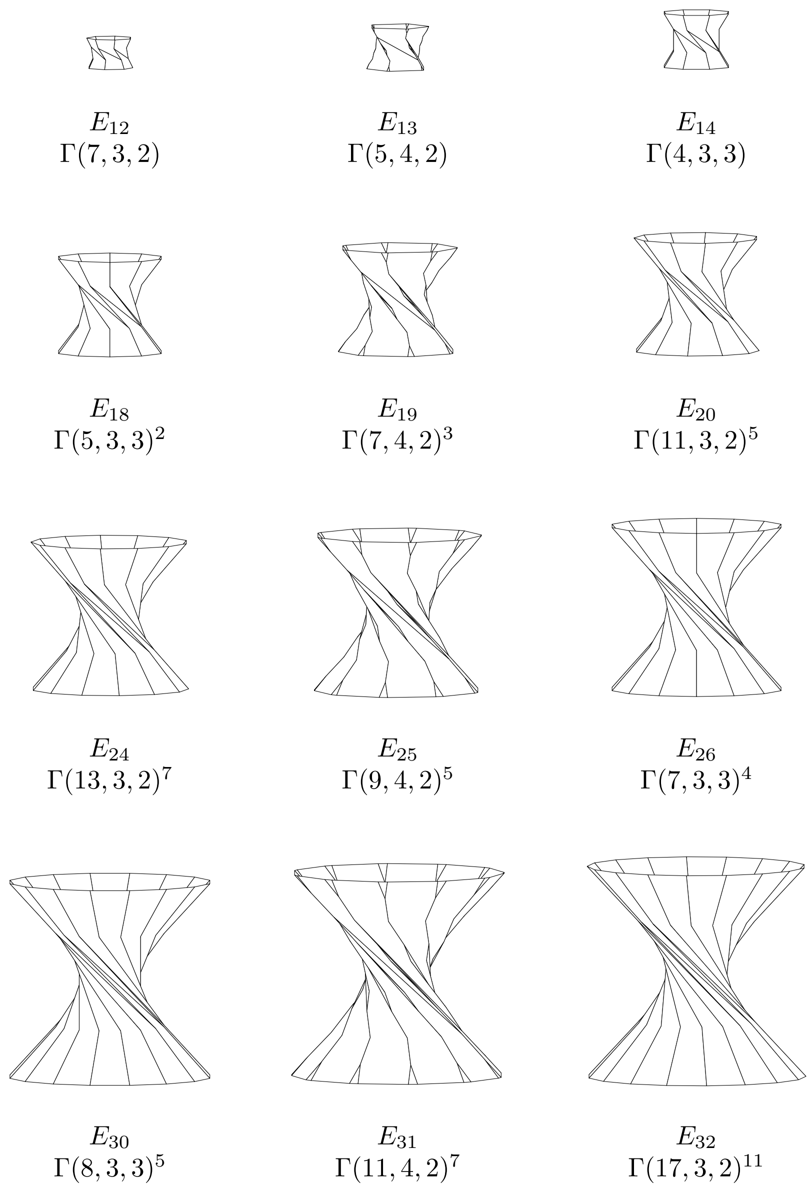

Figure 4: Fundamental domains for the beginning of the E-series

The faces shaded in the same way are identified. Arrows on the edges of shaded faces indicate the identified flags (face, edge, vertex).

7. Concluding Remarks

1) The construction of fundamental domains in the flat Lorentz case using crooked planes by T. Drumm (see papers [DG95], [DG99] by T. Drumm and W. Goldman and references therein) is the only other fundamental domain construction

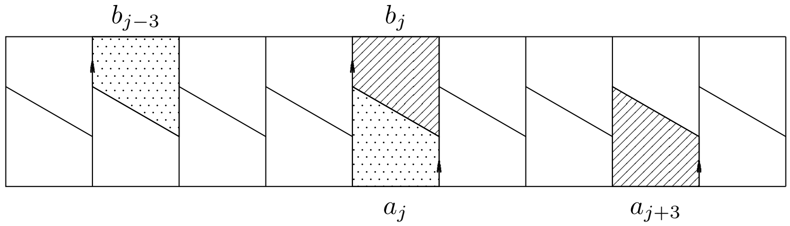

Figure 5: Identification scheme for E_{10+2n}

in the non-Riemannian pseudo-Riemannian case we are aware of, besides the construction for $\widetilde{\mathrm{SU}}(1,1)$ presented in this paper and the construction for $\mathrm{PSU}(1,1)$ studied before in [Fis92], [KNRS96], [BKN+98].

2) The idea of projection of an affine construction with half-planes onto a quadric is also used in the algorithmic construction of Voronoi diagrams for (finite) point sets in the Euclidean and hyperbolic plane, compare J.-D. Boissonnat and M. Yvinec [BY98], Part V.

3) The Lorentz space forms of the form $\tilde{G}/\Gamma$, where $\tilde{G}=\widetilde{\mathrm{SU}}(1,1)$ and Γ is a discrete subgroup of $\tilde{G}$, are standard. A Lorentz space form is called standard if it is a quotient of $\tilde{G}=\widetilde{\mathrm{SU}}(1,1)$ by a discrete subgroup of $\mathrm{Isom}(\tilde{G})$ conjugate to a subgroup of $J=\langle J_0,\eta\rangle$, where
$$J_0=(\tilde{G}\times K)/\Delta_Z\subset \mathrm{Isom}_0(\tilde{G})$$
and $K=\{r_0(t)\}_{t\in\mathbb{R}}$. We recall that
$$\mathrm{Isom}(\tilde{G})=\langle \mathrm{Isom}_0(\tilde{G}),\eta,\varepsilon\rangle$$
and
$$\mathrm{Isom}_0(\tilde{G})\cong(\tilde{G}\times\tilde{G})/\Delta_Z,$$
see section 3. We can think of a discrete subgroup Γ of $\tilde{G}$ acting by left translations as a discrete subgroup of $(\tilde{G}\times\{1\})/\Delta_Z\subset J_0$, so the Lorentz space forms studied in this paper are standard.

We would like to generalize our fundamental domain construction for the case of other Lorentz space forms, at least for standard ones.

The standard Lorentz space forms were studied by R.S. Kulkarni and F. Raymond [KR85]. Examples of non-standard Lorentz space forms were found by W. Goldman [Gol85], É. Ghys [Ghy87], and recently by F. Salein [Sal00]. The survey [BZ04] of Th. Barbot and A. Zeghib and the paper [Fra05] of Ch. Frances are good references for the reader interested in group actions on Lorentz manifolds.

An interesting class of subgroups of $\mathrm{Isom}_0(\tilde{G})$ are the subgroups of the form $(\Gamma\times\Phi)/\Delta_Z$, where Γ is a discrete subgroup of $\tilde{G}$ and Φ is a discrete subgroup of K. Our fundamental domain construction can be modified to work for the subgroups of this special form. This modified construction will be studied in a forthcoming paper.

On the other hand this class of subgroups corresponds to an interesting class of singularities. The is a 1-1-correspondence between the subgroups from this class and quasi-homogeneous $\mathbb{Q}$-Gorenstein surface singularities. This result is proved in [Pra05] and generalizes the work of I. Dolgachev [Dol83] on the correspondence between the subgroups of the form $\Gamma \times \{1\}$ and quasi-homogeneous Gorenstein surface singularities.

4) One can also think of generalizations of the fundamental domain construction in the cases of other (pseudo-)Riemannian quadrics in pseudo-Euclidean spaces. In section 2 we discussed the analogous constructions for two one-dimensional cases. It would be interesting to generalize the constructions for higher dimensional quadrics.

8. Some facts from general topology

Lemma 10. *Let X be a topological space. Let A and B be closed subsets of X. Then*

(i) $\operatorname{Int}\operatorname{Cl}\operatorname{Int} A = \operatorname{Int} A$,
(ii) $\operatorname{Int} A \cap \operatorname{Cl}\operatorname{Int} B \neq \varnothing \Rightarrow \operatorname{Int}(A \cap B) \neq \varnothing$.

Proof. The assertion (i) is a well known fact, a part of the proof of Kuratowski's Closure-Complement Problem. We now give the proof of the assertion (ii). Let us assume that the set $\operatorname{Int} A \cap \operatorname{Cl}\operatorname{Int} B$ is not empty and choose a point p in this set. The point p is contained in $\operatorname{Int} A$, hence there is an open neighbourhood U of p contained in A. Moreover p is contained in $\operatorname{Cl}\operatorname{Int} B$, hence any neighbourhood of p intersects $\operatorname{Int} B$, in particular $U \cap \operatorname{Int} B \neq \varnothing$. Let us consider a point $q \in U \cap \operatorname{Int} B$. The point q is contained in $\operatorname{Int} B$, hence there is an open neighbourhood V of q contained in B. Hence the point q has an open neighbourhood $U \cap V$, which is contained in $A \cap B$. This means that the point q is contained in $\operatorname{Int}(A \cap B)$, i.e. this set is not empty. □

References

[BKN+98] Ludwig Balke, Alexandra Kaess, Ute Neuschäfer, Frank Rothenhäusler, and Stefan Scheidt, *Polyhedral fundamental domains for discrete subgroups of* $\mathrm{PSL}(2,\mathbb{R})$, Topology **37** (1998), 1247–1264.

[BPR03] Egbert Brieskorn, Anna Pratoussevitch, and Frank Rothenhäusler, *The Combinatorial Geometry of Singularities and Arnold's Series E, Z, Q*, Moscow Mathematical Journal **3** (2003), no. 2, 273–333, the special issue dedicated to Vladimir I. Arnold on the occasion of his 65th birthday.

[BY98] Jean-Daniel Boissonnat and Mariette Yvinec, *Algorithmic geometry*, Cambridge University Press, Cambridge, 1998, Translated from the 1995 French original by Hervé Brönnimann.

[BZ04] Thierry Barbot and Abdelghani Zeghib, *Group actions on Lorentz spaces, mathematical aspects: a survey*, The Einstein equations and the large scale behavior of gravitational fields, Birkhäuser, Basel, 2004, pp. 401–439.

[Cox69] H. S. M. Coxeter, *Introduction to geometry*, second ed., John Wiley & Sons Inc., New York, 1969.

[DG95] Todd A. Drumm and William M. Goldman, *Crooked planes*, Electron. Res. Announc. Amer. Math. Soc. **1** (1995), no. 1, 10–17 (electronic).

[DG99] ———, *The geometry of crooked planes*, Topology **38** (1999), no. 2, 323–351.

[Dol83] Igor V. Dolgachev, *On the Link Space of a Gorenstein Quasihomogeneous Surface Singularity*, Math. Ann. **265** (1983), 529–540.

[Fis92] Thomas Fischer, *Totalgeodätische Polytope als Fundamentalbereiche von Bewegungsgruppen der dreidimensionalen Minkowskischen Pseudosphäre*, Ph.D. thesis, Universität Bonn, 1992.

[Fra05] Charles Frances, *Lorentzian kleinian groups*, Comment. Math. Helv. **80** (2005), 883–910.

[Ghy87] Étienne Ghys, *Flots d'Anosov dont les feuilletages stables sont différentiables*, Ann. Sci. École Norm. Sup. (4) **20** (1987), no. 2, 251–270.

[Gol85] William M. Goldman, *Nonstandard Lorentz space forms*, J. Differential Geom. **21** (1985), no. 2, 301–308.

[KNRS96] Alexandra Käss, Ute Neuschäfer, Frank Rothenhäusler, and Stefan Scheidt, *Fundamentalbereiche diskreter, cokompakter Untergruppen von* $\mathrm{PSU}(1,1)$, Diploma thesis, Universität Bonn, 1996, (joint work combining four Diploma thesises).

[KR85] Ravi S. Kulkarni and Frank Raymond, *3-dimensional Lorentz space forms and Seifert fiber spaces*, Journal of Differential Geometry **21** (1985), 231–268.

[Mil75] John Milnor, *On the 3-dimensional Brieskorn manifolds* $M(p,q,r)$, Knots, groups and 3-manifolds (L. P. Neuwirth, ed.), Annals of Math. Studies, vol. 84, Princeton University Press, Princeton, 1975, pp. 175–225.

[Möh] Konrad Möhring, *Numerical invariants and series of quasihomogeneous singularities*, preprint.

[Neu77] Walter D. Neumann, *Brieskorn complete intersections and automorphic forms*, Invent. Math. **42** (1977), 285–293.

[Neu83] ———, *Geometry of quasihomogeneous surface singularities*, Proceedings of the summer institute on singularities held at Humboldt State University, Arcata, Calif., 1981 (P. Orlik, ed.), Proceedings of Symposia in Pure Mathematics, vol. 40, 2, American Math. Soc., Providence, 1983, pp. 245–258.

[Pra01] Anna Pratoussevitch, *Polyedrische Fundamentalbereiche diskreter Untergruppen von* $\widetilde{\mathrm{SU}}(1,1)$, Bonner Mathematische Schriften [Bonn Mathematical Publications], 346, Universität Bonn, Mathematisches Institut, Bonn, 2001, Dissertation, Rheinische Friedrich-Wilhelms-Universität Bonn, Bonn, 2001.

[Pra05] ———, *On the Link Space of a* $\mathbb{Q}$*-Gorenstein Quasi-Homogeneous Surface Singularity*, to appear in the Proceedings of the VIII São Carlos Workshop in Real and Complex Singularities in Luminy, 2005.

[Rot01] Frank Rothenhäusler, *Fundamentalpolyeder zu Fuchsschen Gruppen der Signatur* $(0,4;p,2,2,2)$, Bonner Mathematische Schriften [Bonn Mathematical Publications], 343, Universität Bonn, Mathematisches Institut, Bonn, 2001, Dissertation, Rheinische Friedrich-Wilhelms-Universität Bonn, Bonn, 2001.

[Sal00] Francois Salein, *Variétés anti-de Sitter de dimension 3 exotiques*, Ann. Inst. Fourier (Grenoble) **50** (2000), no. 1, 257–284.

Mathematisches Institut, Universität Bonn, Beringstrasse 1, 53115 Bonn

E-mail address: anna@math.uni-bonn.de