THE COMBINATORIAL GEOMETRY OF SINGULARITIES AND ARNOLD'S SERIES E, Z, Q

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To Vladimir Igorevich Arnold
With affection and admiration

Etenim cum multae figurae solidae, quae varias superficies habeant, cogitatione fingi possint, imprimis tamen respiciendae sunt eae quae ordinatae esse videntur.

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ABSTRACT. We consider discrete subgroups Γ of the simply connected Lie group $\widetilde{\mathrm{SU}}(1,1)$ of finite level. This Lie group has the structure of a 3-dimensional Lorentz manifold coming from the Killing form. Γ acts on $\widetilde{\mathrm{SU}}(1,1)$ by left translations. We want to describe the Lorentz space form $\Gamma\backslash\widetilde{\mathrm{SU}}(1,1)$ by constructing a fundamental domain F for Γ. We want F to be a polyhedron with totally geodesic faces. We construct such F for all Γ satisfying the following condition: The image $\bar{\Gamma}$ of Γ in $\mathrm{PSU}(1,1)$ has a fixed point u in the unit disk of order larger than the level of Γ. The construction depends on Γ and Γu.

For co-compact Γ the Lorentz space form $\Gamma\backslash\widetilde{\mathrm{SU}}(1,1)$ is the link of a quasi-homogeneous Gorenstein singularity. The quasi-homogeneous singularities of Arnold's series E, Z, Q are of this type. We compute the fundamental domains for the corresponding group. They are represented by polyhedra in Lorentz 3-space shown on tables 1–13. Each series exhibits a regular characteristic pattern of its combinatorial geometry related to classical uniform polyhedra.

1. INTRODUCTION

1.1 Between 1972 and 1976 Vladimir Igorevich Arnold published a very important series of articles on the classification of singularities of functions. The series began with a beautiful paper in Функциональный анализ и его приложения entitled *Normal forms of functions near degenerate critical points, the Weyl groups of A_k, D_k, E_k and Lagrangian singularities*, [1]. In this paper Arnold introduced the notion of a simple singularity. A simple singularity is one which does not have moduli. It has a normal form not involving any continuous parameters. The main result of the paper was the classification of all simple singularities of functions. The

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classification was given in the form of a complete list of normal forms as follows:

$$\begin{aligned}
A_k: &\quad f = \pm x_1^{k+1} \pm x_2^2 + Q, && k \geqslant 1,\\
D_k: &\quad f = x_1^2 x_2 \pm x_2^{k-1} + Q, && k \geqslant 4,\\
E_6: &\quad f = x_1^3 \pm x_2^4 + Q,\\
E_7: &\quad f = x_1^3 + x_1 x_2^3 + Q,\\
E_8: &\quad f = x_1^3 + x_2^5 + Q.
\end{aligned}$$

where Q is a standard nondegenerate quadratic form in the remaining variables $x_3, \ldots, x_n$. These are real normal forms. In the complex analytic case one can ignore the signs, so that there is just one normal form for each type A_k, D_k, E_k.

At the time when Arnold published this list of normal forms for the simple singularities which he had just introduced in 1972, these forms, or at least some of them, were exactly 100 years old. They first occur in a paper by H. A. Schwarz which appeared in 1872 in Crelles Journal [82]. The title was: *Ueber diejenigen Fälle, in welchen die Gaussische hypergeometrische Reihe eine algebraische Function ihres vierten Elementes darstellt.* The problem indicated in the title and solved by Schwarz goes back to Riemann. In a manuscript about minimal surfaces written around 1860 and published in 1867 after Riemann's death, Riemann not only pointed to the relevance of the problem, but also indicated how to solve the analytic problem by means of geometry, [77], p. 296. The quotient $s = y_1/y_2$ of two linearly independent solutions of a hypergeometric differential equation defines a multivalued map from the Riemann sphere to the Riemann sphere. The upper half-plane is mapped to a spherical triangle. Its angles are $\pi(1-c)$, $\pi(a+b-c)$ and $\pi(a-b)$, where a, b and c are the parameters of the hypergeometric differential equation. The lower half-plane is mapped to a reflected triangle, and the whole range of the function s is covered by the triangles obtained by iterated reflections, which are permuted by the monodromy group of the differential equation.

The function s is algebraic if and only if this covering is finite. The interesting cases where this occurs are those where the triangles are bounded by symmetry planes of a regular polyhedron inscribed in the sphere. We consider the case where they are fundamental triangles for the full symmetry group. So they are spherical triangles with angles π/p, π/q, π/r, where (p,q,r) equals $(2,3,3)$, $(2,3,4)$ and $(2,3,5)$ for the tetrahedron, octahedron and icosahedron respectively.

The inverse map from the triangle to the half-plane is described by a rational function of s invariant under the triangle group of orientation preserving symmetries. There are three natural relative invariants of this group, namely the polynomials whose zeroes are the orbit of a vertex of the triangle. In case of the icosahedron, these are absolute invariants. In view of their degree, Schwarz denotes them by φ_{12}, φ_{20} and φ_{30}. There is a basic relation between these three invariants, written by Schwarz in the following form:

$$[\varphi_{20}(s)]^3 - 4^3 \cdot 3^3 \cdot [\varphi_{12}(s)]^5 = [\varphi_{30}(s)]^2.$$

This is essentially the equation of the E_8-singularity of Arnold's list for the case of three variables, and we see that from the very beginning there was a close relation between these singularities and the symmetry of regular polyhedra.

1.2 In the next years, the subject was carried on by Felix Klein in a series of articles and in his famous book *Vorlesungen über das Ikosaeder*, which appeared in 1884.

There is a very nice new edition of this book with an introduction and commentaries by Peter Slodowy [51]. Klein considered not only the symmetry groups of the regular polyhedra, but also the corresponding binary polyhedral groups obtained by passing from $\mathrm{PSU}(2)$ to its universal cover $\mathrm{SU}(2)$. The finite subgroups of $\mathrm{SU}(2)$ are the cyclic groups, the binary dihedral groups and the binary tetrahedral, octahedral and icosahedral groups $\mathbf{T}$, $\mathbf{O}$ and $\mathbf{I}$. In Chapter III, § 1 of his book Klein determined the polynomials in two variables invariant under these groups. He found that for any of these groups Γ, the ring of invariant polynomials $\mathbb{C}[u,v]^\Gamma$ is of the form $\mathbb{C}[x,y,z]/(f)$. The polynomial f describing the basic relation between the three generators x, y, z is exactly one of Arnold's list, or it is easily transformed into one of Arnold's normal forms for $n=3$. The correspondence is as follows: cyclic groups correspond to A_k, binary dihedral groups to D_k, and binary tetrahedral, octahedral and icosahedral groups $\mathbf{T}$, $\mathbf{O}$, $\mathbf{I}$ to E_6, E_7, E_8.

Klein's result was rediscovered around 1960 as a result of an exchange of ideas between Friedrich Hirzebruch and Patrick DuVal, see [28]. In geometric terms it means that the affine algebraic surface described by the equation $f(x,y,z)=0$ is the quotient surface $\mathbb{C}^2/\Gamma$.

1.3 Therefore, the link of the singular point of this surface has the structure of the 3-dimensional spherical space form $\Gamma\backslash\,\mathrm{SU}(2)=\Gamma\backslash\mathbb{S}^3$, where we identify $\mathrm{SU}(2)$ with the group $\mathbb{S}^3$ of unit quaternions. It is natural to describe these spherical space forms by means of a fundamental domain for Γ acting on $\mathbb{S}^3$ by left translations. This has been done by Seifert and Threlfall in a paper [87] on 3-dimensional spherical space forms published in two parts in 1930 and 1932. Perhaps the simplest way of stating their result would be to say that the Dirichlet cell of the unit element of $\Gamma\subset\mathbb{S}^3$ is a fundamental domain for Γ. It is a spherical polyhedron with totally geodesic faces which Seifert and Threlfall determine explicitly for each of the groups Γ. However, this way of stating the result does not suggest how to pass from the spherical geometry of $\mathrm{SU}(2)$ to the Lorentz geometry of $\mathrm{SU}(1,1)$, and it also does not do justice to the beautiful classical geometry of the spherical case.

Recall that in the years 1850–1852 Ludwig Schläfli wrote a most remarkable treatise entitled *Theorie der vielfachen Kontinuität* which, alas, was published only six years after his death in 1901, [81]. In section 17 of that treatise Schläfli classified the 4-dimensional regular convex polytopes. There are six of them. Their Schläfli symbols are:

$$(3,3,3),\quad (3,3,4),\quad (4,3,3),$$
$$(3,4,3),\quad (3,3,5),\quad (5,3,3).$$

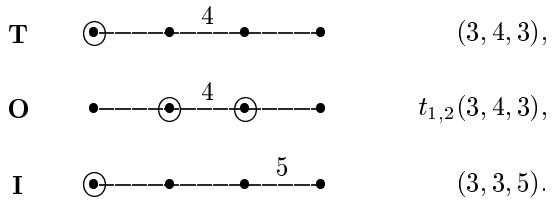
The first three of them are the analogues of the tetrahedron, octahedron and cube, which exist in every dimension. The other three are particular for dimension 4. Two of them, $(3,3,5)$ and $(5,3,3)$ may be seen as analogues of the icosahedron and dodecahedron. Their maximal faces are as follows: $(3,3,5)$ has 600 tetrahedra, and $(5,3,3)$ has 120 dodecahedra. They are dual to each other.

For $(3,4,3)$ and $(3,3,5)$ the vertices can be taken to be the elements of one of the finite groups $\Gamma\subset\mathbb{S}^3$. The polytope $(3,4,3)$ has vertex set $\mathbf{T}$, and $(3,3,5)$ has vertex set $\mathbf{I}$. The dual circumscribed polytopes of type $(3,4,3)$ and $(5,3,3)$ have $\mathbf{T}$ and $\mathbf{I}$ as sets of centres of their octahedral and dodecahedral faces. Thus it is clear that by central projection onto $\mathbb{S}^3$ we get a tiling of $\mathbb{S}^3$ by Dirichlet cells which are spherical octahedra or dodecahedra.

In order to deal with the binary octahedral group **O** we have to consider not only regular, but also semi-regular polytopes, in the same way as Greek mathematicians like Pappus of Alexandria admitted not only Platonic, but also Archimedian solids. Their generalization to higher dimensions may be defined as follows. A convex polytope is uniform, if it satisfies the following two conditions:

(a) The symmetry group acts transitively on the set of vertices.
(b) All facets are uniform.

To start the induction, one has to say what (b) means for the lowest dimensions: a convex polygon is uniform if it is regular. Some authors call uniform polytopes synonymously Archimedian. In dimension 3 the uniform convex polytopes are the 5 Platonic solids, the 13 Archimedian solids and the regular prisms and anti-prisms added to this list by Johannes Kepler in his wonderful book *Harmonice mundi* [49], p. 73. In dimensions larger than 4, there is no complete classification. In dimension 4 the uniform convex polytopes were enumerated by J. H. Conway in joint work with M. T. J. Guy [20]. Most of them can be obtained by applying Wythoff's construction to the 4-dimensional reflection groups as described by H. S. M. Coxeter [21], [22]. In particular, this applies to the convex polytope with vertices **O**. It is obtained by mutual truncation from the two 24-cells of type $(3,4,3)$ whose vertices are the two cosets of **T** in **O**. Here are the Wythoff constructions for the three Archimedian solids with vertex sets **T**, **O** and **I**:



The maximal faces of $t_{1,2}(3,4,3)$ are Archimedian polyhedra obtained from a cube, truncated by an octahedron. Their faces are regular octagons and triangles. The three Archimedian polyhedra belonging to **I**, **O** and **T**, i.e. to E_8, E_7, E_6 are shown in the left column of Table 1. The other figures of that table indicate what we intend to show in this paper.

We intend to show that the tilings of the spherical space $\mathrm{SU}(2)$ coming from the three Archimedian polytopes described above are at the root of three infinite series of tilings of the Lorentz manifold $\widetilde{\mathrm{SU}}(1,1)$ related to Arnold's series E, Z, Q.

1.4 The idea to try something of this kind occurred to one of us many years ago. In 1974 the beautiful results of Arnold and his students were to be presented at the ICM in Vancouver, [4]. Since Arnold was not allowed to travel for political reasons, the task of presenting his work fell to E. Brieskorn. Since that time, Arnold's discoveries have been a source of inspiration for him and his students and coworkers as well as many other mathematicians.

Let us very briefly recall some of the results presented in Vancouver. For details, we have to refer to the series of three articles in Успехи математических наук [3], [5], [6], which also show the rich mathematical context in which this work has evolved. Some part of the history preceding Arnold's work, especially the establishment of the relation between the simple singularities and the simple complex Lie groups of type A_k, D_k, E_6, E_7, E_8 has been described in [18]. For further reading

on this subject, we refer to the literature quoted in Peter Slodowy's foreword to the new edition of the lectures on the icosahedron and to [19]. We also refer to Arnold's article in Inventiones [7] and to the two books [8], [10].

Arnold classified singularities of functions up to right equivalence, and terms such as *number of moduli*, *0-modular* or *1-modular* refer to classes in this sense. Arnold found that the classification of singularities with a small number of moduli is "nice". This applies in particular to the 0-modular and 1-modular singularities, where several possible aspects contribute to the impression that we understand these classes of singularities. One of these aspects is the arithmetic of the quadratic form of the Milnor fibres associated to these singularities. The Milnor fibration of a complex hypersurface singularity is an important part of the differential topology of such singularities. It was introduced by John Milnor in 1966 and published in [55] in the course of developments described in [18]. Another aspect refers to ways of generating or constructing the singularities. It turned out that all the 0- and 1-modular singularities which Arnold found by analyzing the defining polynomial forms have constructions involving discrete groups of transformations of complex curves and surfaces. It is this relation to beautiful classical mathematics which Arnold must have had in mind when, after describing the relation between Platonic solids, simple Lie groups and simple singularities, he wrote in [4]:

> As we will see now, the classification of more and complex singularities provides new wonderful coincidences, where Lobatchevski triangles and automorphic forms take part.

Arnold's classification of 0- and 1-modular singularities is summarized in the following theorem.

Theorem.

(1) *The 0-modular singularities are the simple singularities* A_k, $k \geqslant 1$ *and* D_k, $k \geqslant 4$ *and* E_6, E_7, E_8.
(2) *The 1-modular singularities are (up to stable equivalence) those listed below:*
   (a) *The simply elliptic singularities* $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$.
   (b) *The cusp singularities* $T_{p,q,r}$, $1/p + 1/q + 1/r < 1$.
   (c) *The fourteen exceptional one parameter families* E_{12}, E_{13}, E_{14}, Z_{11}, Z_{12}, Z_{13}, Q_{10}, Q_{11},Q_{12}, S_{11}, S_{12}, W_{12}, W_{13}, U_{12}.

Arnold describes these singularities by normal forms of the corresponding functions. The normal forms of the 0-modular singularities and of the 1-modular singularities of type (a) are quasi-homogeneous. Those of type (b) are not quasi-homogeneous. The normal forms of type (c) are semi-quasihomogeneous. They are forms $f = g + ah$, where g is quasi-homogeneous of a certain integral weight d, h is a monomial of weight $d + 2$ and a is a real or complex parameter. Thus each of the 14 families contains exactly one quasi-homogeneous singularity, the one with $a = 0$.

Note that for some singularities we use symbols different from those originally introduced by Arnold. For the simply elliptic ones we use the symbols $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$ introduced 1974 by K. Saito in [80]. Arnold's symbols are $T_{3,3,3}$, $T_{2,4,4}$, $T_{2,3,6}$ or P_8, X_9, J_{10}. Our exceptional E_{12}, E_{13}, E_{14} were originally denoted by K_{12}, K_{13}, K_{14}, but in 1975 Arnold adopted himself the new notation E_{12}, E_{13}, E_{14}. For the singularities which Arnold denotes by $J_{k,0}$ we shall use $E_{k,0}$, because our work shows that they fit into the E-series in the same way as $Z_{k,0}$ fits into the Z-series and $Q_{k,0}$ into the Q-series.

The distinction between the various cases in the theorem above is reflected by properties of the quadratic form on the second homology group of the Milnor fibre of the corresponding complex surface singularity. Let μ be the rank of the lattice, i.e. the Milnor number. One has $\mu = \mu_+ + \mu_- + \mu_0$, where μ_0 is the rank of the radical and μ_+ the rank of a maximal positive definite sublattice. The result of work of several authors as summarized by Arnold is as follows.

Theorem. *Complex surfaces of embedding dimension 3 with $\mu_+ + \mu_0 \leqslant 2$ are classified as follows.*

(1) Those with $(\mu_+, \mu_0) = (0, 0)$ are the simple singularities.
(2a) Those with $(\mu_+, \mu_0) = (0, 2)$ are the simply elliptic singularities.
(2b) Those with $(\mu_+, \mu_0) = (1, 1)$ are the cusps $T_{p,q,r}$.
(2c) Those with $(\mu_+, \mu_0) = (2, 0)$ and number of moduli equal to 1 are the 14 exceptional 1-modular singularities.

G. N. Tjurina and V. I. Arnold called these singularities in case (1) elliptic, in case (2a) parabolic and in case (2b) hyperbolic.

1.5 The signature (μ_+, μ_-, μ_0) describes only the real quadratic form. Actually much more can be said about the Milnor lattices of these singularities. A. M. Gabrielov has described distinguished bases of vanishing cycles for these singularities [40], [41]. They may be characterized by a triple of integers which Arnold called Gabrielov numbers. From the arithmetic point of view, a very thorough investigation of these lattices was carried out by one of us in [17], supplemented by B. Stoppok [85]. This was closely related to a description of the base space of the semi-universal unfolding of exceptional 1-modular singularities in terms of arithmetic quotients of bounded symmetric domains [16].

These investigations gave us reasons to focus on a particular part of the deformation hierarchy of 1-modular singularities, which was called "boundary layer" in [16]. Today we see this as a layer of transition from spherical to Lorentz geometry. If we add the elliptic singularities, we get the following pattern of 12 singularities:

E_{12}	Z_{11}	Q_{10}	exceptional layer,
$T_{2,3,7}$	$T_{2,4,5}$	$T_{3,3,4}$	hyperbolic layer,
$\tilde{E}_8$	$\tilde{E}_7$	$\tilde{E}_6$	parabolic layer,
E_8	E_7	E_6	elliptic layer.

The singularities of the three unimodular layers may be characterized as follows:

(a) Every non-simple singularity deforms into a singularity of the parabolic layer.
(b) Every non-simple non-parabolic singularity deforms into a singularity of the hyperbolic layer.
(c) Every non-simple, non-parabolic, non-hyperbolic singularity deforms into a singularity of the exceptional layer.

The deformation relations of singularities above the boundary layer are very complicated, also with respect to singularities in the boundary layer and below [12], [13], [15], [42]. But within the boundary layer the situation is simple: the only deformation relations are those of going downward in the vertical columns. We take this as an indication that these three "stems" with "roots" in E_8, E_7, E_6 and continuation by Arnold's series E, Z, Q are very particular objects which deserve special attention.

1.6 In 1973, Arnold had published the *classification of unimodal critical points of functions* [2]. Some of his normal forms for the exceptional 1-modular singularities have a long history. The form for E_{12} occurs already in 1878 in Klein's paper *Über die Transformationen siebenter Ordnung der elliptischen Funktionen* [50] p. 652. The equation for E_{13} occurs in *Vorlesungen über die Theorie der automorphen Functionen* by Fricke and Klein, [38], volume II, p. 652 and is related to the simple group of order 360, [36]. The equation for E_{14} was found in 1880 by Klein's student W. Dyck in his dissertation, [30], [29].

The normal form of Arnold for the quasi-homogeneous singularity E_{12} in three variables is

$$x^3 + y^7 + z^2.$$

Mathematical objects related to this form were important as examples preceding the development of a general theory of automorphic functions by Klein and Poincaré. The same objects have been the starting point of the work of I. V. Dolgachev to which Arnold was referring when he spoke about the wonderful coincidences with Lobatchevsky triangles and automorphic functions.

We consider $\mathrm{PSU}(1,1)$ as group of automorphisms of the unit disk $\mathbb{D}$. In this 3-dimensional Lie group, we consider discrete co-compact subgroups Γ. In particular, we consider triangle groups $\Gamma(p,q,r)$ belonging to hyperbolic triangles with angles

$$\frac{\pi}{p}\,, \quad \frac{\pi}{q} \quad \text{and} \quad \frac{\pi}{r}\,, \qquad \text{where} \quad \frac{1}{p}+\frac{1}{q}+\frac{1}{r} < 1.$$

The smallest triangle is the one with $(p,q,r) = (2,3,7)$. In $\Gamma = \Gamma(2,3,7)$ there is a unique largest normal subgroup Γ'. For a suitable representation of Γ as an arithmetic group, the group Γ' can be described as a certain congruence subgroup. The quotient Γ/Γ' is the simple group G_{168} of order 168. This is the second smallest simple group of composite order. It comes next after the icosahedral group G_{60}. There are isomorphisms $G_{60} \cong \mathrm{PSL}(2,\mathbb{F}_5)$ and $G_{168} \cong \mathrm{PSL}(2,\mathbb{F}_7)$. The analogy between these two cases has been noted by Klein.

The group Γ' acts on $\mathbb{D}$ without fixed points. It has a fundamental domain which is a regular hyperbolic 14-gon consisting of $2 \cdot 168$ hyperbolic triangles with angles $\pi/2$, $\pi/3$, $\pi/7$. This is the *Hauptfigur* of Felix Klein [50], p. 126. The surface $\Gamma'\backslash\mathbb{D}$ is a Riemann surface of genus $g = 3$ with an automorphism group of the maximal possible order $84(g-1)$.

The surface $X = \Gamma'\backslash\mathbb{D}$ is non-hyperelliptic of genus $g > 2$. Therefore it has a canonical embedding $X \subset \mathbb{C}\mathrm{P}^{g-1}$ into the projective space which belongs to the space $\mathbb{C}^g$ dual to the space of holomorphic 1-forms. This canonical curve in $\mathbb{C}\mathrm{P}^2$ is the Klein quartic given by the homogeneous equation

$$x_0^3 x_1 + x_1^3 x_2 + x_2^3 x_0 = 0.$$

The finite group G_{168} acts linearly on the space of holomorphic 1-forms. Therefore it acts on $\mathbb{C}^3$ and on $\mathbb{C}\mathrm{P}^2$ leaving invariant $X \subset \mathbb{C}\mathrm{P}^2$ and the cone $C_X \subset \mathbb{C}^3$. Calculations of invariants by Klein and Gordan imply:

$$\left[\mathbb{C}[x_0,x_1,x_2]/(x_0^3x_1 + x_1^3x_2 + x_2^3x_0)\right]^{G_{168}} \cong \mathbb{C}[x,y,z]/(x^3+y^7+z^2).$$

This algebraic result can be interpreted geometrically as follows: The affine algebraic surface defined by the equation

$$x^3 + y^7 + z^2 = 0$$

is the quotient of the cone C_X over the canonical curve X by the group $G_{168} = \Gamma/\Gamma'$.

This was generalized in 1974 by I. V. Dolgachev [23]. Dolgachev introduced the notion of a *quotient-conical singularity*. Let $X \subset \mathbb{C}\mathrm{P}^{n-1}$ be a smooth projectively normal curve. This means that the cone $C_X \subset \mathbb{C}^n$ over C is a normal affine surface with an isolated singular point. Let $G \subset \mathrm{GL}(n, \mathbb{C})$ be a finite group leaving C_X invariant. The singularity of the quotient surface C_X/G corresponding to the vertex of the cone is called a quotient-conical singularity. If $X \subset \mathbb{C}\mathrm{P}^{g-1}$ is a canonical curve and G a subgroup of $\mathrm{Aut}(X)$, the resulting quotient conical singularity is called *canonical* of type (X, G).

For any hyperbolic triangle group Γ one can find normal subgroups Γ' acting freely on $\mathbb{D}$, Mennicke [54]. Dolgachev proved that the canonical quotient conical singularity of type $(\Gamma'\backslash D, \Gamma/\Gamma')$ depends only on Γ. So there is a unique canonical triangle singularity for each hyperbolic triangle group Γ. Dolgachev characterized these triangle singularities by their resolution graph. He proved the following theorem.

Theorem. *There are exactly 14 canonical triangle singularities which can be embedded in $\mathbb{C}^3$. They are the complex surface singularities corresponding to the 14 1-modular exceptional quasi-homogeneous normal forms of Arnold.*

We note in passing that most of these triangles occur in the work of Fricke and Klein when they describe arithmetic triangle groups. A complete enumeration of all arithmetic triangle groups was given by K. Takeuchi [86].

1.7 The results which we are going to present in this paper are to be seen within the context described in this introduction.

In section 2 we recall work of Dolgachev describing the links of all Gorenstein quasi-homogeneous surface singularities as quotients $\Gamma\backslash G$ of a 3-dimensional simply connected Lie group G by a discrete co-compact subgroup. The groups G that occur are $\mathrm{SU}(2)$, $\widetilde{\mathrm{SU}}(1,1)$ and the Heisenberg group. We describe the groups $\Gamma \subset \widetilde{\mathrm{SU}}(1,1)$ corresponding to Arnold's singularities E_k, Z_k, Q_k and $E_{3,0}$, $Z_{1,0}$, $Q_{2,0}$.

In section 3 we consider more generally discrete subgroups $\Gamma \subset \widetilde{\mathrm{SU}}(1,1)$ of finite level. The level is the index of $\Gamma \cap Z$ in the centre Z of $\widetilde{\mathrm{SU}}(1,1)$. Discrete co-compact subgroups are of finite level by a general result of André Weil [92] on discrete co-compact subgroups of connected semi-simple Lie groups without compact components. We describe a construction of fundamental domains for all discrete subgroups of finite level with a fixed point in $\mathbb{D}$ of order larger than the level. This fundamental domain is a polyhedron in the Lorentz manifold $\widetilde{\mathrm{SU}}(1,1)$ with totally geodesic faces. It is modeled on a polyhedron in Lorentz 3-space.

In section 4 these fundamental domains are determined explicitely for the infinite series E_k, Z_k, Q_k.

Section 5 is devoted to the description of fundamental domains for $E_{3,0}$, $Z_{1,0}$, $Q_{2,0}$. Although these cases have been analyzed completely, we cannot present all details in this exposition.

Section 6 describes fundamental domains for the subgroups Γ of the Heisenberg group corresponding to $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$.

The results are illustrated by tables 1–13. Tables 5–8 correspond to section 4, tables 9–12 to section 5 and table 13 to section 6. Tables 1–4 offer a synopsis of the different cases. They reveal a coherent combinatorial pattern for each of Arnold's series E, Z, Q united with the three stems of the boundary layer. In particular

table 1 shows the transition from the classical elliptic layer to the exceptional layer via the parabolic layer. The tables show that the polyhedra in Lorentz 3-space which we construct are true analogues of the three classical uniform polyhedra in Euclidean 3-space which belong to the binary groups **T**, **O**, **I** of the tetrahedron, octahedron and icosahedron.

1.8 The work presented in this paper has evolved during a period of more than 12 years. It began with the thesis of Thomas Fischer [35]. Fischer found the beautiful construction of fundamental domains for canonical quotient-conic singularities and calculated the first three cases E_{12}, Z_{11}, Q_{10}. His work was carried on by A. Käss, U. Neuschäfer, F. Rothenhäusler and S. Scheidt [48]. Up to now their joint paper with L. Balke [11] published in Topology has been the only publication on this kind of work which has appeared in a journal. Further progress was made in [74] by the second author. At last, the final construction presented in section 3 was found by A. Pratoussevitch [75]. The analysis of E_k, Z_k, Q_k in section 4 is also her work. The analysis of $E_{3,0}$, $Z_{1,0}$, $Q_{2,0}$ in section 5 is the work of F. Rothenhäusler [78]. The observations on $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$ are due to E. Brieskorn and were made many years ago in discussions with Thomas Fischer.

We wish to thank Ludwig Balke and Claus Hertling for discussions with them on our work in progress. We would like to thank Ilya Dogolazky for his help in producing the figures. We also thank the referee for pointing out that any lattice in a semi-simple Lie group without compact factors intersects its center in a subgroup of finite index.

2. Link spaces of quasi-homogeneous singularities

2.1 Let R be the ring $\mathbb{C}[x_1, \ldots, x_n]$ of polynomials in n variables with complex coefficients. Let $q = (q_1, \ldots, q_n)$ be a system of n positive integers, called weights. Then R is a positively graded $\mathbb{C}$-algebra R_q if we define x_i to be homogeneous of degree q_i. A monomial $x_1^{i_1} \cdots x_n^{i_n}$ has degree $d = i_1 q_1 + \cdots + i_n q_n$. The monomials of degree d form a basis for the vector space R_d of homogeneous polynomials of degree d. This terminology is used in commutative algebra. When we deal with singularities, we have to consider many different systems of weights. We shall then call such polynomials *quasi-homogeneous* or *weighted homogeneous* of degree d with weights $(q_1, \ldots, q_n)$.

An ideal $I \subset R_q$ is homogeneous if it is generated by homogeneous elements. An affine algebraic variety $V \subset \mathbb{C}^n$ is *quasi-homogeneous* with weights $(q_1, \ldots, q_n)$ if its defining ideal I in R_q is homogeneous. Its affine coordinate ring R_q/I is a finitely generated positively graded $\mathbb{C}$-algebra.

To a system of weights we associate a $\mathbb{C}^*$-action on $\mathbb{C}^n$:

$$t(z_1, \ldots, z_n) := (t^{q_1} z, \ldots, t^{q_n} z_n).$$

A variety $V \subset \mathbb{C}^n$ is invariant with respect to this action iff the defining ideal in R_q is homogeneous. There is a contravariant equivalence between complex affine algebraic varieties with good $\mathbb{C}^*$-action and finitely generated positively graded $\mathbb{C}$-algebras.

Let (X, x) be a complex analytic singularity, i.e. the germ of a complex space X at a point x. We call (X, x) *quasi-homogeneous* if there is an isomorphism $(X, x) \cong (V, 0)$, where $V \subset \mathbb{C}^n$ is an affine variety which is quasi-homogeneous for some system of weights and $0 \in \mathbb{C}^n$ is the origin. There may be several possible

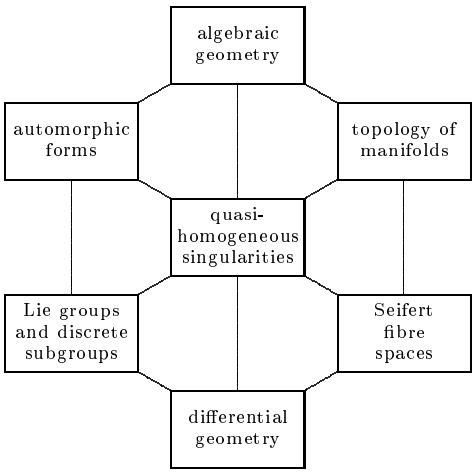


Figure 1: Quasi-homogeneous singularities in mathematics

systems of weights. However, the following result about uniqueness was proved by Saito [79]. Let (X,x) be an isolated quasi-homogeneous hypersurface singularity and $(X,x) \cong (V,0)$, where $(V,0)$ is an affine hypersurface defined by a quasi-homogeneous polynomial of degree d with weights $q_1,\dots,q_n$. The weights can be chosen so that $(q_i,d)=1$ and $2q_i \leqslant d$. Then up to permutations the weights are uniquely determined.

2.2 Quasi-homogeneous singularities are interesting objects. Two-dimensional quasihomogeneous singularities are even more interesting, because they are at the centre of a net of relations between different fields of mathematics, as shown by figure 1. We cannot explain all relations between these fields, but we want to mention those which place our work in its proper context.
The relations between automorphic forms, algebraic geometry and the theory of invariants existed from the beginning of the theory of automorphic functions and are obvious in the writings of Fricke and Klein, [37], [38], [52].

The relation between algebraic geometry and the topology of manifolds which we have in mind is also very old. It goes back to the turn of the century around 1900. The relation is established as follows. Let $V \subset \mathbb{C}^n$ be an m-dimensional complex algebraic variety with an isolated singularity at the origin. Let B_ε^{2n} be the $2n$-ball of radius ε with centre 0. The boundary of the ball is a $(2n-1)$-sphere S_ε^{2n-1}. Consider the intersections

$$V_\varepsilon = V \cap B_\varepsilon^{2n} \quad \text{and} \quad M_\varepsilon = V \cap S_\varepsilon^{2n-1}.$$

For ε sufficiently small M_ε is a compact oriented differentiable manifold of dimension $2m-1$ smoothly embedded in S_ε^{2n-1}. The diffeomorphism type of the pair $(S_\varepsilon^{2n-1}, M_\varepsilon^{2m-1})$, ε small, depends only on the singularity $(V,0)$. Moreover, there is a homeomorphism between the pair $(B_\varepsilon, V_\varepsilon)$ and the pair $(B_\varepsilon, CM_\varepsilon)$, where CM_ε is the cone over M_ε with vertex 0. When V is analytically irreducible at 0, M_ε is connected. Otherwise, it will have several components which may be linked. Therefore, the boundary M_ε of the neighbourhood V_ε of 0 is also called the *link* of the singularity $(V,0)$.

For varieties V of complex dimension 1 and 2 these constructions go back to W. Wirtinger, P. Heegard and H. Tietze and are closely related to the early history of knot theory, M. Epple [34], chapter 8. Around 1960 work of D. Mumford [63] and F. Hirzebruch [47] showed that there is a close link between singularities of complex surfaces and the topology of 3-manifolds established by the link construction. Further developments described in [18] led to interesting relations between links of higher dimensional quasi-homogeneous singularities and differential topology [14], [47], [55], [44]. For example consider the link M^{2n-3} of the quasi-homogeneous affine hypersurface singularity given by the E_8-equation

$$x_1^3 + x_2^5 + x_3^2 + \cdots + x_n^2 = 0.$$

The curve $M^1 \subset \mathbb{S}^3$ is the $(3,5)$-torus knot. $M^3 \subset \mathbb{S}^5$ is the link of the icosahedral singularity. So M^3 is the spherical dodecahedral space obtained from a spherical dodecahedron by identifying opposite faces by a screw motion with angle $\pi/5$, and so M^3 can be identified with the famous Poincaré homology sphere, [91]. For $n = 4$, the link space M^5 is a knotted 5-sphere in $\mathbb{S}^7$. Finally, the link space M^7 in $\mathbb{S}^9$ is the exotic 7-sphere of Milnor, which Hirzebruch constructed as boundary of an 8-manifold obtained by glueing 8 copies of the tangent disc bundle of $\mathbb{S}^4$ according to the Coxeter-Dynkin diagram E_8.

The results mentioned above led to investigations on the topology of quasi-homogeneous singularities such as [44], [57], [56], [65], [68], [88]. At the same time, together with other developments, they led to the first systematic treatment of quasi-homogeneous surface singularities as objects of algebraic geometry by P. Orlik and Ph. Wagreich [72], [71].

2.3 The links of quasi-homogeneous singularities carry additional structures. One structure is obvious. When M is the link of an isolated singularity of a quasi-homogeneous variety V with good $\mathbb{C}^*$ action, this action induces an action of $\mathbb{S}^1 \subset \mathbb{C}^*$ on M. A closely related structure is the orbit decomposition of M associated to the action of $\mathbb{S}^1$. This is a fibration of M by circles which may have exceptional fibres, if the action of $\mathbb{S}^1$ has nontrivial isotropy groups. Such fibrations are called Seifert fibre spaces, since they were first studied in 1933 by H. Seifert as an additional structure on 3-manifolds [83]. Since then this extra structure was used as a condition which makes the topology of 3-manifolds more accessible. Around 1967 investigations on the topology of Seifert fibre spaces such as [70] and closely related work on $\mathbb{S}^1$-actions on 3-manifolds such as [69] merged with the new results quoted above and led to systematic investigations on quasi-homogeneous singularities.

2.4 Two-dimensional quasi-homogeneous singularities are particular because the corresponding graded affine coordinate rings can be identified with graded rings of generalized automorphic forms. This was found in 1975–1977 by Dolgachev, Milnor, Neumann and Pinkham, [24], [25], [56], [65], [73]. Let us recall their results.

Definition. A *negative unramified automorphy factor* $(U, L, \bar{\Gamma})$ is a complex line bundle L on the simply connected Riemann surface U, $U = \mathbb{C}\mathrm{P}^1$ or $\mathbb{C}$ or $\mathbb{D}$, together with a discrete co-compact subgroup $\bar{\Gamma} \subset \mathrm{Aut}(U)$ acting compatibly on U and the line bundle L, such that the following two conditions are satisfied:

(i) The action of $\bar{\Gamma}$ is free on L_0, the complement of the zero-section of L.

(ii) Let $\bar{\Gamma}' \lhd \bar{\Gamma}$ be a normal subgroup of finite index which acts freely on U, and let $L' \to Y$ be the complex line bundle $L' = \bar{\Gamma}' \backslash L$ over the compact Riemann surface $Y = \bar{\Gamma}' \backslash U$. Then L' is a negative line bundle.

Since L' is negative, one can contract the zero-section of L' and get a complex surface with an isolated singularity corresponding to the zero-section. There is a canonical action of the finite group $\bar{\Gamma}/\bar{\Gamma}'$ on this surface. The quotient is a complex surface $X(L, \bar{\Gamma})$ with an isolated singular point 0, which depends only on the automorphy factor $(U, L, \bar{\Gamma})$.

Theorem. *The surface $X(L, \bar{\Gamma})$ associated to a negative unramified automorphy factor $(U, L, \bar{\Gamma})$ is a quasi-homogeneous affine algebraic surface with a normal isolated singularity. Its affine coordinate ring is the graded $\mathbb{C}$-algebra of generalized $\bar{\Gamma}$-invariant automorphic forms*

$$A = \bigoplus_{m \geqslant 0} H^0(U, L^{-m})^{\bar{\Gamma}}.$$

All normal isolated quasi-homogeneous surface singularities (X, x) are obtained in this way, and the automorphy factors with $(X(L, \bar{\Gamma}), 0)$ isomorphic to (X, x) are uniquely determined by (X, x) up to isomorphism.

2.5 In a sense it is an abuse of language to call an element of $H^0(U, L^{-m})^{\bar{\Gamma}}$ a generalized automorphic form. It is an automorphic form of integral weight m in the classical sense when $U = \mathbb{D}$ and $L = T_{\mathbb{D}}$, the tangent bundle of $\mathbb{D}$, on which $\bar{\Gamma} \subset \mathrm{Aut}(\mathbb{D})$ acts in the canonical way. As a generalization which is closer to the classical case one may introduce automorphic forms with fractional weight. This was done by Milnor in [56]. An elegant way of defining such forms is the following definition of Dolgachev [27].

Definition. A *Gorenstein automorphy factor* is an unramified negative automorphy factor $(U, L, \bar{\Gamma})$ such that there is an integer k and an isomorphism of $\bar{\Gamma}$-bundles L^k and T_U, where T_U is the tangent bundle of U. Moreover, for $U = \mathbb{C}$ the group $\bar{\Gamma}$ must be contained in the translation subgroup of $\mathrm{Aut}(\mathbb{C})$. The integer k is called the *exponent* or the *level* of the automorphy factor.

Possible values of the exponent are $k = -1$ or -2 for $U = \mathbb{C}\mathrm{P}^1$, whereas $k = 0$ for $U = \mathbb{C}$ and $k > 0$ for $U = \mathbb{D}$.

The name *Gorenstein* for these automorphy factors was chosen because of their relation with Gorenstein singularities. A Gorenstein singularity is a singularity whose local ring is a Gorenstein local ring. We shall not give the definitions of this notion coming from commutative algebra. Instead, we give the definition used by Dolgachev. An isolated singularity of dimension n is a Gorenstein singularity if its local ring is a Cohen-Macaulay ring and if there is a nowhere vanishing holomorphic n-form on a punctured neighbourhood of x. All isolated singularities of complete intersections are Gorenstein singularities. In particular, the theory applies to the surfaces in $\mathbb{C}^3$ which we are going to study. In [27] Dolgachev proved the following theorem obtained independently by W. Neumann (see also [26]).

Theorem. *The quasi-homogeneous surface singularity $(X(L, \bar{\Gamma}), 0)$ associated to a negative unramified automorphy factor $(U, L, \bar{\Gamma})$ is a Gorenstein singularity iff $(U, L, \bar{\Gamma})$ is a Gorenstein automorphy factor.*

2.6 The next problem is to determine the Gorenstein automorphy factors for a given $\bar{\Gamma}$, if they exist. The following proposition proved in [74] is an answer for $\bar{\Gamma} \subset \mathrm{Aut}(\mathbb{D})$.

Theorem. *Let $\bar{\Gamma} \subset \mathrm{PSU}(1,1)$ be a discrete co-compact subgroup with signature $(g; \alpha_1, \ldots, \alpha_r)$. Let $b = 2(g-1) + r$. There exists a Gorenstein automorphy factor $(\mathbb{D}, L, \bar{\Gamma})$ of level $k > 0$ iff k satisfies the following divisibility conditions:*

(i) $(k, \alpha_i) = 1$ *for all* $i = 1, \ldots, r$;

(ii) k *divides* $[\alpha_1, \ldots, \alpha_r] \cdot (b - \sum_{i=1}^{r} \alpha_i^{-1})$.

If these conditions are satisfied, there exist exactly k^{2g} Gorenstein automorphy factors for $\bar{\Gamma}$. In particular, there is a unique one if $g = 0$.

2.7 The affine coordinate ring of a quasi-homogeneous affine algebraic surface has two alternative descriptions. On one hand it is a graded $\mathbb{C}$-algebra R/I, where R is a polynomial ring and I an ideal generated by quasi-homogeneous polynomials with certain degrees for a given system of weights $(q_1, \ldots, q_n)$. On the other hand, it is a graded $\mathbb{C}$-algebra of automorphic forms for a certain discrete group $\bar{\Gamma}$ with a certain signature $(g; \alpha_1, \ldots, \alpha_r)$. Comparison of these two descriptions leads to relations between the two sets of data. Such arguments were used by Ph. Wagreich and other authors to describe and classify certain algebras of automorphic forms with few generators, [84], [89], [90]. Recently K. Möhring has used similar arguments and K. Saito's paper [79] for proving a theorem which allows to calculate the signature and the level of the Gorenstein automorphy factors from the weights and degree for all isolated quasi-homogeneous surface singularities of embedding dimension 3.

Theorem. *Let $V \subset \mathbb{C}^3$ be a quasi-homogeneous affine surface with an isolated singularity. Let (q_1, q_2, q_3) be the weights and d the degree of a polynomial defining V. Let k be the level and $(g; \alpha_1, \ldots, \alpha_r)$ be the signature of the Gorenstein automorphy factor associated to V. These data are related as follows.*

(1) $k = d - q_1 - q_2 - q_3$,

(2) $\{\alpha_1, \ldots, \alpha_r\}$ *is contained in the union of the two disjoint sets* $\{q_i \mid q_i \nmid d\}$ *and* $\{(q_i, q_j) \neq 1 \mid i < j\}$. *The α_s in the first set occur with multiplicity one. The α_s in the second set occur with multiplicity m_{ij}, where $m_{ij} + 1$ is the number of solutions of the equation $xq_i + yq_j = d$ by nonnegative integers x, y.*

(3) The genus g is determined by the relation

$$q_1 q_2 q_3 \cdot \Big(2g - 2 + r - \sum_{i=1}^{r} \alpha_i^{-1}\Big) = k \cdot d.$$

Remark. Put $\varepsilon_i = 0$ if $q_i \mid d$ and $\varepsilon_i = 1$ otherwise. Then Möhring proves:

$$m_{ij} = (d - \varepsilon_i q_j - \varepsilon_j q_i)/[q_i, q_j].$$

2.8 Using his theorem quoted in 2.7 Möhring has calculated the exponents and signatures of the automorphy factors for all quasi-homogeneous polynomials in three variables in the well-known classes I–VII. In particular, table 19 in [58] gives these data for Arnold's series E, Z, Q, W, S, U. The results for E, Z, Q are as follows.

Theorem. *The Gorenstein automorphy factors $(U, L, \bar{\Gamma})$ for the series E, Z, Q are of hyperbolic type, i.e. $U = \mathbb{D}$ and $\bar{\Gamma} \subset \mathrm{PSU}(1,1)$. Let k be the exponent and $(g; \alpha_1, \ldots, \alpha_r)$ the signature of $\bar{\Gamma}$. These data are given in the following two tables.*

In the first table the signature is given by $(\alpha_1, \alpha_2, \alpha_3)$*, since* $r = 3$ *and* $g = 0$ *for all* E_n*,* Z_n*,* Q_n*.*

Type	n mod 4	k	$(\alpha_1, \alpha_2, \alpha_3)$
E_n	0	$(n-10)/2$	$(2, 3, k+6)$
	2	$(n-10)/4$	$(3, 3, k+3)$
	1, 3	$(n-10)/3$	$(2, 4, k+4)$
Z_n	3	$(n-9)/2$	$(2, 3, 2k+6)$
	1	$(n-9)/4$	$(3, 3, 2k+3)$
	0, 2	$(n-9)/3$	$(2, 4, 2k+4)$
Q_n	2	$(n-8)/2$	$(2, 3, 3k+6)$
	0	$(n-8)/4$	$(3, 3, 3k+3)$
	1, 3	$(n-8)/3$	$(2, 4, 3k+4)$

Type	n mod 2	k	$(g; \alpha_1, \ldots, \alpha_r)$
$E_{n,0}$	1	$n-2$	$(0; 2, 2, 2, n)$
	0	$(n-2)/2$	$(1; n/2)$
$Z_{n,0}$	1	n	$(0; 2, 2, 2, 2(n+1))$
	0	$n/2$	$(1; n+1)$
$Q_{n,0}$	0	$n-1$	$(0; 2, 2, 2, 3n-1)$
	1	$(n-1)/2$	$(1; (3n-1)/2)$

The groups which belong to E_n, Z_n, Q_n are triangle groups $\Gamma(\alpha_1, \alpha_2, \alpha_3)$. They will be investigated in section 4. The groups for $E_{3,0}$, $Z_{1,0}$ and $Q_{2,0}$ are the "quadrangle groups" $\Gamma(2, 2, 2, p)$, where $p = 3, 4, 5$. They will be the subject of section 5.

2.9 Dolgachev's paper [27] shows how to pass from Gorenstein automorphy factors $(U, L, \bar{\Gamma})$ to quotients $\tilde{\Gamma}\backslash G$ of 3-dimensional Lie groups G by discrete co-compact subgroups $\tilde{\Gamma}$. This is done case by case for $U = \mathbb{C}\mathrm{P}^1$, $\mathbb{C}$ and $\mathbb{D}$. We recall the arguments for the case $U = \mathbb{D}$.

(1) The universal covering group $\widetilde{\mathrm{SU}}(1,1)$ of $\mathrm{PSU}(1,1)$ has an infinite cyclic centre Z. For each natural number k there is a unique cyclic covering $G_k \to G_1$ of $G_1 = \mathrm{PSU}(1,1)$ defined by $G_k = \widetilde{\mathrm{SU}}(1,1)/kZ$.

(2) For any complex line bundle and any natural number k there is a canonical ramified covering map $L \to L^k$ defined by $v \mapsto v \otimes \cdots \otimes v$. The restriction to the complements of the 0-sections is a cyclic unramified covering $L_0 \to L_0^k$ of degree k. Let L^k have a hermitian metric. Then there is a hermitian metric on L, such that we get an unramified covering map for the corresponding unit circle bundles:

$$SL \longrightarrow SL^k.$$

(3) The group $G_1 = \mathrm{Aut}(\mathbb{D})$ acts canonically on the circle bundle $ST_{\mathbb{D}}$ of unit tangent vectors in the tangent bundle $T_{\mathbb{D}}$. The action is simply transitive. Choosing a basepoint $v_0 \in T_{\mathbb{D},0}$ we get a G_1-invariant bijection $G_1 \to ST_{\mathbb{D}}$, where G_1 acts on itself by left translations. This is an $\mathbb{S}^1$-bundle isomorphism, where G_1 is fibred by the cosets of the isotropy group of $0 \in \mathbb{D}$.

(4) Now let $(\mathbb{D}, L, \bar{\Gamma})$ be a Gorenstein automorphy factor of level k. Then there is a $\bar{\Gamma}$-equivariant bundle isomorphism $L^k \cong T_{\mathbb{D}}$. This induces a hermitian metric on L^k and a $\bar{\Gamma}$-equivariant isomorphism of $\mathbb{S}^1$-bundles $ST_{\mathbb{D}} \cong SL^k$. Altogether we get a $\bar{\Gamma}$-invariant isomorphism of $\mathbb{S}^1$-bundles $\varphi : G_1 \to SL^k$. This can be lifted to the k-fold cyclic coverings:

$$\begin{array}{ccc} G_k & \xrightarrow{\psi} & SL \\ \downarrow & & \downarrow \\ G_1 & \xrightarrow{\varphi} & SL^k \end{array}$$

The bijection ψ is determined up to multiplication with a root of unity. So we get a well defined action of $\bar{\Gamma}$ on G_k covering the action of $\bar{\Gamma}$ on G_1 by left translation. The $\bar{\Gamma}$-orbit of the unit element in G_k is a discrete co-compact subgroup $\tilde{\Gamma}$ of G_k. The covering map $\tilde{\Gamma} \to \bar{\Gamma}$ is an isomorphism identifying the actions of $\check{\Gamma}$ and $\bar{\Gamma}$ on G_k, where $\check{\Gamma}$ acts by left multiplication. We call $\tilde{\Gamma}$ a *lifting* of $\bar{\Gamma}$.

This leads to the following theorem proved by Dolgachev in [27] (see also the related earlier results of J. Milnor [56], W. Neumann [65], [66], and F. Raymond and A. T. Vasqez [76] quoted by Dolgachev).

Theorem. *To every Gorenstein automorphy factor $(\mathbb{D}, L, \bar{\Gamma})$ of level k corresponds a lifting $\tilde{\Gamma} \subset G_k$ of $\bar{\Gamma} \subset G_1$. The link of the Gorenstein quasi-homogeneous surface singularity $(X(L, \bar{\Gamma}), 0)$ identifies with $\tilde{\Gamma}\backslash G_k$. Conversely every lifting $\tilde{\Gamma} \subset G_k$ of $\bar{\Gamma} \subset G_1$ gives rise to a Gorenstein automorphy factor $(\mathbb{D}, L, \bar{\Gamma})$ of level k.*

The discrete groups $\tilde{\Gamma} \subset G_k$ obtained as liftings of discrete co-compact groups in G_1 are those discrete co-compact subgroups of G_k which do not intersect the centre of G_k. We may also describe them as follows. Let $\Gamma \subset \widetilde{\mathrm{SU}}(1,1)$ be a discrete co-compact subgroup of level k. The image $\tilde{\Gamma}$ of Γ in G_k is a lifting of the image $\bar{\Gamma}$ of Γ in G_1. Therefore we may rephrase the results quoted above as follows.

Corollary. *The links of quasi-homogeneous Gorenstein surface singularities of hyperbolic type identify with quotient spaces $\Gamma\backslash\widetilde{\mathrm{SU}}(1,1)$, where Γ is a discrete co-compact subgroup in the simply connected 3-dimensional Lie group $\widetilde{\mathrm{SU}}(1,1)$.*

By "hyperbolic type" we mean that the singularity comes from an automorphy factor $(\mathbb{D}, L, \bar{\Gamma})$ for the hyperbolic plane $\mathbb{D}$.

2.10 In view of the results quoted above it is interesting to discuss the relations between quasi-homogeneous singularities and differential geometry. The links of quasi-homogeneous surface singularities may be given different kinds of geometric structures.

One structure that always exists on links of isolated singularities is the CR-structure obtained immediately from the construction of the link. This CR-structure determines the complex analytic singularity. 3-dimensional compact locally homogeneous nondegenerate CR-manifolds (i.e. CR-space forms) have been classified by F. Ehlers, J. Scherk and W. D. Neumann [33]. They also classified the normal complex surface singularities whose link is a CR-space form: Dolgachev's quasi-homogeneous Gorenstein singularities, cusp singularities, and quotients of them by involutions.

Another possibility is to ask for a geometric structure on the link of a surface singularity in the sense that it should carry a locally homogeneous Riemannian metric. This leads to the well-known 8 geometries of Thurston. W. Neumann has discussed the question which of these geometric structures occur on links of surface singularities [67]. In particular he proved the following theorem. We consider a closed orientable 3-manifold M endowed with a geometric structure which admits a Seifert fibration with negative Euler number, and we exclude lens spaces. Then there is a one-to-one correspondence between isometry classes of such structures on M and biholomorphic equivalence classes of quasi-homogeneous surface singularities with link homeomorphic to M.

For Gorenstein quasi-homogeneous singularities there is a third possibility. In the hyperbolic case their links identify with quotients $\Gamma\backslash\widetilde{\mathrm{SU}}(1,1)$, and so they are Lorentz space forms. Here we do not restrict the notion of space form to Riemannian space forms. A space form is any complete pseudo-Riemannian manifold with constant curvature. The group $\widetilde{\mathrm{SU}}(1,1)$ has a Lorentz metric of constant curvature coming from the Killing form. So the links of quasi-homogeneous Gorenstein surface singularities are closed 3-dimensional Lorentz space forms. Closed 3-dimensional Lorentz space forms have been characterized by R. S. Kulkarni and F. Raymond [53]. Such space forms are orientable Seifert fibre spaces with hyperbolic base and nonzero Euler number. Of course, the relation with Seifert fibrations is very important. However, we want to plead for another perspective which has a long tradition in the case of spherical and hyperbolic space forms, but has not been explored in the realm of Lorentz space forms. We propose to represent such space forms $\Gamma\backslash\widetilde{\mathrm{SU}}(1,1)$ by constructing a polyhedral totally geodesic fundamental domain F for Γ in the Lorentz manifold $\widetilde{\mathrm{SU}}(1,1)$ together with the corresponding pairing of faces. The construction given in section 3 shows that this is possible, and the examples analyzed in section 4 and 5 show that this combination of differential and combinatorial geometry reveals subtle features of the theory of representations of discrete groups in $\widetilde{\mathrm{SU}}(1,1)$ and is related to the structure of series of singularities as defined by Arnold.

3. The construction of fundamental domains

3.1 In this section we shall construct fundamental domains for a large class of discrete subgroups Γ of $\widetilde{\mathrm{SU}}(1,1)$. The Lorentz geometry of $\widetilde{\mathrm{SU}}(1,1)$ is not as simple as the spherical geometry of $\mathrm{SU}(2)$. Therefore, the construction cannot be as simple as in the spherical case. So the beautiful construction of fundamental domains for subgroups of level 1 discovered by Thomas Fischer was something really new. We shall generalize this construction to subgroups of $\widetilde{\mathrm{SU}}(1,1)$ of any finite level k.

There is one feature of Fischer's construction which is similar to the construction in the spherical case as presented in section 1.2. The spherical fundamental domains were not constructed directly in the 3-sphere $\mathrm{SU}(2)$. They were obtained from a 4-dimensional polytope constructed in the ambient Euclidean 4-space. The boundary of this polytope was projected onto the sphere by central projection from the origin, where we view Euclidean 4-space as a cone over $\mathrm{SU}(2)$ with vertex at the origin. In Fischer's construction $\mathrm{SU}(1,1)$ is embedded as a Lorentz manifold in a 4-dimensional linear space with a flat pseudo-metric of signature $(2,2)$. Fischer constructs a 4-dimensional polytope in the cone over $\mathrm{SU}(1,1)$. The boundary of

this polytope is projected onto $\mathrm{SU}(1,1)$, and the fundamental domains are the projections of the faces. The new idea of Fischer was the remarkable construction of the 4-dimensional polytope.

Since we want to generalize the construction to arbitrary levels, we pass to the universal cover. This will be done in sections 3.2 to 3.4 for $\mathrm{SU}(1,1)$ as well as its cone. Sections 3.5 to 3.6 contain some elements of the construction. Finally the construction itself is presented in 3.7 and visualized in 3.8.

3.2 We consider the complex vector space $\mathbb{C}^2$ with the standard hermitian form of signature $(1,1)$. The real part is a symmetric real bilinear form of signature $(n_+, n_-) = (2,2)$. The associated quadratic form is

$$q(z_1, z_2) = z_1\bar{z}_1 - z_2\bar{z}_2.$$

The group $\mathrm{SU}(1,1)$ acts on $\mathbb{C}^2$ preserving q. The action is free on the complement of the isotropic cone. Let L_0 be the component containing $v_0 = (0,1)$, i.e.

$$L_0 = \big\{(z_1, z_2) \in \mathbb{C}^2 \mid z_1\bar{z}_1 < z_2\bar{z}_2\big\}.$$

There is a canonical bijective map from $\mathrm{SU}(1,1)$ to its orbit

$$G = \mathrm{SU}(1,1)v_0 = \big\{(z_1, z_2) \in \mathbb{C}^2 \mid z_1\bar{z}_1 - z_2\bar{z}_2 = -1\big\} \subset L_0.$$

The space G with the pseudo-metric induced from the pseudo-metric on $\mathbb{C}^2$ is a complete homogeneous Lorentz manifold of signature $(n_+, n_-) = (2,1)$ with constant curvature -1, in other words G is a pseudo-hyperbolic space. The map $\mathrm{SU}(1,1) \to G$ is equivariant with respect to the action of $\mathrm{SU}(1,1)$ on G and the action on itself by left translation. The pseudo-metric induced on $\mathrm{SU}(1,1)$ agrees with the biinvariant metric defined by the Killing form up to multiplication with a scalar factor 8. Henceforth we identify $\mathrm{SU}(1,1)$ with G.

The group $\mathrm{SU}(1,1)$ acts on the hermitian hyperbolic space $\mathbb{D}$ by fractional linear transformations

$$z \longmapsto \frac{az+b}{cz+d},$$

where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is a matrix in $\mathrm{SU}(1,1)$, i.e. $c = \bar{b}$, $a = \bar{d}$ and $ad - bc = 1$. The isotropy group of $0 \in \mathbb{D}$ is $S\big(U(1) \times U(1)\big)$. We denote the corresponding group in G by H. Then we have canonical identifications of homogeneous spaces

$$\mathrm{SU}(1,1)/S\big(U(1) \times U(1)\big) = G/H = \mathbb{D},$$

where the map $G \to G/H$ is given by $(z_1, z_2) \mapsto z_1/z_2$. There is a corresponding map on L_0 defined by $(z_1, z_2) \mapsto z_1/z_2$

$$L_0 \longrightarrow \mathbb{D}.$$

This is a principal $\mathbb{C}^*$-bundle, where the action of $\lambda \in \mathbb{C}^*$ is defined by

$$\lambda(z_1, z_2) := (\lambda^{-1}z_1, \lambda^{-1}z_2).$$

The associated complex line bundle is denoted by

$$L \longrightarrow \mathbb{D}.$$

The group $\mathrm{SU}(1,1)$ acts on this bundle. In order to identify the action, we trivialize L mapping L_0 to $\mathbb{D} \times \mathbb{C}^*$ by $(z_1, z_2) \mapsto (z_1 z_2^{-1}, z_2^{-1})$. This induces the following action of $\mathrm{SU}(1,1)$ on $\mathbb{D} \times \mathbb{C}^*$:

$$(z, v) \longmapsto \left(\frac{az+b}{cz+d}\,, \frac{1}{cz+d}v\right).$$

This identifies the line bundle L^2 as the complex tangent bundle $T_{\mathbb{D}}$ of the hermitian symmetric space $\mathbb{D}$. The action of $\mathrm{SU}(1,1)$ induces the canonical action of $\mathrm{Aut}(\mathbb{D}) = \mathrm{PSU}(1,1)$ on $T_{\mathbb{D}}$. The action of $\mathrm{PSU}(1,1)$ on the hermitian line bundle $T_{\mathbb{D}}$ is simply transitive on the unit circle bundle $ST_{\mathbb{D}}$. Choosing a base point we may identify $\mathrm{PSU}(1,1)$ with $ST_{\mathbb{D}}$. The double covering $\mathrm{SU}(1,1)$ is identified with the circle bundle $G \subset L_0 \subset L$. We may view G as the boundary of a disk bundle in L which is a neighbourhood of the zero section. This is the reason why the locally homogeneous spaces $\Gamma\backslash\widetilde{\mathrm{SU}}(1,1)$ are links of quasi-homogeneous Gorenstein singularities. For a Gorenstein automorphy factor $(\mathbb{D}, L, \bar{\Gamma})$ of level 1 or 2, the punctured singularity $X(L,\bar{\Gamma}) \smallsetminus \{0\}$ equals $\tilde{\Gamma}\backslash L_0$, where $\tilde{\Gamma} \subset \mathrm{SU}(1,1)$ is a lifting of $\bar{\Gamma}$ for level 2, and the inverse image of $\bar{\Gamma}$ for level 1. Note that we have a commutative diagram of maps

$$\begin{array}{ccc} L_0 & & \\ {\scriptstyle\psi}\downarrow & \searrow^{\varphi} & \\ G & \xrightarrow[\chi]{} & \mathbb{D} \end{array}$$

The map φ is the principal $\mathbb{C}^*$-bundle described above, χ is the restriction of φ to $G \subset L_0$ and ψ is the central projection given by

$$\psi(z_1, z_2) = \big((z_2\bar{z}_2 - z_1\bar{z}_1)^{-\frac{1}{2}} z_1, (z_2\bar{z}_2 - z_1\bar{z}_1)^{-\frac{1}{2}} z_2\big).$$

The $\mathbb{C}^*$-action on L_0 induces actions of $\mathbb{R}_+ \subset \mathbb{C}^*$ on L_0 and of $\mathbb{S}^1 \subset \mathbb{C}^*$ on G, so that ψ is a principal $\mathbb{R}_+$-bundle and χ is a principal $\mathbb{S}^1$-bundle.

3.3 Now we shall consider universal coverings. In view of the identification of $\mathrm{SU}(1,1)$ with $G \subset L_0$, the universal covering $\widetilde{\mathrm{SU}}(1,1) \to \mathrm{SU}(1,1)$ identifies with the universal covering $\tilde{G} \to G$. Denote by $\tilde{L}_0$ the induced $\mathbb{R}_+$-bundle over $\tilde{G}$. We have a commutative diagram

$$\begin{array}{ccc} \tilde{L}_0 & \xrightarrow{\pi} & L_0 \\ {\scriptstyle\tilde{\psi}}\downarrow & & \downarrow{\scriptstyle\psi} \\ \tilde{G} & \xrightarrow{\pi'} & G \end{array}$$

The maps $\tilde{\psi}$ and ψ are the projection maps of $\mathbb{R}_+$-bundles, and π and π' are universal covering maps. $\tilde{L}_0$ inherits a pseudo-Riemannian metric of signature $(n_+, n_-) = (2,2)$ from L_0. Both bundles have canonical sections $\tilde{G} \subset L_0$ and $\tilde{G} \subset \tilde{L}_0$. So we might describe them by a canonical trivialization.

However, we find it more convenient to work with another description of $\tilde{L}_0$ obtained as follows. L_0 is contained in

$$\big\{(z_1, z_2) \mid z_2 \neq 0\big\} = \mathbb{C} \times \mathbb{C}^*.$$

We may view L_0 as a bundle of punctured discs imbedded in the $\mathbb{C}^*$-bundle defined by the projection m on the first factor. Consider the universal covering $\pi : \mathbb{C} \times \mathbb{R} \times \mathbb{R}_+ \to \mathbb{C} \times \mathbb{C}^*$ defined by $(z, \alpha, r) \mapsto (z, re^{i\alpha})$. The inverse image of L_0 identifies with the universal covering $\pi : \tilde{L}_0 \to L_0$, where

$$\tilde{L}_0 = \big\{(z, \alpha, r) \in \mathbb{C} \times \mathbb{R} \times \mathbb{R}_+ \mid |z| < r\big\}$$

and $\pi(z,\alpha,r) = (z, re^{i\alpha})$. Moreover $\tilde{G} \subset \tilde{L}_0$ has the following description

$$\tilde{G} = \{(z,\alpha,r) \in \mathbb{C} \times \mathbb{R} \times \mathbb{R}_+ \mid r^2 = 1 + |z|^2\}.$$

The map $\tilde{\psi} : \tilde{L}_0 \to \tilde{G}$ is described as follows:

$$\tilde{\psi}(z,\alpha,r) = \left(\frac{z}{\sqrt{r^2 - |z|^2}}\ , \alpha\ , \frac{r}{\sqrt{r^2 - |z|^2}} \right).$$

3.4 The universal cover $\widetilde{\mathrm{SU}}(1,1)$ of $\mathrm{PSU}(1,1) = \mathrm{Aut}(\mathbb{D})$ acts on $\mathbb{D}$. For $x \in \mathbb{D}$ there is a unique 1-parameter subgroup

$$\rho_x : \mathbb{R} \longrightarrow \widetilde{\mathrm{SU}}(1,1)$$

such that $\rho_x(t)$ acts on $\mathbb{D}$ as rotation through angle t with centre x. It is easy to see that $\rho_0 : \mathbb{R} \to \tilde{G}$ is given by

$$\rho_0(2t) = (0, -t, 1).$$

Moreover, multiplication by $\rho_0(2t)$ from the left is given by

$$\rho_0(2t)(z,\alpha,r) = (e^{it}z, \alpha - t, r).$$

The two generators of the infinite cyclic centre Z of $\widetilde{\mathrm{SU}}(1,1)$ are $\rho_0(\pm 2\pi) = (0, \mp\pi, 1) = \rho_x(\pm 2\pi)$ for all $x \in \mathbb{D}$.

Let k be a natural number. The subgroup of index k in Z has generators $(0, \pm k\pi, 1)$. Given a level k and a natural number p relatively prime to k, we define

$$d := k/p \quad \text{and} \quad r_d := (0, -\pi d, 1) = \rho_0(2\pi k/p).$$

The image of r_d in $\mathrm{PSU}(1,1)$ generates a cyclic group of order p.

Now let $\Gamma \subset \widetilde{\mathrm{SU}}(1,1)$ be a discrete subgroup of level k. Let $\bar{\Gamma}$ be the image of Γ in $\mathrm{PSU}(1,1)$. Assume that $\bar{\Gamma}$ has a fixed point $x \in \mathbb{D}$ of order p. We assume $x = 0$ without loss of generality. Moreover, we make the following assumption which is important for our construction:

$$p > k.$$

Because of 2.6 we have $(k,p) = 1$. Therefore the isotropy group of $0 \in \mathbb{D}$ in Γ is the infinite cyclic group generated by r_d, $d = k/p$.

We shall now start presenting the elements of the construction of a fundamental domain for Γ.

3.5 The advantage of embedding the Lorentz manifold $\mathrm{SU}(1,1)$ as a submanifold G of L_0 in the affine space $\mathbb{C}^2$ with its pseudo-metric comes from the fact that $\mathbb{C}^2$ is flat. The maximal geodesics in $\mathbb{C}^2$ are the real affine lines. The maximal totally geodesic submanifolds are the real affine linear subspaces. Their intersections with L_0 are maximal totally geodesic submanifolds of L_0. The maximal totally geodesic submanifolds of G are the connected components of the intersections of G with real affine linear subspaces of $\mathbb{C}^2$ containing the origin.

We shall use the affine linear geometry in $L_0 \subset \mathbb{C}^2$ in order to define certain totally geodesic hypersurfaces in $\tilde{L}_0$ corresponding to affine tangent hyperplanes of G in L_0. Let g be any element $g \in \tilde{G}$ and $\bar{g}$ its image $\bar{g} = \pi(g)$ in G. The affine tangent hyperplane of $G \subset L_0$ at $\bar{g}$ is

$$\bar{E}_{\bar{g}} = \{y \in L_0 \mid \langle g, y\rangle = -1\}$$

where $\langle \cdot , \cdot \rangle$ is the real part of the hermitian form on $\mathbb{C}^2$. The totally geodesic hypersurface $\bar{E}_{\bar{g}}$ decomposes L_0 into two half-spaces, an "inner" half-space

$$\bar{I}_{\bar{g}} = \big\{ y \in L_0 \mid \langle g, y \rangle \leqslant -1 \big\}$$

and an "outer" half-space $\bar{H}_{\bar{g}}$. The spaces $\bar{E}_{\bar{g}}$ and $\bar{I}_{\bar{g}}$ are simply connected and even contractible. Hence their preimages under $\pi : \tilde{G} \to G$ consist of infinitely many components, one of them containing g.

Definition. For $g \in \tilde{G}$, the spaces $E_g, I_g, H_g \subset \tilde{L}_0$ are defined as follows:

(i) E_g is the component of $\pi^{-1}(\bar{E}_{\bar{g}})$ containing g.
(ii) I_g is the component of $\pi^{-1}(\bar{I}_{\bar{g}})$ containing g.
(iii) $\tilde{L}_0 \backslash E_g$ has two connectedness components. I_g is the closure of one of them. H_g is the closure of the other one. $E_g = I_g \cap H_g$ is the common boundary.

Note that I_g maps bijectively onto $\bar{I}_{\bar{g}}$, whereas H_g is the union of $\pi^{-1}(\bar{H}_{\bar{g}})$ and $\pi^{-1}(\bar{I}_{\bar{g}}) \backslash I_g$.

In terms of the description of $\tilde{L}_0$ given in 3.3, the spaces defined above for any $g \in G$ have the following concrete and simple description for the unit element $e = (0, 0, 1)$

$$\bar{I}_{\bar{e}} = \big\{ (z_1, z_2) \in \mathbb{C}^2 \mid \operatorname{Re}(z_2) \geqslant 1, \ \ |z_1| < |z_2| \big\},$$
$$\bar{E}_{\bar{e}} = \big\{ (z_1, z_2) \in \mathbb{C}^2 \mid \operatorname{Re}(z_2) = 1, \ \ |z_1| < |z_2| \big\}.$$

The boundary of $\bar{E}_{\bar{e}}$ is a rotational hyperboloid of one sheet decomposing the 3-space $\operatorname{Re}(z_2) = 1$ into two components, and $\bar{E}_{\bar{e}}$ is the component containing the axis of rotation.

The corresponding subsets of $\tilde{L}_0$ are as follows:

$$\pi^{-1}(\bar{I}_{\bar{e}}) = \big\{ (z, \alpha, r) \in \mathbb{C} \times \mathbb{R} \times \mathbb{R}_+ \mid r \cos\alpha \geqslant 1, \ \ |z| < r \big\},$$
$$I_e = \big\{ (z, \alpha, r) \in \mathbb{C} \times \mathbb{R} \times \mathbb{R}_+ \mid r \cos\alpha \geqslant 1, \ \ |z| < r, \ \ |\alpha| < \pi/2 \big\},$$
$$E_e = \big\{ (z, \alpha, r) \in \mathbb{C} \times \mathbb{R} \times \mathbb{R}_+ \mid r \cos\alpha = 1, \ \ |z| < r, \ \ |\alpha| < \pi/2 \big\}.$$

We may visualize these sets by means of a projection to the (α, r)-half-plane $\mathbb{R} \times \mathbb{R}_+$. The common boundary E_e of I_e and H_e projects to the curve defined by $|\alpha| < \frac{\pi}{2}$ and $r = 1/\cos\alpha$. This curve decomposes the half-plane into two components. Their closures are

$$X_e = \big\{ (\alpha, r) \in \mathbb{R} \times \mathbb{R}_+ \mid r \leqslant 1/\cos\alpha \ \text{ or } \ |\alpha| \geqslant \pi/2 \big\}$$
$$Y_e = \big\{ (\alpha, r) \in \mathbb{R} \times \mathbb{R}_+ \mid r \geqslant 1/\cos\alpha \ \text{ and } \ |\alpha| < \pi/2 \big\}$$

The condition $|z| < r$ defines H_e as an open disk bundle over X_e and I_e as a disc bundle over Y_e. If we consider I_e and H_e as differentiable manifolds, they are obviously 4-dimensional half-spaces.

The sets I_g and H_g associated to any other element $g \in \tilde{G}$ are obviously obtained from I_e and H_e by the operation of $\tilde{G}$ on $\tilde{L}_0$, i.e. $H_g = gH_e$ etc.

3.6 Suppose we are given positive integers k and p without common divisor. Put $d = k/p$ and consider the infinite cyclic subgroup $\Gamma_d \subset \tilde{G}$ generated by the element $r_d = (0, -\pi d, 1)$ as in section 3.4. This group acts on $\tilde{G}$ by left multiplication. Consider the set

$$Q(d) := \bigcap_{g \in \Gamma_d} H_g.$$

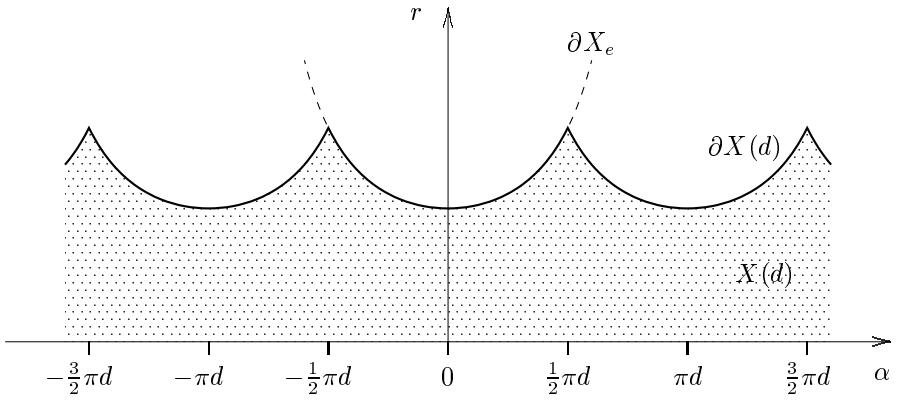


Figure 2: The image $X(d)$ of $Q(d)$ in the (r, α) half-plane

How does it look like? The generator r_d acts as follows:

$$r_d(z, \alpha, r) = (e^{i\pi d} z, \alpha - \pi d, r).$$

It acts on the (α, r)-half-plane by the translation τ_d mapping (α, r) to $(\alpha - \pi d, r)$. In view of $gH_e = H_g$, the images of the sets H_g, $g \in \Gamma_d$ are the translates $\tau_d^n(X_e)$ of the image X_e of H_e described in 3.5. Therefore we see that $Q(d)$ is a disc bundle over the set

$$X(d) := \bigcap_{n \in \mathbb{Z}} \tau_d^n(X_e).$$

Obviously, the nature of this set is very different for the two cases $d < 1$ and $d \geqslant 1$. For instance, in the case $d < 1$, the boundary is connected, whereas for $d \geqslant 1$ there are infinitely many boundary components. Figure 2 shows the case $d < 1$, i.e. $p > k$. The shaded area is the image $X(d)$ of $Q(d)$.

The manifolds $gQ(d)$ play a central role in our construction. So it is important that the reader should understand the geometric nature of these objects. We have described $Q(d)$ as a disc bundle over the set $X(d)$ in the (α, r)-half-plane $\mathbb{R} \times \mathbb{R}_+$. We may describe $Q(d) \subset \tilde{L}_0 \subset \mathbb{C} \times \mathbb{R} \times \mathbb{R}_+$ as

$$Q(d) = (\mathbb{C} \times X(d)) \cap L_0.$$

The reader should think of $X(d)$ as a universal covering of a punctured plane polygon. Consider the following diagram of covering maps

$$\begin{array}{ccc} \mathbb{R} \times \mathbb{R}_+ & \xrightarrow{\pi'} & \mathbb{C}^* \\ & \searrow^{\pi} & \downarrow \pi'' \\ & & \mathbb{C}^* \end{array}$$

where $\pi(\alpha, r) = re^{i\alpha}$ and $\pi'(\alpha, r) = r^{1/k} e^{i\alpha/k}$ and $\pi''(z) = z^k$. Consider the curve $\pi(\partial X(d))$. It is easy to see that this is a regular star polygon $\left\{\frac{2p}{k}\right\}$ when k is odd and a regular star polygon $\left\{\frac{p}{k}\right\}$ when k is even. Therefore the curve $\pi'(\partial X(d))$ is a curvilinear $2p$-gon covering the star polygon once or twice. Let $P' \subset \mathbb{C}$ and $P = P(d) \subset \mathbb{C}$ be the plane areas bounded by the curvilinear polygon $\pi'(\partial X(d))$ and by the star polygon $\pi(X(d))$. The images of $X(d)$ are the punctured plane

polygons $\pi'(X(d)) = P' \setminus \{0\}$ and $\pi(X(d)) = P \setminus \{0\}$. We think of the product $\mathbb{C} \times P'$ as a 4-dimensional $2p$-gonal *prism*. $\mathbb{C} \times X(d)$ is the universal covering of the pierced prism $\mathbb{C} \times (P' \setminus \{0\})$. The product $\mathbb{C} \times P \subset \mathbb{C}^2$ might be considered as a 4-dimensional *"star prism"*. Its axis $\mathbb{C} \times \{0\}$ does not meet $L_0 \subset \mathbb{C} \times \mathbb{C}^*$. Therefore the universal covering $\pi : \tilde{L}_0 \to L_0$ maps $Q(d)$ to the intersection of L_0 with the star prism:

$$\pi(Q(d)) = L_0 \cap (\mathbb{C} \times P(d)).$$

3.7 Let $\Gamma \subset \widetilde{\mathrm{SU}}(1,1)$ be a discrete subgroup of finite level k. Its image $\bar{\Gamma}$ in $\mathrm{PSU}(1,1)$ is a discrete subgroup of $\mathrm{Aut}(\mathbb{D})$. We assume that $u \in \mathbb{D}$ is a fixed point of $\bar{\Gamma}$ of order $p > k$. Set $d = k/p$. The construction of a fundamental domain for the action of Γ on $\widetilde{\mathrm{SU}}(1,1)$ depends on the choice of u.

Let $\Gamma_u \subset \Gamma$ be the isotropy subgroup of u and $\Gamma(u) \subset \mathbb{D}$ the Γ-orbit of u. For $x \in \Gamma(u)$, let $T(x)$ be the left coset of Γ_u

$$T(x) := \big\{ g \in \Gamma \mid g(u) = x \big\}.$$

Definition.

$$Q_x := \bigcap_{g \in T(x)} H_g.$$

The $H_g \subset \tilde{L}_0$ are the "half-spaces" constructed in 3.5. Note that obviously

$$Q_{gu} = gQ_u.$$

The geometry of Q_u has been described in 3.6. We assume without loss of generality $u = 0$. Then

$$Q_u = Q(d),$$

where $d = k/p$ and $Q(d)$ is the universal prismatic set described in 3.6. So all Q_x are obtained from such a prismatic set by the action of Γ on $\tilde{L}_0$.

Definition.

$$P := \bigcup_{x \in \Gamma(u)} Q_x.$$

Now we can state the main result.

Theorem. *The boundary of P is invariant with respect to the action of Γ on $\tilde{L}_0$. For any $g \in \Gamma$ the subset*

$$F_g := \mathrm{Cl}_{\partial P}(\mathrm{Int}_{\partial P}(\partial H_g \cap \partial P))$$

is a fundamental domain for the action of Γ on ∂P. The projection $\tilde{\psi} : \tilde{L}_0 \to \tilde{G}$ induces a Γ-equivariant homeomorphism $\partial P \to \tilde{G}$. The image

$$\mathcal{F}_g := \tilde{\psi}(F_g)$$

is a fundamental domain for the action of Γ on $\tilde{G}$, the universal covering of $\mathrm{SU}(1,1)$. The family $(\mathcal{F}_g)_{g \in \Gamma}$ is a locally finite Γ-equivariant tiling of $\tilde{G}$. For every pair of different elements $g, h \in \Gamma$ the intersection $\mathcal{F}_g \cap \mathcal{F}_h$ lies in a totally geodesic submanifold of $\tilde{G}$. If Γ is co-compact, then F_g and $\mathcal{F}_g$ are compact.

The proof is given in [75].

3.8 The construction of the 4-dimensional polytope P and the Γ-equivariant tiling of its boundary ∂P by the fundamental domains F_g was done in the universal covering $\tilde{L}_0$ of L_0. It descends to the quotient of $\tilde{L}_0$ by the subgroup of index k in the centre of $\tilde{G}$, but in general not to L_0. However, the individual fundamental

domains F_g and $\mathcal{F}_g$ have models $\pi(F_g)$ and $\pi(\mathcal{F}_g)$ in $L_0 \subset \mathbb{C}^2$. Without loss of generality we consider only F_e, since $\pi(\mathcal{F}_g) = \psi(\pi(F_e))$ by radial projection. We also assume that the fixed point is $u = 0$, so that $Q_{gu} = gQ(d)$.

By definition F_e lies in $\partial H_e = E_e$. Recall that $\pi : E_e \to \bar{E}_{\bar{e}}$ is a homeomorphism onto a solid rotational hyperbola lying in the affine tangent space of $G = \mathrm{SU}(1,1)$ at the neutral element $\bar{e}$. Therefore π maps F_e bijectively onto a domain

$$\pi(F_e) \subset \bar{E}_{\bar{e}}$$

lying in that solid hyperbola. Moreover F_e is contained in the intersection of E_e and $Q(d)$. Therefore the image $\pi(F_e)$ lies in $\pi(E_e \cap Q(d))$. This is a piece of the solid hyperbola cut out by two parallel planes orthogonal to the rotational axis. In terms of coordinates $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ we have

$$\pi(E_e \cap Q(d)) = \\ \big\{(z_1, z_2) \in \mathbb{C}^2 \mid x_2 = 1,\ \ x_1^2 + y_1^2 - y_2^2 < 1,\ \ |y_2| \leqslant \tan(\pi d/2)\big\}.$$

The set F_e is obtained from $E_e \cap Q(d)$ by removing the interior of its intersection with the other prismatic sets $Q_{gu} = gQ(d)$. Therefore $\pi(F_e)$ is obtained from the piece of the solid hyperbola $\pi(E_e \cap Q(d))$ by removing those parts of its intersections with the star prisms $\pi(gQ(d))$ that are images of $E_e \cap Q(d) \cap gQ(d)$.

This shows that for a discrete co-compact group $\Gamma \subset \widetilde{\mathrm{SU}}(1,1)$ the image $\pi(F_e)$ of the fundamental domain F_e is a compact polyhedron with flat faces in the three-dimensional flat Lorentz space tangent to G at $\bar{e}$. Thus we have inside the flat Lorentz space $\mathfrak{su}(1,1)$ a polyhedral model for the curved fundamental domains $\mathcal{F}_g$. This polyhedron represents the Lorentz space form $\Gamma \backslash \widetilde{\mathrm{SU}}(1,1)$.

Figure 3 shows how the polyhedron $\pi(F_e)$ is carved from the solid hyperbola by removing intersections with prisms. The example shown in the figure is a fundamental domain for Arnold's exceptional singularity E_{14}. The tables given in 2.8 show that E_{14} has an automorphy factor of level $k = 1$ and signature $(3, 3, 4)$. The fundamental domain is constructed for the fixed points of order 4. Because of $k = 1$ the star prisms are honest prisms, and the order 4 leads to prisms with an octagonal base. Figure 3 is a slightly improved version of figure 4 in [11].

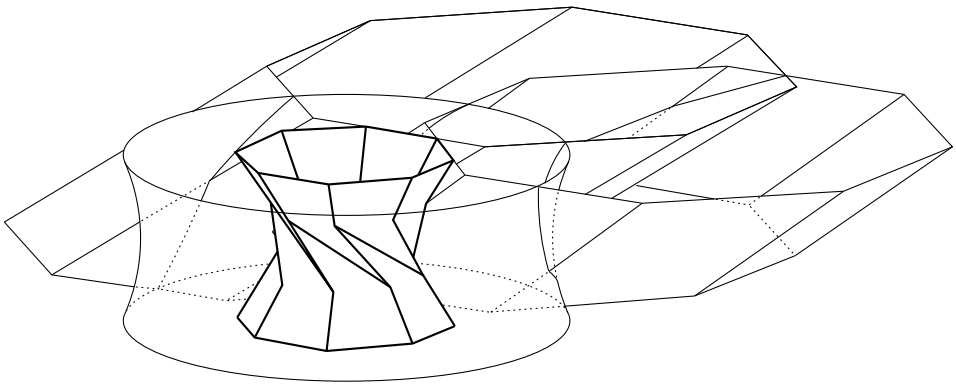


Figure 3: The construction in the case E_{14}

4. Fundamental domains for E_m, Z_m, Q_m

4.1 Anybody who has come to know the construction of fundamental domains described in the last section will want to see examples. But if he tries to do some examples, he is going to discover that there is a remarkable contrast between the elegance of the general construction and the hard work required for the explicit determination of the fundamental domains for a given class of discrete groups. The examples presented in this section were calculated in [75] on more than hundred pages and could not be done on less. The examples in the next section were done in [78], and the analysis of those three examples needed about two hundred pages without preceding preparations.

There is an obvious explanation for these difficulties. The definition of the polytope P and the fundamental domain F_e given in 3.7 involves all prismatic sets Q_x for the infinitely many points $x \in \Gamma(u)$ in the orbit of the fixed point u of Γ chosen for the construction. When Γ is co-compact, only finitely many Q_x are needed in the construction of F_e. However, there is no reasonable a priori estimate to tell us up to which distance from u points $x \in \Gamma(u)$ have to be taken into account. In fact in section 5 we shall see an example where the number of essential prisms Q_x varies in the Teichmüller space of Γ and goes to infinity when we approach the boundary.

4.2 The choice of the examples presented in this paper was motivated by two kinds of experiences. One motivation has been described in the introduction. It is the belief that the series E, Z, Q play a distinguished role. The second motivation comes from the previous experience with calculations of fundamental domains in [35], [48], [74]. The authors of [35] and [48] calculated the fundamental domains of the Fischer construction for all 14 triangle groups of Arnold's 14 exceptional unimodular quasi-homogeneous singularities and for all choices of fixed points except those of order two. Altogether, these are 27 examples of fundamental domains. The experience with these examples shows two things. First, the choice of the fixed point of highest order leads to the fundamental domain with the highest degree of symmetry. As we will shortly see, this is not too surprising. Secondly, the choice of the fixed point of the highest order seems to be suitable for the arrangement of singularities in series. This assumption was confirmed in [74] by the calculation of fundamental domains of the six triangle groups of level 2 which correspond to bimodular exceptional quasi-homogeneous singularities.

We do believe that all these fundamental domains are interesting and that more calculations for other series both for highest order of the fixed points and lower orders would lead to new insight into the nature of Arnold's series and the relations between the series. However, we have decided to adhere to the principle stated by Pappus of Alexandria quoted at the beginning of this paper. Pappus uses this principle when he introduces the Archimedian polyhedra coming right after the Platonic solids because of their regularity. So we have chosen to calculate the fundamental domains for the series E_m, Z_m, Q_m as well as for the three cases $E_{3,0}$, $Z_{1,0}$, $Q_{2,0}$, and in all cases we have chosen the fixed point of highest order. In our opinion, the results confirm our expectations. In particular, the series of polyhedra for E_m with m even, for Z_m with m odd, and for Q_m with m even, is simple, regular and beautiful. Other parts of the results are more subtle and will be discussed later on.

Before we go on, the reader should contemplate the figures of tables 5–7 showing the fundamental domains for the series E_m, Z_m, Q_m. We refer to the legend preceding the tables for explanations concerning the drawing of the figures. We emphasize that the figures are strictly accurate representations of precisely calculated polyhedra.

4.3 The group $\Gamma \subset \tilde{G}$ acts on $\tilde{G}$ by left multiplication, and this action extends to an action on $\tilde{L}_0$ by isometries. This induces an action on P and on ∂P and finally an action of Γ on the tiling $(\mathcal{F}_g)_{g\in G}$ which is simply transitive. However, there may be other isometries of $\tilde{L}_0$ which act on the tiling. Those of these isometries which map a particular $\mathcal{F}_g$ onto itself will be called symmetries of $\mathcal{F}_g$. We are interested in the group of these symmetries or subgroups of this group. It suffices to describe these symmetries for the linear model of F_e.

The group $\tilde{G}$ acts on itself by left multiplications and also by right multiplications. Any isometry in the connected component of the identity is a product of a left multiplication and a right multiplication. In particular, we have the subgroup $\bar{G} \cong \mathrm{PSU}(1,1)$ of inner automorphisms and its adjoint representation on $\mathfrak{su}(1,1)$, the space containing the linear model of the fundamental domain. The isometry group of $\tilde{G}$ has four connected components. They may be described as follows. The element $\varepsilon \in \mathrm{Isom}(\tilde{G})$ is defined by $\varepsilon(g) = g^{-1}$. The isometry $\eta \in \mathrm{Isom}(\tilde{G})$ is the involutive automorphism defined by $\pi(\eta(g)) = \overline{\pi(g)}$. We have

$$\mathrm{Isom}(\tilde{G}) = \mathrm{Isom}(\tilde{G})_0 \rtimes \{1, \varepsilon, \eta, \varepsilon\eta\},$$
$$\mathrm{Isom}^+(\tilde{G}) = \mathrm{Isom}(\tilde{G})_0 \rtimes \{1, \eta\}.$$

The isometries of $\tilde{G}$ lift to isometries of $\tilde{L}_0$. The symmetry groups of our polyhedra F_e will be dihedral groups of the form

$$\langle\kappa\rangle \rtimes \langle\eta\rangle,$$

where κ is an inner automorphism of finite order.

Now let $\Gamma \subset \tilde{G}$ be a discrete co-compact subgroup of level k, such that $0 \in \mathbb{D}$ is a fixed point of order p for $\bar{\Gamma} \subset \mathrm{PSU}(1,1)$, and let F_e be the fundamental domain for Γ with this fixed point. As before, let $\rho_0 : \mathbb{R} \to \tilde{G}$ be the 1-parameter subgroup such that $\rho_0(t)$ acts on $\mathbb{D}$ by the rotation $\zeta \mapsto e^{it}\zeta$. Let $\kappa(t) \in \mathrm{Isom}(\tilde{G})_0$ be the conjugation by $\rho_0(t)$. This isometry acts on $\tilde{L}_o$ as follows:

$$\kappa(t)(z, \alpha, r) = (e^{it}z, \alpha, r).$$

The isometry $\kappa(2\pi/p)$ comes from conjugation with a generator of the isotropy group Γ_0. Thus Γ, Γ_0, P, ∂P and F_e are invariant under $\kappa(2\pi/p)$. Therefore the symmetry group of F_e contains at least the cyclic group $\langle\kappa(2\pi/p)\rangle$ of order p. However, there may be more rotational symmetry. Suppose for example that $\bar{\Gamma}$ is a triangle group $\Gamma(p,q,r)$, where $0 \in \mathbb{D}$ is the fixed point of order p. If $q = r$, the normalizer of $\bar{\Gamma}$ in $\mathrm{Aut}(\mathbb{D}) = \mathrm{Isom}^+(\mathbb{D})$ contains the rotation by the angle π/p with centre 0, whereas the isotropy group $\bar{\Gamma}_0$ is generated by the rotation by the angle $2\pi/p$. Therefore, in this case the symmetry group of F_e contains the cyclic group $\kappa(\pi/p)$ of order $2p$.

Other symmetries may occur when there is a reflection in $\mathrm{Isom}(\mathbb{D})$ which normalizes $\bar{\Gamma}$ and the isotropy group $\bar{\Gamma}_0$. Without loss of generality we may assume that the reflection is given by $\zeta \mapsto \bar{\zeta}$. When the signature of $\bar{\Gamma}$ has genus 0, the group $\Gamma \subset \tilde{G}$ is uniquely determined by Γ. In this case it is obvious that $\eta(\Gamma) = \Gamma$

and $\eta(\Gamma_0) = \Gamma_0$. Therefore, in this case, the involutive isometry η is a symmetry of the fundamental domain F_e. In particular, these arguments apply to all triangle groups, since they are those normal subgroups of index 2 in the group generated by reflections in the sides of the triangles which consist of orientation preserving isometries. We assume without loss of generality that the triangle group is normalized by the reflection $\zeta \mapsto \bar{\zeta}$.

4.4 A discrete co-compact subgroup Γ of level k in $\widetilde{\mathrm{SU}}(1,1)$ such that the image in $\mathrm{PSU}(1,1)$ is a triangle group with signature $(\alpha_1, \alpha_2, \alpha_3)$ will be denoted by $\Gamma(\alpha_1, \alpha_2, \alpha_3)^k$. We assume without loss of generality that $\alpha_1 \leqslant \alpha_2 \leqslant \alpha_3$ and that $0 \in \mathbb{D}$ is a fixed point of order α_3. When we consider a fundamental domain F_e of Γ, we always mean the fundamental domain for the fixed point 0. Moreover, we assume without loss of generality that Γ is normalized by η.

Definition. The symmetry index $q(\Gamma)$ of $\Gamma = \Gamma(\alpha_1, \alpha_2, \alpha_3)^k$ is defined by

$$q(\Gamma) = \begin{cases} \alpha_3, & \text{if } \alpha_1 < \alpha_2, \\ 2\alpha_3, & \text{if } \alpha_1 = \alpha_2. \end{cases}$$

In 2.8 we have given tables showing the groups $\Gamma(\alpha_1, \alpha_2, \alpha_3)^k$ corresponding to singularities of the series E_m, Z_m, Q_m. Whenever it is convenient, we shall denote these groups by the symbols E_m, Z_m, Q_m.

The following two tables list all singularities E_m, Z_m, Q_m and show the symmetry index of their group Γ. In both tables n is a positive integer. In the table on the left n is not divisible by 3. We shall say that singularities or groups listed on the left are of type I and those on the right of type II.

Γ	$q(\Gamma)$
E_{10+2n}	$n+6$
Z_{9+2n}	$2n+6$
Q_{8+2n}	$3n+6$

Type I

Γ	$q(\Gamma)$
E_{7+6n}	$2n+3$
Z_{6+6n}	$4n+2$
Q_{5+6n}	$6n+1$

Type II

Theorem. *Let $\Gamma = \Gamma(\alpha_1, \alpha_2, \alpha_3)^k$ be the group corresponding to one of the singularities of the series E_m, Z_m, Q_m. The fundamental domain F_e of Γ has the symmetry group*

$$\mathrm{Sym}(F_e) = \langle \kappa(2\pi/q(\Gamma)) \rangle \rtimes \langle \eta \rangle.$$

This is a dihedral group of order $2q(\Gamma)$, where $q(\Gamma)$ is the symmetry index of Γ.

The inclusion $\langle \kappa \rangle \rtimes \langle \eta \rangle \subset \mathrm{Sym}(F_e)$ is obvious. The arguments for the other inclusion are given in [11] Proposition 8 and [75], p. 41.

4.5 The fundamental domains for the groups Γ of type I are sufficiently simple so that we can describe them in this expository paper. For those of type II we refer to the figures of the tables and to [75].

There are two different levels of precision in the description of the fundamental polyhedra $\pi(F_e)$. A precise description has to determine such a polyhedron as a certain subspace of the affine Lorentz space. This may be done by giving all vertices and the partially ordered structure of the facets. Or we may present the polyhedron by some construction beginning with half-spaces and applying the operations of union and intersection. This is what we shall do.

The second and lower level of precision is the purely combinatorial description of the partially ordered structure of the facets. There is a systematic way of describing these data for a group acting on a tiling, which was developed by A. Dress. Not withstanding the advantages of such a systematic approach, we prefer a simpler and naive description of the combinatorial structure which is adequate for the tilings which we want to describe. This approach is also suitable for the analysis of the tilings in the next section, where the combinatorial structure is not constant on the Teichmüller space.

We shall now indicate a precise construction for the model fundamental domains of type I. These polyhedra live in the flat Lorentz space of signature $(n_+, n_-) = (2, 1)$. However, such a polyhedron has a distinguished rotational axis of symmetry. The direction of this axis is negative definite, and the orthogonal complement is positive definite. Changing the sign of the pseudo-metric in the direction of the axis of rotation transforms Lorentz space into a well-defined Euclidean space. In this way, the model fundamental domain is transformed into a polyhedron in Euclidean space with dihedral symmetry. We are going to give a construction, or rather two constructions for such polyhedra in $\mathbb{R}^3$.

Let κ_q be the rotation of $\mathbb{R}^3$ around the z-axis by the angle $2\pi/q$. Let η be the rotation around the x-axis by the angle π. These rotations generate the dihedral group $\langle\kappa_q\rangle \rtimes \langle\eta\rangle$ of order $2q$. Let H^+ be an half-space bounded by a plane which is not parallel to a coordinate axis, and let H^- be the half-space $H^- = \eta H^+$. We assume that the wedge $H^+ \cap H^-$ does not meet the z-axis. The wedge meets the (x, y)-plane in a certain sector with some angle α. We assume that $0 < \alpha - 2\pi/q < \pi$. Let ω be some positive real number. We define the following subset of $\mathbb{R}^3$:

$$P(H, q, \omega) := \big(\mathbb{R}^3 \setminus \bigcup_{i=1}^{q} \kappa_q^i (H^+ \cap H^-)^0\big) \cap \big(\mathbb{R}^2 \times [-\omega, \omega]\big).$$

This is a compact polyhedron with symmetry group $\langle\kappa_q\rangle \rtimes \langle\eta\rangle$. We shall call it a *polyhedron of type Ia*. We can modify the construction replacing the wedge by a blunted wedge where the edge has been cut off by a plane parallel to the edge and to the z-axis. We call polyhedra obtained by this modified construction of *type Ib*.

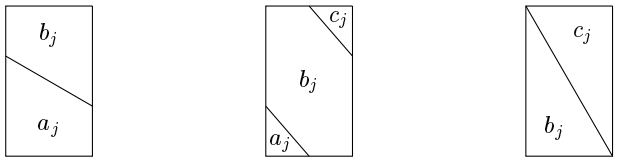
Much of the labour in calculations of the fundamental domains consists in reducing their theoretical construction given in 3.7 to an explicit description such as the one given in the following theorem.

Theorem. *Let $\Gamma \subset \widetilde{\mathrm{SU}}(1, 1)$ be a discrete co-compact subgroup which belongs to one of the series E_m, Z_m, Q_m. Let $q = q(\Gamma)$ be its symmetry index. Suppose that Γ is of type I. Then the fundamental domain for Γ is a polyhedron in Lorentz space whose symmetry group is a dihedral group of order $2q$. It is of type Ia for the series E_m and Q_m and of type Ib for the series Z_m.*

4.6 Once we have obtained a description of the fundamental domain F_e as in 4.5 where the faces are identified as components of intersections $F_e \cap F_g$, it is easy to deduce the following description of the combinatorial structure.

We shall describe the identification of faces of F_e by pairings of flags (f, e) and (f', e'), where f is a face and e is an edge of the face. Such a pairing is enough to describe the identification of f and f', since the identification reverses the orientation. When f_1 and f_2 are adjacent faces with common edge $e = f_1 \cap f_2$, the flag (f_1, e) will be denoted by $(f_1; f_2)$.

Let P_q be a regular q-gonal prism. The rectangular faces are numbered in cyclic order. For each of the types E, Z, Q define $P_q(E)$, $P_q(Z)$, $P_q(Q)$ as the prism P_q together with a subdivision of the rectangular faces which is equivariant with respect to the dihedral group of orientation preserving symmetries of P_q. The subdivision of the j-th face is described by figure 4 together with a notation for the faces. The q-gonal faces on top and bottom of the prism are denoted by d_+ and d_-.



Type E Type Z Type Q

Figure 4: The subdivision of the j-th face of the prism P_q

Theorem. *Let $\Gamma \subset \widetilde{\mathrm{SU}}(1,1)$ be a group of type I belonging to one of the series E_m, Z_m, Q_m. Let $q = q(\Gamma)$ be its symmetry index. The fundamental domain for Γ constructed in 3.7 has the same combinatorial type as $P_q(E)$, $P_q(Z)$ and $P_q(Q)$ respectively. The face identification is equivariant with respect to the dihedral symmetry of these prisms. It is given by the following table of pairs of flags.*

Γ	$q(\Gamma)$	pairings
E_{10+2n}	$n+6$	$(a_j; b_j) \longleftrightarrow (b_{j-3}; d_+)$ $\quad (d_+; b_j) \longleftrightarrow (d_-; a_{j-n})$
Z_{9+2n}	$2n+6$	$(a_j; b_j) \longleftrightarrow (c_{j-3}; d_+)$ $\quad (d_+; b_j) \longleftrightarrow (d_-; a_{j-n})$ $(b_j; c_j) \longleftrightarrow (b_{j-3-n}; d_+)$
Q_{8+2n}	$3n+6$	$(b_j; c_j) \longleftrightarrow (c_{j-3-n}; d_+)$ $\quad (d_+; c_j) \longleftrightarrow (d_-; b_{j-n})$

These identifications of faces are illustrated on table 8.

The results of 4.5 and 4.6 cover 6 of the 9 cases in the first table in 2.8. The remaining three cases with signature $(2,4,p)$ are considerably more complicated. We have calculated fundamental domains for all these cases, as illustrated in tables 5–7, in [75]. However, at present it is not proved for all p that these fundamental domains coincide with those constructed in 3.7. We are convinced that this is true.

5. Fundamental domains for $E_{3,0}$, $Z_{1,0}$, $Q_{2,0}$

5.1 The second table in 2.8 shows that the automorphy factors for $E_{3,0}$, $Z_{1,0}$ and $Q_{2,0}$ have level 1 and signature $(0; 2, 2, 2, p)$, where $p = 3$, 4 and 5 respectively. Since the level is 1, it is enough to consider Fuchsian groups of signature $(0; 2, 2, 2, p)$ in $\mathrm{PSU}(1,1)$ and their preimages in $\mathrm{SU}(1,1)$. The construction of fundamental domains in $\mathrm{SU}(1,1)$ can be carried out within the framework of the original construction of Thomas Fischer.

We begin with a description of the real analytic Teichmüller space of Fuchsian groups with signature $(0; 2, 2, 2, p)$. The essential idea is the use of Fricke coordinates and goes back to Fricke [37], p. 335–341 and [38], p. 296–299.

Let Γ_p be the group defined by the following presentation:

$$\Gamma_p := \langle \gamma_0, \gamma_1, \gamma_2, \gamma_3 \mid \gamma_0^p = \gamma_1^2 = \gamma_2^2 = \gamma_3^2 = \gamma_0\gamma_1\gamma_2\gamma_3 = 1 \rangle.$$

The group of orientation preserving automorphisms of Γ_p is defined as

$$\mathrm{Aut}^+(\Gamma_p) := \{\varphi \in \mathrm{Aut}(\Gamma_p) \mid \exists \alpha \in \Gamma \ \ \varphi(\gamma_0) = \alpha\gamma_0\alpha^{-1}\}.$$

The group of inner automorphisms is a subgroup, and the modular groups are defined as

$$\mathrm{Mod}^+(\Gamma_p) := \mathrm{Aut}^+(\Gamma_p)/\mathrm{Inn}(\Gamma_p) \cong \mathrm{PSL}(2,\mathbb{Z})$$
$$\mathrm{Mod}\ \ (\Gamma_p) := \mathrm{Aut}\ \ (\Gamma_p)/\mathrm{Inn}(\Gamma_p) \cong \mathrm{PGL}(2,\mathbb{Z})$$

The representation space $\mathcal{R}(\Gamma_p)$ and the Teichmüller space $\mathbf{T}(\Gamma_p)$ are defined as follows:

$$\mathcal{R}(\Gamma_p) := \{d \ \in \mathrm{Hom}(\Gamma_p, \mathrm{PSU}(1,1)) \mid d \text{ injective and } d(\Gamma) \text{ discrete}\}.$$
$$\mathbf{T}(\Gamma_p) := \mathrm{Aut}(\mathrm{PSU}(1,1)) \setminus \mathcal{R}(\Gamma_p).$$

The moduli space and the reduced moduli space for Fuchsian groups with signature $(0; 2, 2, 2, p)$ are the quotients

$$\mathbf{T}(\Gamma_p)/\mathrm{Mod}^+(\Gamma_p) \quad \text{and} \quad \mathbf{T}(\Gamma_p)/\mathrm{Mod}(\Gamma_p).$$

We shall construct an isomorphism

$$\Phi : \mathbf{T}(\Gamma_p) \longrightarrow \mathcal{T}_p$$

of the real analytic Teichmüller space with a real analytic variety $\mathcal{T}_p$ which is a connectedness component of the real cubic hypersurface $\mathcal{V}_p$ in $\mathbb{R}^3$ given by the following equation:

$$t_1^2 + t_2^2 + t_3^2 - t_1t_2t_3 - 4\sin^2(\pi/p) = 0.$$

The cubic $\mathcal{V}_p$ has tetrahedral symmetry and has five connectedness components separated by the planes $t_i = \pm 2$. The component $\mathcal{T}_p$ is defined as follows:

$$\mathcal{T}_p = \{(t_1, t_2, t_3) \in \mathcal{V}_p \mid t_1, t_2, t_3 > 2\}.$$

We define three special elements $\delta_i \in \Gamma_p$ as follows:

$$\delta_1 := \gamma_0\gamma_1, \quad \delta_2 := \gamma_2\gamma_0, \quad \delta_3 := \gamma_1\gamma_2.$$

The coordinate functions Φ_i of the map Φ are the Fricke coordinates defined by

$$\Phi_i(d) = |\mathrm{trace}\ d(\delta_i)|, \quad i = 1,\ 2,\ 3.$$

The canonical action of the modular group $\mathrm{Mod}(\Gamma_p)$ on $\mathbf{T}(\Gamma_p)$ is transferred to $\mathcal{T}_p$ via Φ. The modular group acts on $\mathcal{T}_p$ as a group generated by reflections S_i and S_i' defined as follows: Let $\{i, j, k\} = \{1, 2, 3\}$. Then S_i permutes the coordinates t_j and t_k, whereas S_i' replaces t_i by

$$t_i' = t_jt_k - t_i.$$

We want to construct a fundamental triangle $\Delta_p \subset \mathcal{T}_p$ such that $\mathrm{Mod}(\Gamma_p)$ is the group generated by the reflections in the sides of Δ_p. The construction is illustrated by figure 5.

The figure shows an image of $\mathcal{T}_p$ obtained by central projection from $0 \in \mathbb{R}^3$ onto the projective plane. The projection maps $\mathcal{T}_p$ one to one onto the equilateral triangle with sides $z_i = 0$, where $i = 1, 2, 3$. The fixed point sets of the S_i are mapped onto the straight lines bisecting the angles of the triangle. The images

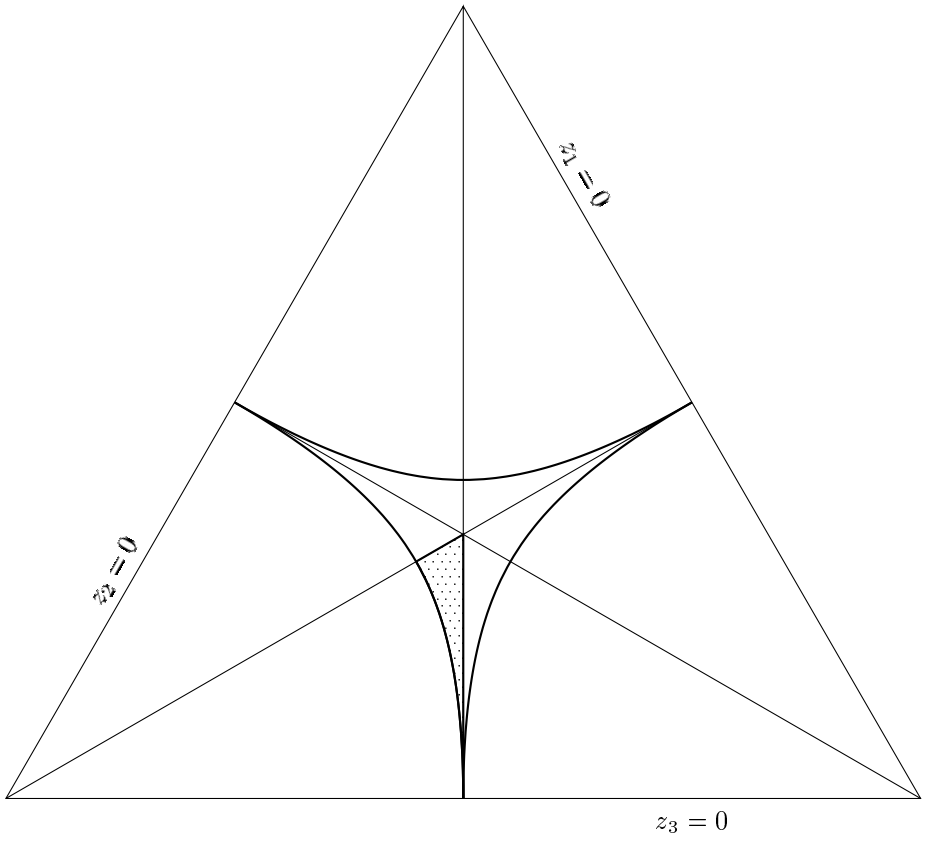


Figure 5: The image of $\mathcal{T}_p$ in the projective plane

of the fixed point sets of the S_i' are three curves which form a curvilinear triangle with cuspidal vertices on the boundary. The bisectors subdivide the curvilinear triangle into six smaller triangles. The preimages of these triangles are the six subsets $X_{ij} \subset \mathcal{T}_p$ defined as follows:

$$X_{ij} = \big\{t \in \mathcal{T}_p \mid t_i \geqslant t_j \geqslant t_k,\ \ t_j t_k \geqslant 2t_i\big\}.$$

The reflections in the sides of X_{ij} are S_i, S_k and S_i'. The shaded triangle in the Figure 5 is X_{12}. We choose

$$\Delta_p = X_{12}$$

as a fundamental domain for the triangle group

$$\mathrm{Mod}(\Gamma_p) = \big\langle S_1, S_3, S_1' \big\rangle.$$

$\mathrm{Mod}^+(\Gamma_p)$ is the subgroup of index two preserving the orientation, and we might choose $X_{12} \cup X_{21}$ as fundamental domain for this group.

5.2 It suffices to study the construction of the fundamental domain $F(d)$ defined in subsection 3.7 as $F(d) = F_e(d(\Gamma_p))$ for representations $d \in \mathcal{R}(\Gamma_p)$ which satisfy the following conditions:

(i) $0 \in \mathbb{D}$ is a fixed point of $d(\gamma_0)$, and $d(\gamma_0)$ is the rotation by the angle $2\pi/p$.
(ii) $\Phi(d) \in \Delta$.

Let us call such d normalized and reduced, and let us denote the subset of these representations by $\mathcal{R}(\Gamma_p)^*$.

We are going to need a precise description of the elements in the preimages $\widetilde{d(\Gamma_p)} \subset \mathrm{SU}(1,1)$ of the groups $d(\Gamma_p) \subset \mathrm{PSU}(1,1)$. Consider the group $\tilde{\Gamma}_p$ presented as follows:

$$\tilde{\Gamma}_p = \langle r_0, r_1, r_2, r_3 \mid r_0^{2p} = r_1^4 = r_2^4 = r_3^4 = r_0 r_1 r_2 r_3 = 1 \rangle.$$

There is a natural way of lifting elements of finite order in $\mathrm{PSU}(1,1)$ to elements of twice that order if we consider these elements as contained in 1-parameter groups of rotations and lift these 1-parameter groups. In this way we get for any $d \in \mathcal{R}(\Gamma_p)^*$ well-defined elements $r_i(d) \in \mathrm{SU}(1,1)$ by lifting $d(\gamma_i)$. Note that $r_0(d) = r_0$ is constant. There is an isomorphism

$$\tilde{d} : \tilde{\Gamma}_p \longrightarrow \widetilde{d(\Gamma_p)}$$

defined by $\tilde{d}(r_i) = r_i(d)$.

5.3 We shall now begin with Fischer's construction of the fundamental domains $F(d)$ for $d \in \mathcal{R}(\Gamma_p)^*$. We recall two elements of that construction. Recall that in 3.5 we have defined for any $g \in G = \mathrm{SU}(1,1) \subset L_0 \subset \mathbb{C}^2$ a certain "half-space" $\bar{I}_g \subset L_0$ bounded by the tangent hyperplane $\bar{E}_g$. Recall also from 3.8 that we have considered a certain intersection $E_e \cap Q(d)$ of a tangent space and a prismatic set. We have described the image $S_e := \pi(E_e \cap Q(d))$ in the tangent space $\bar{E}_e$ of $e \in G$ as a certain piece of a solid rotational hyperbola. Using these elements we define the following polyhedron in the tangent space $\bar{E}_e$ of G at e

$$F_0(d) = S_e \cap \bigcap_{m=0}^{p-1} \bigcap_{i=1}^{3} \bigcup_{n=p+1}^{2p-1} \bar{I}_{r_0^m r_i(d) r_0^{n-m}}.$$

This is not yet the Fischer domain $F(d)$. But $F(d)$ will be constructed by intersecting a finite number of polyhedra of this type. In order to get them, we define the following automorphism $\chi \in \mathrm{Aut}^+(\Gamma)$:

$$\chi(\gamma_0, \gamma_1, \gamma_2, \gamma_3) = (\gamma_0, \gamma_1 \gamma_2 \gamma_1^{-1}, \gamma_1, \gamma_3).$$

The first main result is the following theorem.

Theorem. *For $p = 3, 4, 5$ and $d \in \mathcal{R}(\Gamma_p)^*$ the following statements hold:*

(i) The Fischer fundamental domain $F(d)$ for the Fuchsian group $d(\Gamma_p)$ of signature $(0; 2, 2, 2, p)$ can be described as follows

$$\boxed{F(d) = \bigcap_{\lambda=-\infty}^{\infty} F_0(d \circ \chi^{-\lambda})}$$

(ii) This intersection is finite. Therefore

$$\boxed{F(d) = \bigcap_{\lambda=\lambda_-(d)}^{\lambda_+(d)} F_0(d \circ \chi^{-\lambda})}$$

with uniquely determined maximal $\lambda_-(d) \in \mathbb{Z}$ and minimal $\lambda_+(d) \in \mathbb{Z}$.

(iii) $\lambda_-(d) \leqslant 0 \leqslant \lambda_+(d)$

(iv) $\lambda_-(d) + \lambda_+(d) \in \{0, 1\}$

(v) $\Lambda(d) = \lambda_+(d) - \lambda_-(d)$ is lower semi-continuous on $\mathcal{R}(\Gamma_p)^*$.

The proof given in [78] is very complex. To some extent it uses the analysis of the combinatorial structure in the individual cases $p = 3, 4, 5$. In the present exposition we take this main theorem as a point of departure for the description of the results in the individual cases which will be given below.

5.4 Note that the element $\chi \in \mathrm{Aut}^+(\Gamma_p)$ defined in 5.3 acts on $\mathcal{T}_p$ as a generator for the infinite cyclic isotropy group of $\mathrm{Mod}^+(\Gamma_p)$ at the cuspidal vertex of Δ, since $\Phi(d \circ \chi) = S_1' \cdot S_3(\Phi(d))$. Therefore, for $d \in \mathcal{R}(\Gamma_p)^*$, the representations $d \circ \chi^n$ in the main theorem have images $\Phi(d \circ \chi^n)$ in the following set

$$\bigcup_{n \in \mathbb{Z}} (S_1' \cdot S_3)^n (X_{12} \cup X_{21}).$$

This is a neighbourhood of the cusp in the Satake-Borel-Bailey topology.

5.5 We shall now state the results of the analysis for the three cases $p = 3$, 4 and 5. There are certain very interesting features which are common to all three cases, but there are also differences so that we prefer to present the individual cases in the order of increasing complexity. We shall deal with $p = 3$ in section 5.6, while $p = 5$ is done in 5.7 and $p = 4$ in 5.8. The results are illustrated on tables 10–12 for the individual cases, table 9 for all three cases and on tables 2–4 in a synopsis of the results of all three authors.

The functions λ_+, λ_- and $\Lambda = \lambda_+ - \lambda_-$ defined in 5.3 induce corresponding functions on Δ_p which we shall denote with the same symbols. Λ is a lower semi-continuous function $\Lambda : \Delta_p \to \mathbb{N}$. For any nonnegative integer n we consider the interior of the corresponding preimage in Δ_p,

$$\Delta_p^{(n)} := \Lambda^{-1}(n)^0.$$

In all three cases the vertices of the fundamental triangle Δ_p will play a special role. We shall denote the vertex with angle $\pi/3$ by v_0, the one with angle $\pi/2$ by v_1. They are the fixed points of $\mathrm{Mod}^+(\Gamma_p)$ in Δ_p of order 3 and 2 respectively. These points correspond to special values of the j-invariant of the quasi-homogeneous singularities. There are several possible normal forms for these singularities (see e.g. [13], p. 191). Consider the following ones:

$$\begin{aligned} E_{3,0}: &\quad x^3 + ax^2y^3 + xy^6 + z^2, \\ Z_{1,0}: &\quad x^3y + ax^2y^3 + xy^5 + z^2, \\ Q_{2,0}: &\quad x^3 + ax^2y^2 + xy^4 + yz^2. \end{aligned}$$

Then the j-invariant is

$$j = \frac{4}{27} \cdot \frac{(a^2-3)^3}{a^2-4}.$$

The values $j = 0$ and $j = 1$ are attained for $a^2 = 3$ and $a = 0$. The point v_0 corresponds to $j = 0$, and v_1 corresponds to $j = 1$. We shall therefore refer to the fundamental domains of groups $d(\Gamma_p)$ with $\Phi(d) = v_0$ or $\Phi(d) = v_1$ as *the fundamental domains for $j = 0$ or $j = 1$*. These fundamental domains are distinguished by special symmetries. Moreover, they are distinguished by a very interesting feature which we shall observe in each of the six cases $p = 3$, 4, 5 and $j = 0$, 1. Namely, each of them fills a well-defined gap in one of the six series E_m, Z_m, Q_m of type I or of type II. If the reader has not yet noticed these gaps, he should look again at the tables of symmetry indices in 4.4 and contemplate the

figures on tables 5–7. Tables 2–4 show how the gaps are filled. The fundamental domains for $j = 0$ and $j = 1$ fit perfectly with respect to combinatorial structure, symmetry and identification of faces.

5.6 For $p = 3$ we can prove $\lambda_- \equiv 0$. Thus Δ_3 decomposes into the two open subsets $\Delta_3^{(0)}$ and $\Delta_3^{(1)}$ and a curve C_3 separating these regions. We have $v_0 \in \Delta_3^{(0)}$ and $v_1 \in \Delta_3^{(1)}$. The curve C_3 is defined by the equation

$$-t_1 + t_2 + t_3 = 2.$$

This decomposition $\Delta_3 = \Delta_3^{(0)} \cup C_3 \cup \Delta_3^{(1)}$ is shown on table 9.

Theorem. *There are three combinatorial types of fundamental domains $F(d)$ for Fuchsian groups $d(\Gamma_3)$. For $d \in \mathcal{R}(\Gamma_3)^*$ the type of $F(d)$ is constant on $\Delta_3^{(0)}$, on C_3 and on $\Delta_3^{(1)}$.*

Figures 1–3 on table 10 show examples for the three combinatorial types for $p = 3$. The numbers of the figures are the same as those of the corresponding points of Δ_3 shown on table 9.

Corollary. *The fundamental domains for $E_{3,0}$ fill the gaps*

(i) for $j = 0$ between E_{14} and E_{18},
(ii) for $j = 1$ between E_{13} and E_{19}.

The corollary is illustrated by table 2.

5.7 For $p = 5$ we can prove $\lambda_- \equiv 0$. Thus Δ_5 decomposes into two open subsets $\Delta_5^{(0)}$ and $\Delta_5^{(1)}$. They are separated by a curve C_5 defined by the equation

$$\begin{aligned} -t_1t_2t_3 + t_2^2t_3 + t_2t_3^2 + t_1^2 - t_1t_2 - t_1t_3 \\ -(1+\tau)t_2t_3 + \tau t_1 + \tau t_2 + \tau t_3 - \tau = 0, \end{aligned}$$

where $\tau = (\sqrt{5} - 1)/2$. We have $v_0 \in \Delta_5^{(0)}$ and $v_1 \in \Delta_5^{(1)}$. We must refine this stratification of Δ_5 in order to get a stratification of Δ_5 by the combinatorial type of fundamental domains. The domain $\Delta_5^{(0)}$ is subdivided into two open domains $\Delta_5^{(0)\prime}$ and $\Delta_5^{(0)\prime\prime}$ by a curve C_5' defined by the equation

$$-t_2^2 + t_1t_2 + \tau t_2t_3 - 2\tau t_1 - t_3 + 2 - \tau = 0.$$

The figure at the right hand on table 9 shows that C_5' runs from the vertex v_0 to the cusp. The vertex v_0 is not considered as a point of the curve. So we have defined a decomposition of Δ_5 into 6 disjoint strata:

$$\Delta_5 = \{v_0\} \cup \Delta_5^{(0)\prime} \cup C_5' \cup \Delta_5^{(0)\prime\prime} \cup C_5 \cup \Delta_5^{(1)}.$$

We have marked one point on each stratum, numbered in this order. Table 11 shows the corresponding fundamental domains with the same numbering.

Theorem. *There are six combinatorial types of fundamental domains $F(d)$ for Fuchsian groups $d(\Gamma_5)$. For $d \in \mathcal{R}(\Gamma_5)^*$ the type of $F(d)$ is constant on the six strata of Δ_5 defined above. In particular, the combinatorial type for $j = 0$ occurs only at the isolated point v_0.*

Corollary. *The fundamental domains for $Q_{2,0}$ fill the gaps*

(i) for $j = 0$ between Q_{12} and Q_{16},
(ii) for $j = 1$ between Q_{11} and Q_{17}.

The corollary is illustrated by table 4.

5.8 The analysis of the case $p = 4$ led to a result which we did not expect at all: the existence of infinitely many different combinatorial types of fundamental domains for Fuchsian groups with signature $(0; 2, 2, 2, 4)$.

For $p = 4$ it turns out that the lower semi-continuous map $\Lambda : \Delta_4 \to \mathbb{N}$ to the nonnegative integers is surjective. Therefore, one gets a decomposition into infinitely many open sets $\Delta_4^{(n)}$, $n \geqslant 0$. It turns out that anyone of these domains is adjacent to its successor $\Delta_4^{(n+1)}$, and that there is a connected curve $C_4^{(n)}$ separating $\Delta_4^{(n)}$ and $\Delta_4^{(n+1)}$. The first of these curves $C_4^{(0)}$ is defined by the following equation:

$$t_2 t_3 - t_1 - t_2 - t_3 + \sqrt{2} = 0.$$

This equation defines a curve C_4 in all of $\mathcal{T}_4$, which intersects Δ_5 in $C_4^{(0)}$. The other curves $C_4^{(n)}$ are obtained from C_4 by applying the reflections S_3 and S_1' by turns. Altogether we get an infinite stratification

$$\Delta_4 = \Delta_4^{(0)} \cup C_4^{(0)} \cup \Delta_4^{(1)} \cup C_4^{(1)} \cup \Delta_4^{(2)} \cup C_4^{(2)} \cup \dots$$

The stratification is illustrated by the figure in the middle of table 9. We have again $v_0 \in \Delta_4^{(0)}$, $v_1 \in \Delta_4^{(1)}$.

Theorem. *There are infinitely many combinatorial types of fundamental domains $F(d)$ for Fuchsian groups $d(\Gamma_4)$. For $d \in \mathcal{R}(\Gamma_4)^*$ the combinatorial type is constant as long as $\Phi(d)$ remains in one of the strata defined above.*

Table 12 shows four fundamental domains corresponding to four points in the first four strata $\Delta_4^{(n)}$, $n = 0, 1, 2, 3$. The four points are shown on table 9.

Corollary. *The fundamental domains for $Z_{1,0}$ fill the gaps*

(i) for $j = 0$ between Z_{13} and Z_{17},
(ii) for $j = 1$ between Z_{12} and Z_{18}.

The corollary is illustrated by table 3.

6. Fundamental domains for $\tilde{E}_8$, $\tilde{E}_7$, $\tilde{E}_6$

6.1 The results of Dolgachev quoted in section 2 imply that the links of singularities of type $\tilde{E}_8$, $\tilde{E}_7$, $\tilde{E}_6$ can be described as $\Gamma\backslash G$, where G is the group of unipotent upper triangular 3×3-matrices and Γ is a discrete co-compact subgroup. Prior to this, Milnor had given such a description for the link as a differentiable manifold, where $\Gamma \subset G \cap \mathrm{SL}(2, \mathbb{Z})$ was the congruence subgroup modulo κ, where $\kappa = 1, 2, 3$ for $\tilde{E}_8$, $\tilde{E}_7$, $\tilde{E}_6$. However, Milnor's description did not involve the moduli of these singularities, and Milnor considered his proof as "rather ad hoc" and wrote "I do not know whether there exists a more natural construction of these diffeomorphisms", [56].

The approach of Dolgachev leads to more natural constructions. But if we want to describe the quotient $\Gamma\backslash G$ by a fundamental domain, in contrast to the spherical case SU(2) and to the Lorentz case SU(1,1) we do not have a natural pseudo-metric coming from the Lie group, and we do not have a general construction such as the classical construction in the spherical case and the generalized Fischer construction developed by A. Pratoussevitch. Nevertheless, we shall make an "ad hoc" construction which is fit to fill the gap between the spherical case and the Lorentz case.

6.2 By the work of K. Saito on simply elliptic singularities [80] it is known that the singularities of type $\tilde{E}_8$, $\tilde{E}_7$, $\tilde{E}_6$ are obtained by contracting the zero section of a line bundle over an elliptic curve with Chern class $-\kappa$, where $\kappa = 1, 2, 3$. Therefore, the links of these singularities are the corresponding $\mathbb{S}^1$-bundles.

Complex line bundles over complex tori are described by the theorem of Appel-Humbert, Mumford [64], p. 20. The specialization to the case of elliptic curves is as follows.

Let $\mathbb{H}$ be the upper half-plane, $\tau = \rho + i\sigma \in \mathbb{H}$ and Γ_τ the lattice $\mathbb{Z} + \mathbb{Z}\tau \subset \mathbb{C}$. The complex line bundles over the elliptic curve $X_\tau = \mathbb{C}/\Gamma_\tau$ are constructed as follows. We define a hermitian form H on $\mathbb{C}$ by

$$H(z, w) = \tfrac{k}{\sigma}(z\bar{w}).$$

Let $\alpha = (\alpha_1, \alpha_2) \in \mathbb{S}^1 \times \mathbb{S}^1$ a pair of complex numbers of absolute value 1. For $u = m + n\tau \in \Gamma_\tau$, define

$$e_u(z) = \alpha_1^m \alpha_2^n \cdot \exp\big(\pi(ikmn + H(z, u) + \tfrac{1}{2}\, H(u, u))\big).$$

The lattice Γ_τ acts on $\mathbb{C} \times \mathbb{C}$ as follows:

$$u(z, \lambda) = (z + u, e_u(z) \cdot \lambda).$$

The projection to the first factor defines a complex line bundle $L_{k,\tau,\alpha} = \mathbb{C} \times \mathbb{C}/\Gamma_\tau$ over X_τ with Chern number k. The theorem of Appel-Humbert says that any complex line bundle over X_τ is isomorphic to a unique $L_{k,\tau,\alpha}$. Two bundles $L_{k,\tau,\alpha}$ and $L_{k,\tau,\beta}$ differ only by a translation. In our case $k = -\kappa$, where $\kappa = 1, 2, 3$.

The link of the singularity obtained by contracting the zero section identifies with $L_{k,\tau,\alpha}/\mathbb{R}_+$, and this identifies with $\mathbb{C} \times \mathbb{S}^1/\Gamma_\tau$, where $u = m + n\tau \in \Gamma_\tau$ acts as $u(z, \lambda) = (z + u, \varepsilon_u(z) \cdot \lambda)$ with

$$\varepsilon_u(z) = \alpha_1^m \alpha_2^n \cdot \exp\big(i\pi(kmn + \operatorname{Im} H(z, u))\big).$$

We evaluate the symplectic form $\omega(z, u) = \operatorname{Im} H(z, u)$. For $u = m + n\tau$ and $z = \xi + \eta\tau$ with real ξ, η we have

$$\omega(z, u) = \kappa(n\xi - m\eta).$$

6.3 We shall now pass to the universal covering $\mathbb{C} \times \mathbb{R} \to \mathbb{C} \times \mathbb{S}^1$ mapping (z, t) to $(z, e^{i\pi t})$. We define a Heisenberg group structure on $\mathbb{C} \times \mathbb{R}$ by means of the symplectic form $\omega = \operatorname{Im} H$:

$$(u, s) \cdot (z, t) := (u + z, s + t + \omega(z, u)).$$

Let us denote $\mathbb{C} \times \mathbb{R}$ with this group structure depending on κ and τ by $\mathcal{H}_{\kappa,\tau}$. We shall describe the links of our singularities as quotients of $\mathcal{H}_{\kappa,\tau}$ by discrete subgroups. We describe these discrete subgroups as representations of an abstract group Π_κ isomorphic to the fundamental group of the link ($\kappa = 1, 2, 3$):

$$\Pi_\kappa = \big\langle a, b, c \mid aba^{-1}b^{-1} = c^\kappa,\ \ ac = ca,\ \ bc = cb \big\rangle.$$

Recall that in 6.2 we used $\alpha = (\alpha_1, \alpha_2) \in \mathbb{S}^1 \times \mathbb{S}^1$ in our construction of a bundle over $\mathbb{C}/\Gamma_\tau$. Passing to the universal covering, we have to use instead a pair of real numbers $\varepsilon = (\varepsilon_1, \varepsilon_2)$, where $\alpha_\nu = e^{i\pi\varepsilon_\nu}$. Now we can define a representation

$$\rho_\varepsilon : \Pi_\kappa \longrightarrow \mathcal{H}_{\kappa,\tau}$$

as follows:

$$\rho_\varepsilon(a) = (1, \varepsilon_1), \quad \rho_\varepsilon(b) = (\tau, \varepsilon_2), \quad \rho_\varepsilon(c) = (0, 2).$$

It is easy to see that ρ_ε is injective and that the image is a discrete co-compact subgroup

$$\rho_\varepsilon(\Pi_\kappa) =: \Gamma_{\kappa,\tau,\varepsilon} \subset \mathcal{H}_{\kappa,\tau}.$$

This discrete group operates on $\mathcal{H}_{\kappa,\tau}$ by left multiplication, and the following proposition follows immediately from the definitions and 6.2.

Proposition. *$\Gamma_{\kappa,\tau,\varepsilon}\backslash\mathcal{H}_{\kappa,\tau}$ identifies with the link of the singularities obtained by contracting the zero section in $L_{-\kappa,\tau,\alpha}$.*

6.4 The parameters (τ,ε) are points in a 4-dimensional space of representations of Π_κ. We shall simplify the analysis by two different reductions. The first reduction is to consider only τ in a fundamental domain $\Delta \subset \mathbb{H}$ of the modular group. Δ is the triangle defined by

$$\Delta := \{\tau \in \mathbb{H} \mid \tau\bar\tau \geqslant 1,\ -1 \leqslant \tau+\bar\tau \leqslant 0\}.$$

The vertices are $v_0 = e^{2\pi i/3}$ and $v_1 = i$ and the cusp at infinity. For $\tau \in \Delta$, we consider the Dirichlet cell D_τ of $0 \in \mathbb{C}^2$ for the lattice Γ_τ. For $\tau \in \Delta$ not on the imaginary axis D_τ is a hexagon. The adjacent Dirichlet cells belong to ±1, $\pm\tau$ and $\pm(1+\tau)$. When τ tends to the imaginary axis, the Dirichlet cell degenerates into a rectangle. D_τ is a regular hexagon for $v = v_0$ and a square for $v = v_1$.

Now consider the prism

$$D_\tau \times [-1,1] \subset \mathcal{H}_{\kappa,\tau}.$$

It is obvious from the definition of $\Gamma_{\kappa,\tau,\varepsilon}$ that we may choose this prism as a fundamental domain for $\Gamma_{\kappa,\tau,\varepsilon}$ acting on $\mathcal{H}_{\kappa,\tau}$ by left multiplication. However, we have to subdivide the rectangular faces in $\partial D_\tau \times [-1,1]$ if we want that the identifications on the boundary of the prism maps faces to faces. The minimal subdivisions satisfying this condition are canonical, and we define

$$P_{\kappa,\tau,\varepsilon} = D_\tau \times [-1,1]$$

as the prism with this subdivision of $\partial D_\tau \times [-1,1]$.

The second reduction is guided by the principle of highest symmetry stated in 4.2. We want that $P_{\kappa,\tau,\varepsilon}$ should have a dihedral symmetry group of order 12 for $\tau = v_0$ and of order 8 for $\tau = 1$. For any $\tau \in \Delta$, the subdivision of a rectangular face of the prism in $\partial D_\tau \times [0,1]$ should be invariant under rotation of the face around its center by 180°. It is easy to see that these conditions are equivalent to the condition $\varepsilon_1, \varepsilon_2 \in \mathbb{Z}$. Therefore, we assume without loss of generality

$$\varepsilon_1, \varepsilon_2 \in \{0,1\}.$$

After this reduction one has to analyze $12 = 3 \cdot 4$ families of fundamental domains $P_{\kappa,\tau,\varepsilon}$, where $\kappa = 1$, 2, 3 and $\varepsilon = (\varepsilon_1,\varepsilon_2)$ and $\tau \in \Delta$. This is an exercise in linear algebra. In each case it is easy to determine the stratification of Δ by combinatorial types and the most symmetric polytopes for the vertices v_0 and v_1 of Δ. We shall be content to state the result pertinent to the main theme of this article. We define

$$\varepsilon(\kappa) = \begin{cases} (1,1), & \text{for } \kappa = 1 \text{ and } 3, \\ (0,0), & \text{for } \kappa = 2. \end{cases}$$

Proposition. *The six fundamental domains $P_{\kappa,\tau,\varepsilon(\kappa)}$ with $\kappa = 1,2,3$ for $\tau = v_0$ and $\tau = v_1$ correspond to the links of the simply elliptic singularities $\tilde E_8$, $\tilde E_7$, $\tilde E_6$*

with j-invariants $j = 0$ and $j = 1$. They are the six fundamental domains shown on table 13.

Corollary.

(i) $P_{1,\tau,(1,1)}$ *for* $j = 0$ *fits between* E_8 *and* E_{12}.
(ii) $P_{2,\tau,(0,0)}$ *for* $j = 0$ *fits between* E_7 *and* Z_{11}.
(iii) $P_{3,\tau,(1,1)}$ *for* $j = 0$ *fits between* E_6 *and* Q_{10}.

The corollary is illustrated by table 1.

7. Concluding Remarks

7.1 We believe that the work of Vladimir Igorevich Arnold on series of singularities and our work on polyhedra representing Lorentz space form for such series foreshadow the existence of some structure as yet invisible. Therefore, we want to conclude with some remarks on open problems, history and future perspectives.

As for open problems there are at least three problems resulting from our article. Problem number one is the analysis of the series $E_{n,0}$, $Z_{n,0}$, $Q_{n,0}$. This may be a formidable task. At least we have done the three first cases. We may expect that fundamental domains for these series fit into the gaps of the series E_n, Z_n, Q_n.

Problem number two is the determination of the complex structure of the Teichmüller spaces $\mathcal{T}_p$. This is the unsolved problem of the accessory parameters. We wish we could calculate the j-invariant of a quadrangle singularity from a given point of $\mathcal{T}_p$. For it is known that singularities with special values of the j-invariant allow exotic deformations. First examples were given by F. Pham and C. T. C. Wall. Afterwards, there was extensive work on this done by our group, [12], [13], [42]. For example the exotic deformations of $E_{3,0}$, $Z_{1,0}$ and $Q_{2,0}$ into combinations of simple singularities occur exactly for $j = 0$ and $j = 1$:

$j = 0$	$j = 1$
$E_{3,0} \longrightarrow E_6 + E_8$	$E_{3,0} \longrightarrow E_7 + E_7$
$Z_{1,0} \longrightarrow E_8 + E_5,\ E_7 + E_6$	$Z_{1,0} \longrightarrow E_7 + D_6$
$Q_{2,0} \longrightarrow E_8 + 2A_2,\ E_6 + E_6$	$Q_{2,0} \longrightarrow E_7 + A_5$

For the other three bimodular quadrangle singularities exotic deformations occur also for other values of j (see [13], p. 56). For example $W_{1,0} \to D_{13}$ occurs for

$$j = \frac{5^3 \cdot 1093^3}{2^{12} \cdot 11^2}.$$

One may wonder about the meaning of these special values of the j-function. Do they have anything to do with special properties of our fundamental domains?

The third problem is to understand the unexpected phenomenon of infinitely many combinatorial types for the signature $(0; 2, 2, 2, 4)$ as opposed to finitely many types for $(0; 2, 2, 2, 3)$ and $(0; 2, 2, 2, 5)$.

7.2 In 1983 Arnold published a list of "Some open problems in the theory of singularities" [9]. In it Arnold posed the problem "A, D, E", which consists in finding a general classification theorem from which one could derive the solutions of the many different problems in which there appear "unexpectedly" the Dynkin diagrams of type A, D, E. It seems to us that such a problem raises questions about the nature of our science. The "unexpected" occurrence of the same combinatorial

structure in solutions of different problems may be due to the fact that in all these problems we are trying to classify objects of a particular simple nature and that in all cases the conditions necessary for their construction or existence reduce to the same simple structure of some combinatorial nature which we do not yet see. However, this structure might be something very abstract of a metamathematical nature. Frequently in the history of mathematics concrete individual objects of a simple and regular nature appear many years before they find a place in the framework of some general structure.

It seems to us that this may still be the status of Arnold's series of singularities. As we have seen, first examples appeared 100 years before Arnold found his series. And yet Arnold himself has to say ([10], Vol. I, p. 243):

> After a series has been found, we can define it. However a general definition of a series of singularities is not known.

7.3 There is no doubt that the series of quasi-homogeneous singularities defined by Arnold are meaningful. Their meaning appears in the context of various mathematical theories, as pointed out in 2.2. The work of many mathematicians has shown regular patterns within individual series or in the relations between series or common to many of them. It would lead us to far to quote all these articles. We would like to mention only a few results of our group apart form those already quoted: The results of W. Ebeling [31] and Ebeling and C. T. C. Wall [32] on quadratic forms and monodromy groups of singularities and on Arnold's "strange duality" between Dolgachev numbers and Gabrielov numbers, the results of C. Hertling on Torelli type theorems for Arnold's unimodular and bimodular singularities and other quasi-homogeneous singularities [45], [46], the results of Greuel, Hertling and Pfister on moduli spaces of semi-quasihomogeneous singularities [43], and the recent work of K. Möhring [58] on numerical invariants and series of quasi-homogeneous singularities which led to the discovery of a certain regular pattern for the system of several of Arnold's series and the introduction of new series which fit into this pattern.

7.4 The approach presented in this paper offers a new perspective on regular patterns related to Arnold's series. Our regularity is that of a combinatorial pattern, the combinatorics and symmetry of the fundamental domain constructed in perfect generality by Anna Pratoussevitch. This pattern can be used as an instrument for the exploration of relations between series of quasi-homogeneous Gorenstein surface singularities. At the same time it is an instrument for the exploration of relations between series of closed Lorentz space forms.

Some aspects of this combinatorial pattern are nice and simple, at least for sufficiently simple examples. Other aspects show surprisingly subtle properties even in the case of simple examples such as the Fuchsian group of signature $(0; 2, 2, 2, 4)$. These subtle phenomena should not be ignored or rejected because of the contrast between their complexity and the apparent simplicity of the normal forms of such singularities.

7.5 The remarks about the appearance of individual nice objects which later become examples of a general theory or construction applies to our construction too. When Thomas Fischer had found his construction, we discovered that one of our combinatorial patterns had appeared many years before, albeit without any realization of a connection with Lorentz space forms. However, there was some contact

with two of the fields mentioned before: space forms and Seifert fibre spaces. Here is the story, as we know it from a letter of H. Seifert, who got it from the diary of W. Threlfall. In 1933 Seifert and Weber had published a joint paper entitled "Die beiden Dodekaederräume" [91]. They constructed a spherical space form and a hyperbolic space form by identifying opposite faces of a dodecahedron by screw motions with angles $\pi/5$ and $3\pi/5$. In [83], p. 209 and [87], I, § 12 Seifert and Threlfall identified the spherical dodecahedral space as the unique closed orientable Seifert fibre space with finite fundamental group different from the sphere. Early in 1938 a student who had written a masters thesis on space groups asked Threlfall for a topic for a PhD-thesis. His name was H. Friedgé. In January 1938 Seifert showed Friedgé the position of the three exceptional fibres of multiplicity 2, 3, 5 in the spherical dodecahedral space. At the end of the year, Friedgé presented his thesis entitled "Verallgemeinerung der Dodekaederräume". It was published in 1940 in Mathematische Zeitschrift [39].

In his thesis Friedgé examines an infinite series of closed 3-manifolds obtained by identification of faces of certain polyhedra. The polyhedra are not realized in some affine space. The construction is purely topological. In essence the polyhedra are the same as our prisms with the subdivision of the rectangular faces described in 4.6, figure 4, type E. And the identification of faces is the same as the one shown on table 8 for the E-series, type I.

Friedgé calculates the fundamental group and homology of his manifolds and notices the period 6 in his series. For those of his manifolds which are homology spheres he constructs a Seifert fibration with his bare hands and calculates the multiplicities of the fibres. They are $(2, 3, 6k \pm 1)$ and agree with the signature $(\alpha_1, \alpha_2, \alpha_3)$ for E_{4m} in our table 2.8. This identifies his manifolds with knot-manifolds obtained as coverings of the sphere ramified over the trefoil knot.

Finally, he notices that there are other schemes for the identification of faces of the same polyhedra leading to other manifolds.

7.6 A similar remark applies to the polyhedra which we have found for the type I Z-series. In 1983/84 E. Molnár has given a combinatorial construction of an infinite series of twice punctured compact hyperbolic manifolds obtained from such polyhedra [59], [60]. Of course, his identification scheme is different from ours.

There is a rich literature on combinatorial constructions of hyperbolic space forms. Combinatorial constructions for Lorentz space forms seem to be rare. But we have found at least one such construction, again by E. Molnár. It was found around 1988 and presented in a short note [61]. Molnár constructs a doubly infinite series of 3-manifolds by identification of the faces of polyhedra obtained from a tetrahedron by a subdivision of the faces depending on two natural numbers m and n. He claims that this is a Seifert fibre space and that the corresponding 2-dimensional orbifold belongs to a triangle group with signature $(2, m, n)$. So for $1/2 + 1/m + 1/n < 1$ the universal covering is $\widetilde{\mathrm{SL}}(2, \mathbb{R})$. The case of signature $(2, 3, a)$, with $a > 6$ is treated in [62], section 3. Again we see the appearance of the simplest possible cases. We do not see whether there is a relation between that combinatorial construction and our construction of fundamental domains, which is not only combinatorial but geometrical in the sense of Lorentz geometry.

7.7 When we see all these different combinatorial constructions of infinite series of polyhedra and space forms of different geometries related to "series" of presentations and representations of discrete groups we may dream of a theory comprising

them all and giving us also a general notion of series of singularities. For the time being we are happy with what we have found. When we asked Seifert about the motivation for the thesis of Friedgé, he replied:

> At that time we were delighted by every new three dimensional manifold.

List of Tables

The image $\pi(F_e)$ of the fundamental domain F_e for a discrete co-compact group $\Gamma \subset \widetilde{\mathrm{SU}}(1,1)$ of finite level is a compact polyhedron in $\mathfrak{su}(1,1)$ with flat faces. The Lie algebra $\mathfrak{su}(1,1)$ is a 3-dimensional flat Lorentz space of signature $(n_+, n_-) = (2,1)$. Such a polyhedron has a distinguished rotational axis of symmetry. The direction of this axis is negative definite, and the orthogonal complement is positive definite. Changing the sign of the pseudo-metric in the direction of the rotational axis transforms Lorentz space into a well-defined Euclidean space. The image $\pi(F_e)$ of the fundamental domain is then transformed into a polyhedron in Euclidean space with dihedral symmetry. Tables 1–7 and 10–13 show the Euclidean polyhedra obtained in this way. The direction of the rotational axis is vertical. The top and bottom faces are removed.

The polyhedra in tables 5–7 are all scaled by the same factor to illustrate the proportions between different fundamental domains. The same is true for tables 10–13. On the contrary the polyhedra in tables 1–4 are scaled by different factors in such a way that all the figures in the same table seem to have the same size.

Table 8 illustrates the identification scheme for E_m, Z_m, Q_m in the equianharmonic case. The face identification is equivariant with respect to the dihedral symmetry of the polyhedron. The faces shaded in the same way are identified. Arrows on the edges of shaded faces indicate the identified flags (face, edge, vertex).

Table 9 shows fundamental domains Δ_p for the group $\mathrm{Mod}(\Gamma_p)$ and their stratifications by curves. Some points in Δ_p are marked and numbered. Their numbers correspond to the numbers of figures in tables 10–12.

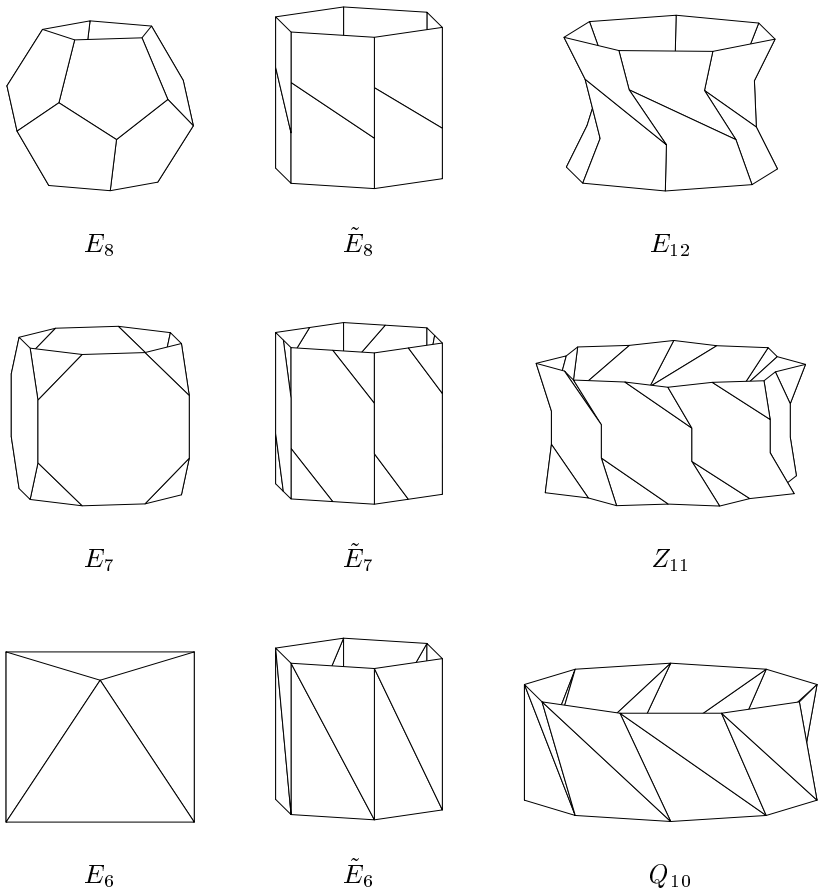
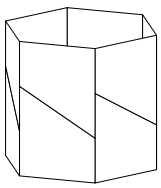
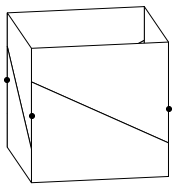


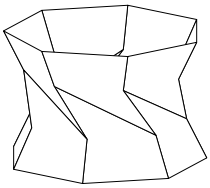
Table 1: Fundamental domains for the boundary layer singularities



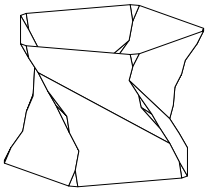
$\tilde{E}_8$, $j = 0$



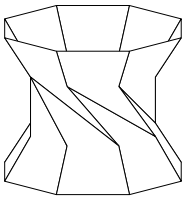
$\tilde{E}_8$, $j = 1$



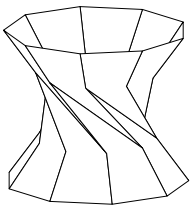
E_{12}



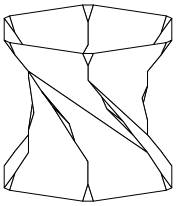
E_{13}



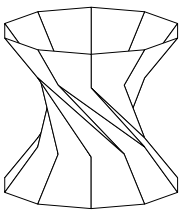
E_{14}



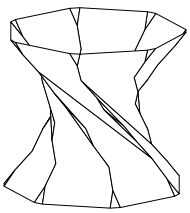
$E_{3,0}$, $j = 0$



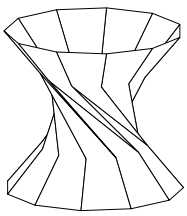
$E_{3,0}$, $j = 1$



E_{18}

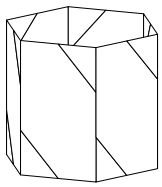


E_{19}

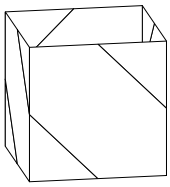


E_{20}

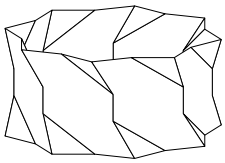
Table 2: Fundamental domains for $\tilde{E}_8$, E_{12}, E_{13}, E_{14}, $E_{3,0}$, E_{18}, E_{19}, E_{20}



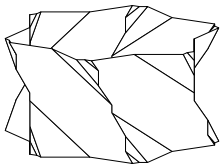
$\tilde{E}_7$, $j = 0$



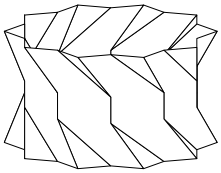
$\tilde{E}_7$, $j = 1$



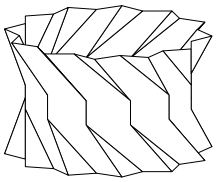
Z_{11}



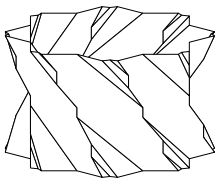
Z_{12}



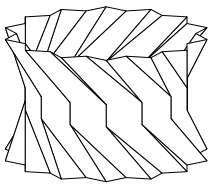
Z_{13}



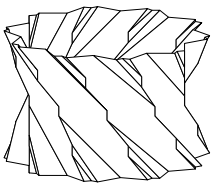
$Z_{1,0}$, $j = 0$



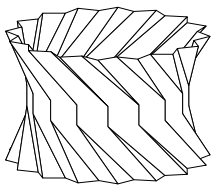
$Z_{1,0}$, $j = 1$



Z_{17}



Z_{18}



Z_{19}

Table 3: Fundamental domains for $\tilde{E}_7$, Z_{11}, Z_{12}, Z_{13}, $Z_{1,0}$, Z_{17}, Z_{18}, Z_{19}

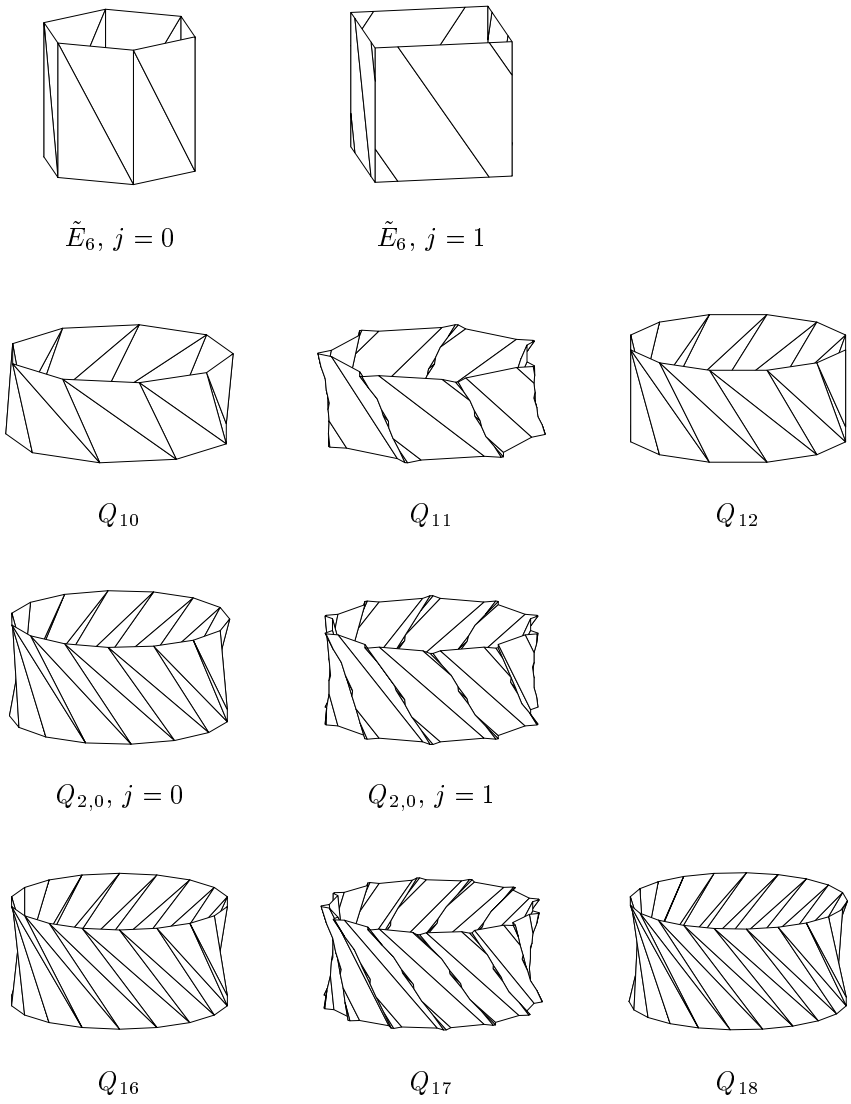
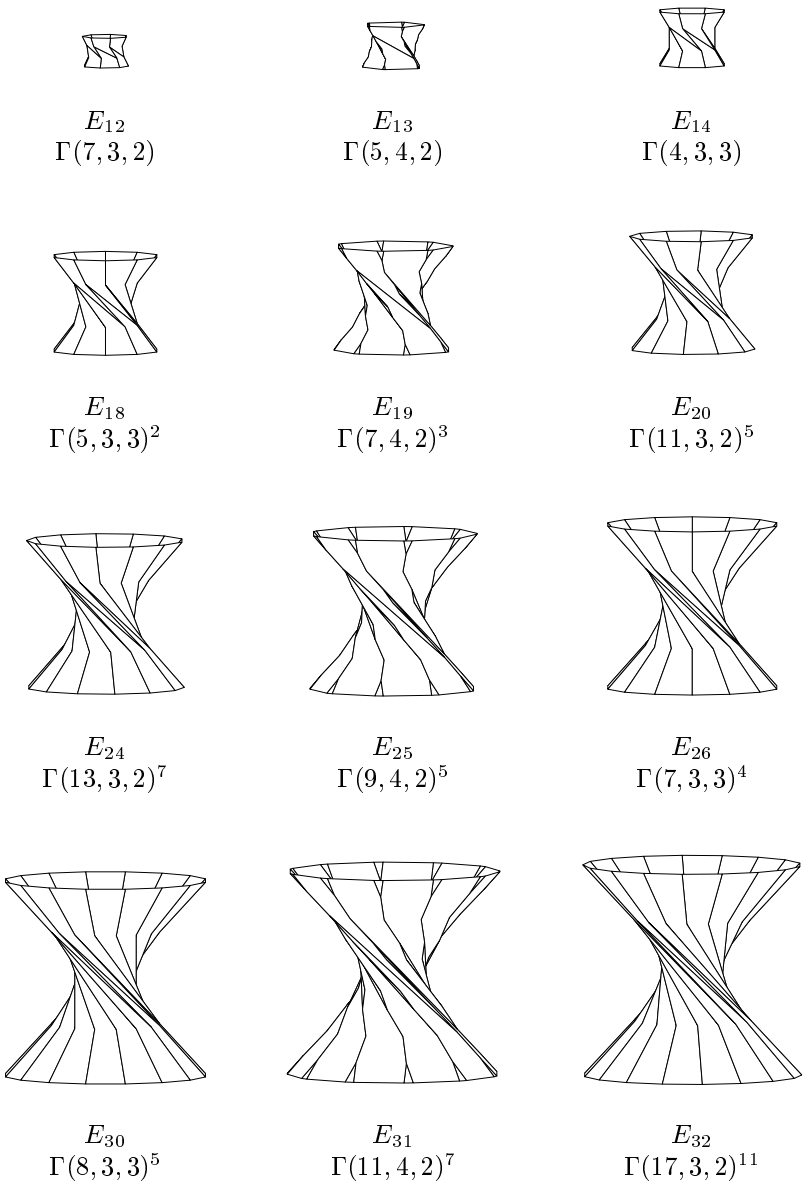


Table 4: Fundamental domains for $\tilde{E}_6$, Q_{10}, Q_{11}, Q_{12}, $Q_{2,0}$, Q_{16}, Q_{17}, Q_{18}



E_{12}
$\Gamma(7,3,2)$

E_{13}
$\Gamma(5,4,2)$

E_{14}
$\Gamma(4,3,3)$

E_{18}
$\Gamma(5,3,3)^2$

E_{19}
$\Gamma(7,4,2)^3$

E_{20}
$\Gamma(11,3,2)^5$

E_{24}
$\Gamma(13,3,2)^7$

E_{25}
$\Gamma(9,4,2)^5$

E_{26}
$\Gamma(7,3,3)^4$

E_{30}
$\Gamma(8,3,3)^5$

E_{31}
$\Gamma(11,4,2)^7$

E_{32}
$\Gamma(17,3,2)^{11}$

Table 5: Fundamental domains for the beginning of the E-series

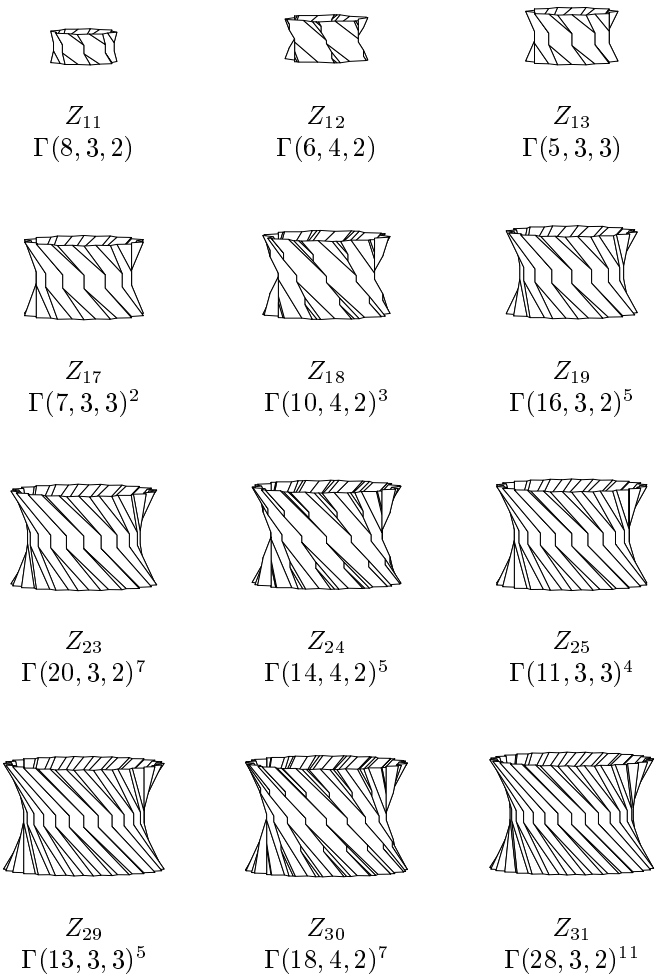


Table 6: Fundamental domains for the beginning of the Z-series

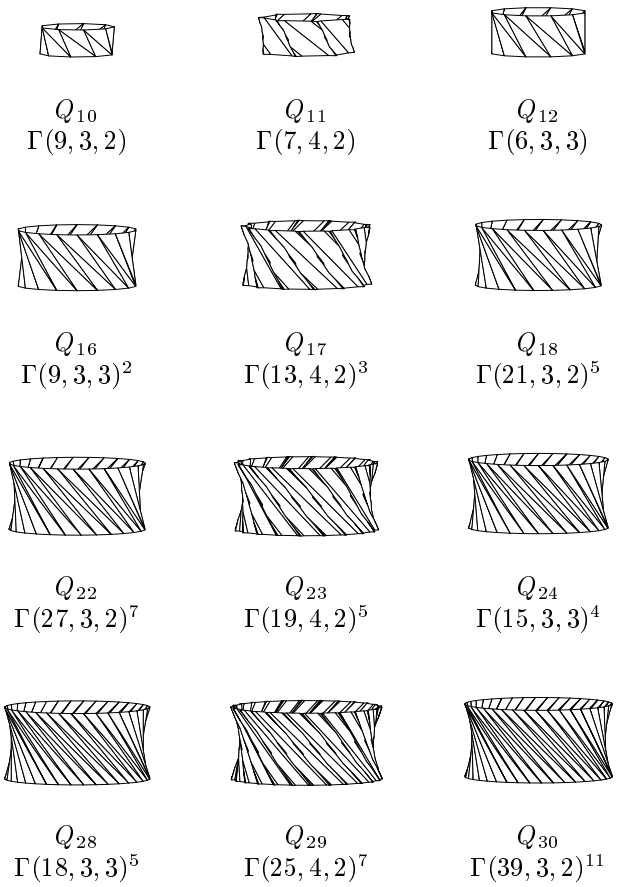
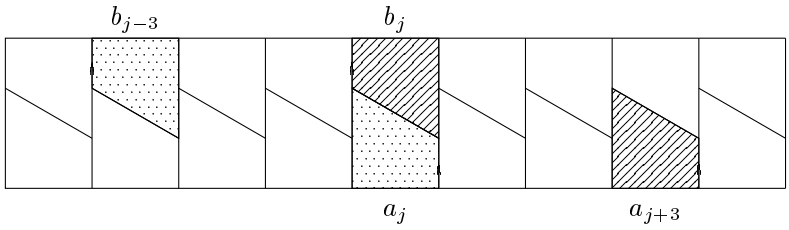
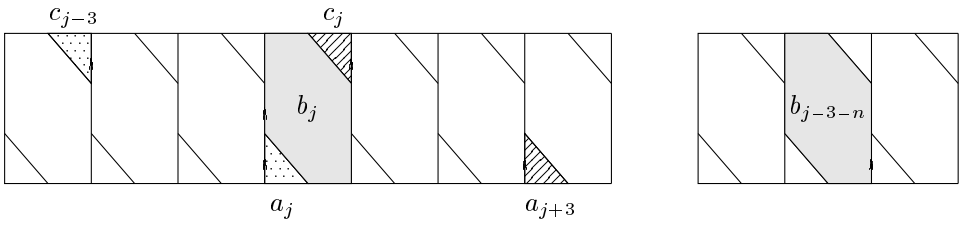


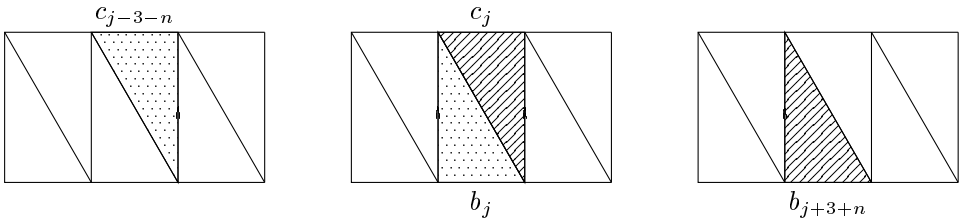
Table 7: Fundamental domains for the beginning of the Q-series



The case E_{10+2n}, i.e. $\Gamma = \Gamma(k+3,3,3)^k$ or $\Gamma = \Gamma(k+6,3,2)^k$



The case Z_{9+2n}, i.e. $\Gamma = \Gamma(2k+3,3,3)^k$ or $\Gamma = \Gamma(2k+6,3,2)^k$



The case Q_{8+2n}, i.e. $\Gamma = \Gamma(3k+3,3,3)^k$ or $\Gamma = \Gamma(3k+6,3,2)^k$

Table 8: Identification scheme for E, Z, Q in the equianharmonic case

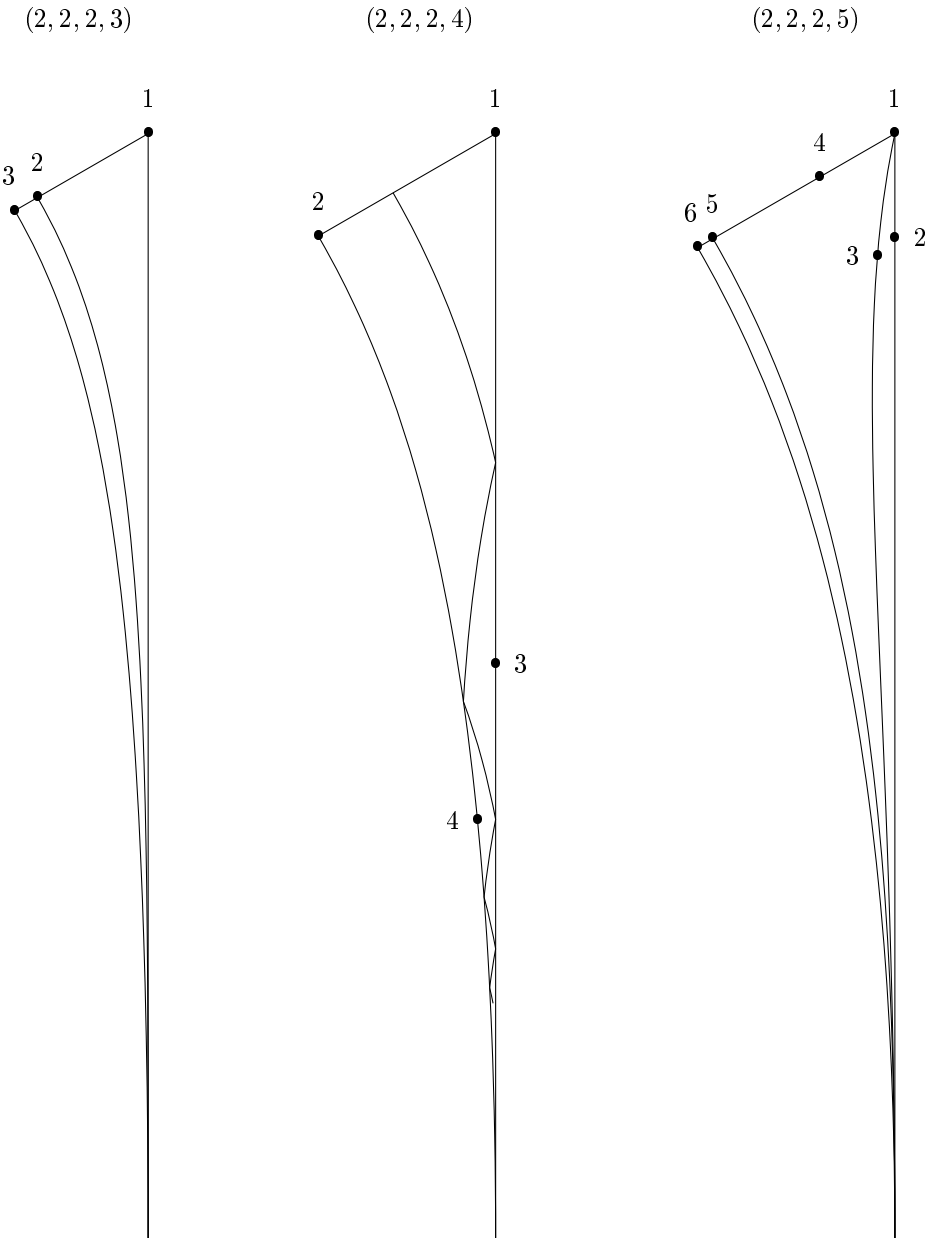


Table 9: Stratification of the fundamental triangles for $(2, 2, 2, p)$, where $p = 3, 4, 5$

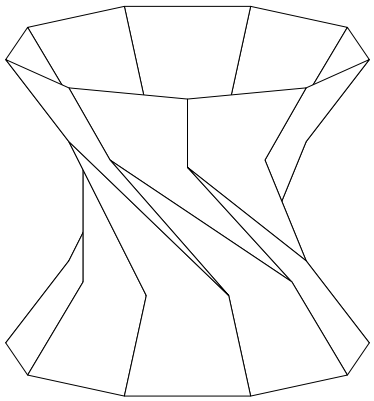


Figure 1

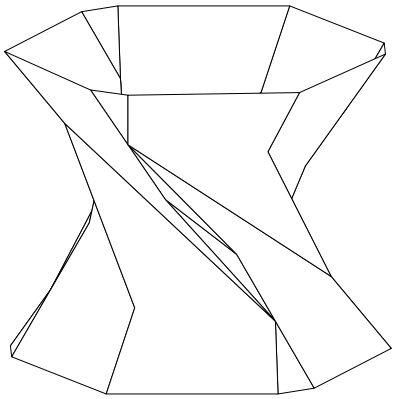


Figure 2

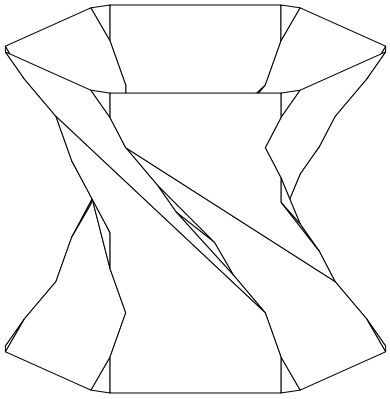


Figure 3

Table 10: The three combinatorial types of fundamental domains for $E_{3,0}$

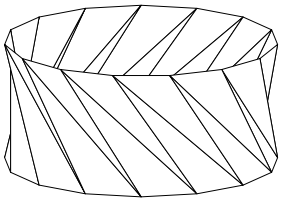


Figure 1

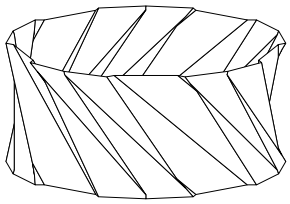


Figure 2

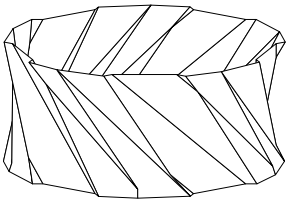


Figure 3

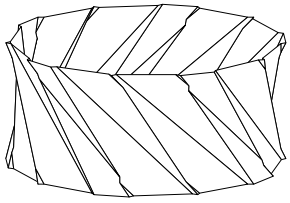


Figure 4

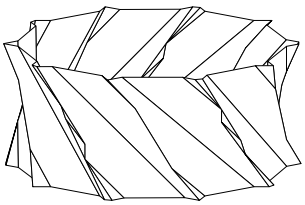


Figure 5

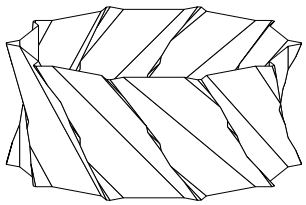


Figure 6

Table 11: The six combinatorial types of fundamental domains for $Q_{2,0}$

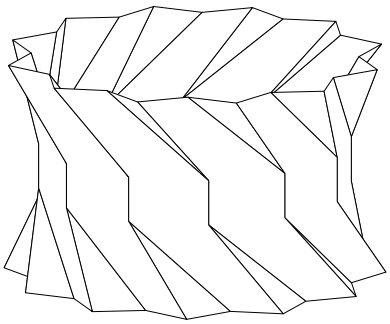


Figure 1

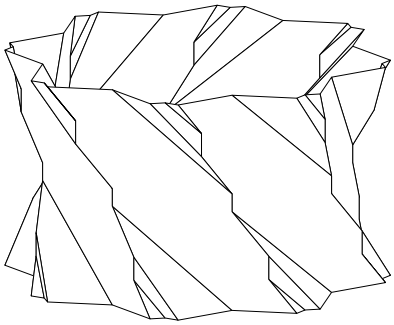


Figure 2

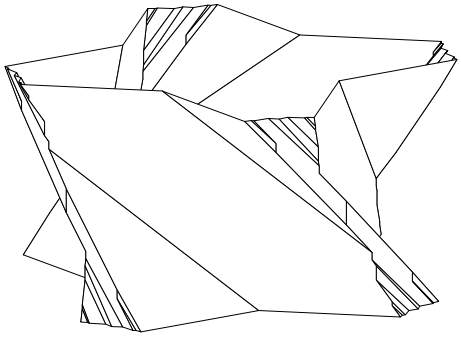


Figure 3

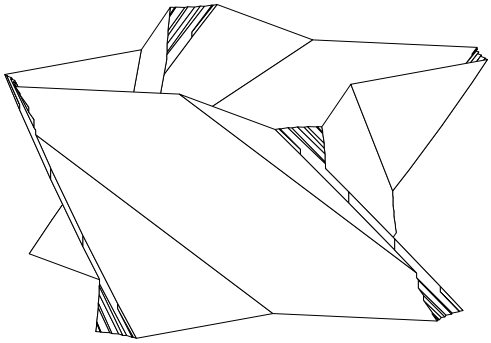


Figure 4

Table 12: Four generic combinatorial types of fundamental domains for $Z_{1,0}$

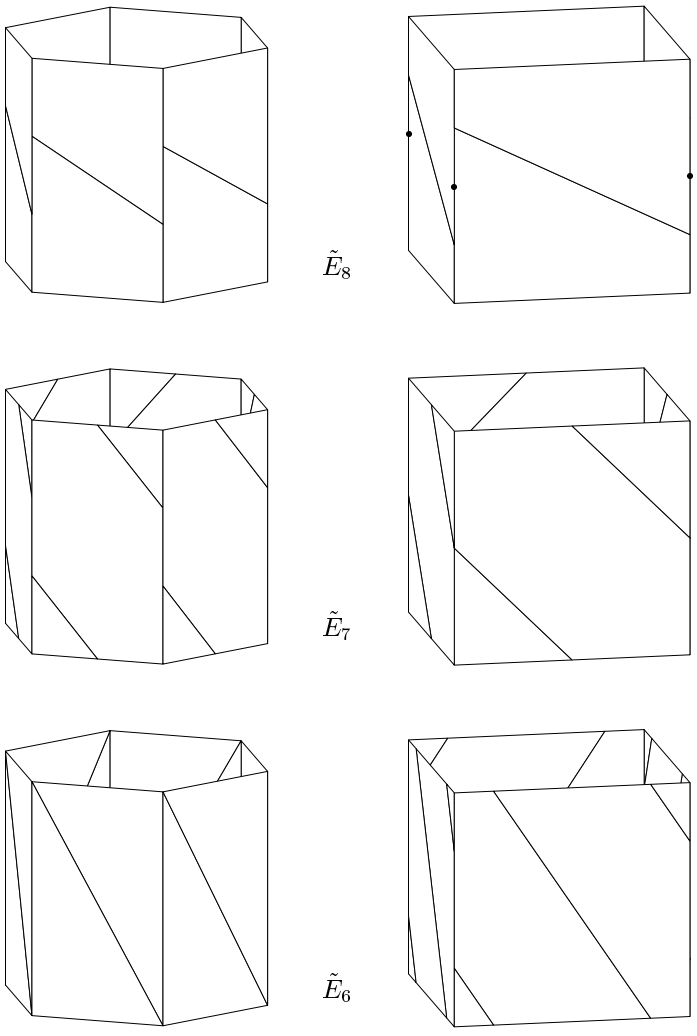


Table 13: Fundamental domains for the simply elliptic singularities

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