ON THE NECESSARY AND SUFFICIENT CONDITION FOR A HOMOGENEOUS MATRIX PENCIL OF A PAIR \((F,G)\) TO HAVE ZERO OR INFINITE ELEMENTARY DIVISORS

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Abstract: This article details with the study of the necessary and sufficient condition for a homogenous matrix pencil of a pair \((F, G)\) to have zero or infinite elementary divisors. This results is obtained by using the \([s,\hat{s}]\) - unimodular equivalent of Smith form, the compound matrices theory and the famous Binet-Cauchy theorem.

In this part of the article some useful notations and definitions from Matrix Pencil theory and Exterior Algebra are introduced.

Definition 1: Let \((F,G) \in \mathbb{C}^{m \times n} \times \mathbb{C}^{m \times n}\) and \((s,\hat{s})\) be a pair of indeterminates. The polynominal matrix \(sF-\hat{s}G \in \mathbb{C}^{m \times n}[s,\hat{s}]\) is defined as the homogeneous matrix pencil of the pair \((F, G)\).

Definition 2: The matrix pencil \(sF-\hat{s}G \in \mathbb{C}^{m \times n}[s,\hat{s}]\) is said to be regular if \(m = n\), i.e. \(F,G \in \mathbb{C}^{n \times n}[s,\hat{s}]\) and \(\text{rank}_{[s,\hat{s}]}(sF-\hat{s}G) = n\).

Theorem 1 (\(\mathbb{C}^{[s,\hat{s}]}\) - unimodular equivalence, Smith form, see [1])
There exist unimodular matrices \(P(s,\hat{s}), Q(s,\hat{s}) \in \mathbb{C}^{m \times n}[s,\hat{s}]\) such that

\[
P(s,\hat{s})(sF-\hat{s}G)Q(s,\hat{s}) = \begin{bmatrix} f_1(s,\hat{s}) & \cdots & f_{n}(s,\hat{s}) \end{bmatrix}
\]

where the \(f_i(s,\hat{s})\) for \(i = 1, 2, ..., n\) are the invariant polynomials over \(\mathbb{C}[s,\hat{s}]\) of \(sF-\hat{s}G\).

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By factoring the invariants polynomials $f_1(s,\hat{s})$ into powers of homogeneous polynomials irreducible over $\mathbb{C}$, it is obtained the set of elementary divisors (e. d.) of the pencil $sF - \hat{s}G$; there are of the following types: $s^p, \hat{s}^q$ and pairs of complex conjugate e. d. $(s - \alpha \hat{s})^\tau$, $(s - \overline{\alpha \hat{s}})^\tau$, where $\alpha, \overline{\alpha} \in \mathbb{C}$. Particularly, the e. d. of the types $\hat{s}^q$ are called infinite elementary divisors (i. e. d.) and of tye types $s^p$ are called zero elementary divisors (z. e. d.).

**Definition 3** Denote $Q_{l,m}$ the set of strictly increasing sequences of $l$-integers, for $1 \leq l \leq n$ chosen from $1, 2, \ldots, n$, i.e. $Q_{l,m} = \{(1,2),(1,3),(2,3),\ldots\}$. Thus, the number of the sequences which belongs to $Q_{l,m}$ is $\binom{n}{l}$

**Definition 4** Suppose $A = [a_{ij}] \in M_{m,n}(\mathbb{C})$, where $M_{m,n}(\mathbb{C})$ denotes the set of $m \times n$ matrices over the field $\mathbb{C}$; let $\mu, \nu$ be positive integers satisfying $1 \leq \mu \leq m$, $1 \leq \nu \leq n$ and let $\alpha = (i_1, \ldots, i_\mu) \in Q_{\mu,m}$ and $\beta = (j_1, \ldots, j_\nu) \in Q_{\nu,n}$. Then det $\begin{bmatrix} A[\alpha|\beta] \end{bmatrix} \in M_{\mu,\nu}(\mathbb{C})$ denotes the sub-matrix of $A$ which contains the rows $i_1, i_2, \ldots, i_\mu$ and the columns $j_1, j_2, \ldots, j_\nu$.

**Definition 5** Let $A \in M_{m,n}(\mathbb{C})$, and $1 \leq l \leq \min \{m, n\}$, then the $l$-compound matrix or $l$-adjugate of $A$ is the $\binom{n}{l} \times \binom{n}{l}$ matrix whose entries are det $\{A[\alpha|\nu]\} \in Q_{l,m}$ arranged lexicographically in $\alpha$ and $\beta$. This matrix will be designated by $C_l(A)$.

**Theorem 2** (Binet - Cauchy Theorem)
If $A \in M_{k,l}(\mathbb{C})$ and $B \in M_{l,m}(\mathbb{C})$, $1 \leq n \leq \min \{k, l, m\}$, then $C_n(AB) = C_n(A)C_n(B)$.

More details about the Binet - Cauchy theorem for compound matrices can be found in [3] and [2].

**Theorem 3** For the homogenous matrix pencil of the pair $(F, G)$, there exist zero elementary divisors, i.e. of the type $s^p$ if and only if $|G| = 0$.

**Proof.** According to Theorem 1, there exist unimodular matrices $P(s,\hat{s}), Q(s,\hat{s}) \in \mathbb{C}^{n \times p}[s,\hat{s}]$ such that

$$
P(s,\hat{s})(sF - \hat{s}G)Q(s,\hat{s}) = \begin{bmatrix}
f_1(s,\hat{s}) \\
f_2(s,\hat{s}) \\
\vdots \\
f_n(s,\hat{s})
\end{bmatrix}
$$

Since the $P(s,\hat{s}), Q(s,\hat{s})$ are unimodular matrices, it is known that there exist non-zero - constant $k$ and $\lambda$ such that the determinant $|P(s,\hat{s})| = k$ and $|Q(s,\hat{s})| = \lambda$, respectively. Thus, by taking the determinant in expression (1), it is obtained

$$
|P(s,\hat{s})||sF - \hat{s}G||Q(s,\hat{s})| = f_1(s,\hat{s})f_2(s,\hat{s})\ldots f_n(s,\hat{s})
$$
Hence \[ |sF - \hat{s}G| = \frac{1}{k\lambda} \prod_{i=1}^{n} f_i(s, \hat{s}) \] 

Equivalently, the matrix pencil \( sF - \hat{s}G = \begin{bmatrix} F & G \end{bmatrix} \begin{bmatrix} \hat{s}I_n \\ -\hat{s}I_n \end{bmatrix} \) \tag{3}

Now, consider then \( n \) - compound matrix for the expression (3), and by taking also into consideration the famous Binet - Cauchy theorem, see Theorem 2, it is derived that

\[ C_n(sF - \hat{s}G) = |sF - \hat{s}G| = C_n \left( \begin{bmatrix} F & G \end{bmatrix} \begin{bmatrix} \hat{s}I_n \\ -\hat{s}I_n \end{bmatrix} \right)^{C-B} = C_n(F \ G)C_n \left( \begin{bmatrix} \hat{s}I_n \\ -\hat{s}I_n \end{bmatrix} \right) \]

\[ \left[ \begin{array}{c} s^n \\ s^{n-1}\hat{s} \\ \vdots \\ (-1)^n \hat{s} \end{array} \right] = |F| s^n + c_1 s^{n-1}\hat{s} + c_2 s^{n-2}\hat{s}^2 + \ldots + (-1)^n |G| \hat{s}^n \]

and using the expression (2) it is obtained

\[ \prod_{i=1}^{n} f_i(s, \hat{s}) = k\lambda \left\{ |F| s^n + c_1 s^{n-1}\hat{s} + c_2 s^{n-2}\hat{s}^2 + \ldots + (-1)^n |G| \hat{s}^n \right\} \]

Consequently, from the important result above it is easily derived there exists zero elementary divisors, i.e. of the types \( \hat{s}^q \) if and only if \( |G| = 0 \).

The following well known, straight forward result is easily derived from the proof of the theorem 3. Hence

**Corollary 1** A matrix pencil \( sF - \hat{s}G \) has infinite elementary divisors, i.e. of the types \( \hat{s}^q \) if and only if \( |F| = 0 \)

**Proof.** According to the proof of theorem 3 it is obtained

\[ \prod_{i=1}^{n} f_i(s, \hat{s}) = k\lambda \left\{ |F| s^n + c_1 s^{n-1}\hat{s} + c_2 s^{n-2}\hat{s}^2 + \ldots + (-1)^n |G| \hat{s}^n \right\} \]

Consequently, there exist infinite elementary divisors, i.e. of the types \( \hat{s}^q \) if and only if and only if \( |F| = 0 \).

The following corollary, it is very important as it can be found in many particular applications.

**Corollary 2** The matrix pencil \( sA - I \), where \( I \) is the \( n \times n \) identical matrix, has not zero elementary divisors.

**Proof.** Since \( G = I \) and the determinant of the identical matrix is non zero, the theorem 3 is valid, thus the relevant matrix pencil has not zero elementary divisors.

**REFERENCES**