

Structured distance-to-singularity problems arising in robust control

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Main Points

- The structured singular value problem
- Quadratic Integer programming problem
- Convex relaxation and duality gap reduction
- General characteristics of approach
- Conclusions and further work

Distance to singularity and ssv - 1

- Let $M \in \mathcal{C}^{n \times n}$, $\det(M) \neq 0$ (w.l.g.). Define $\Delta \subseteq \mathcal{C}^{n \times n}$:

$$\Delta = \{\text{diag}[\delta_1 I_{r_1}, \dots, \delta_S I_{r_S}, \Delta_{S+1}, \dots, \Delta_{S+F}] :$$

$$\delta_i \in \mathcal{C}, \Delta_{S+j} \in \mathcal{C}^{m_j \times m_j}, 1 \leq i \leq S, 1 \leq j \leq F\}$$

where $\sum_{i=1}^S r_i + \sum_{j=1}^F r_j = n$.

- Define the “*structured distance to singularity*” of M :

$$r_{\Delta}(M) = \min\{\|\Delta\| : \Delta \in \Delta, \det(I - \Delta M) = 0\}$$

and the “*structured singular value*” of M :

$$\mu_{\Delta}(M) = r_{\Delta}^{-1}(M)$$

Distance to singularity and ssv - 2

Problem has applications in:

- *Robust control*: Structured distance problems to stability, controllability radius, etc.
- *Numerical Analysis*: Structured condition number, approximate GCD of polynomials, pseudo-spectra of operators, etc.

Example: Suppose that $a(s)$ and $b(s)$ are monic relative-prime polynomials with $\partial a(s) = m$ and $\partial b(s) = n$. What is the minimal magnitude perturbation in their coefficients such that the perturbed polynomials have a common root?

Distance to singularity and ssv - 3

Define Sylvester matrix:

$$R_{m,n}(a,b) = \begin{pmatrix} 1 & \alpha_{m-1} & \alpha_{m-2} & \cdots & \alpha_0 & 0 & \cdots & 0 \\ 0 & 1 & \alpha_{m-1} & \cdots & \alpha_1 & \alpha_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & \alpha_{m-1} & \cdots & \alpha_1 & \alpha_0 \\ 1 & \beta_{n-1} & \beta_{n-2} & \cdots & \beta_0 & 0 & \cdots & 0 \\ 0 & 1 & \beta_{n-1} & \cdots & \beta_1 & \beta_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & \beta_{n-1} & \cdots & \beta_1 & \beta_0 \end{pmatrix}$$

Then:

$$(a,b) \text{ co-prime} \Leftrightarrow \text{Rank}(R_{m,n}(a,b)) = m + n$$

Distance to singularity and ssv - 4

“Perturbed” Sylvester matrix:

$$R_{m,n}(a, b) + \begin{pmatrix} 0 & \delta_{m-1} & \delta_{m-2} & \cdots & \delta_0 & 0 & \cdots & 0 \\ 0 & 0 & \delta_{m-1} & \cdots & \delta_1 & \delta_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \delta_{m-1} & \cdots & \delta_1 & \delta_0 \\ 0 & \epsilon_{n-1} & \epsilon_{n-2} & \cdots & \epsilon_0 & 0 & \cdots & 0 \\ 0 & 0 & \epsilon_{n-1} & \cdots & \epsilon_1 & \epsilon_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \epsilon_{n-1} & \cdots & \epsilon_1 & \epsilon_0 \end{pmatrix}$$

and hence the uncertainty in this case has “structure”. Hence original problem is equivalent to the computation of a s.s.v:

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Theorem: Define $\gamma = \max\{ |\delta_0|, \dots, |\delta_{m-1}|, |\epsilon_0|, \dots, |\epsilon_{n-1}| \}$ and $\gamma^* = \min\{\gamma : (a, b) \text{ have a common root}\}$. Then,

$$\gamma^* = \frac{1}{\mu_{\Delta}(-ZR_{m,n}^{-1}(a, b)\Theta)}$$

where

$$\Delta = \{\text{diag}(\delta_{m-1}I_n, \dots, \delta_0I_n, \epsilon_{n-1}I_m, \dots, \epsilon_0I_m)\}$$

$$\Theta = \left(\begin{array}{ccc|ccc} I_n & \cdots & I_n & O_{n,m} & \cdots & O_{n,m} \\ O_{m,n} & \cdots & O_{m,n} & I_m & \cdots & I_m \end{array} \right)$$

$$Z' = \left((Z_{nm}^0) \cdots (Z_{nm}^{m-1}) \mid (Z_{mn}^0) \cdots (Z_{mn}^{n-1}) \right)$$

and

$$Z_{nm}^k = \left(O_{n,k+1} \quad I_n \quad O_{n,m-k-1} \right)'$$

for $k = 0, 1, \dots, m-1$.

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- In general, the computation of μ_{Δ} is a non-convex NP-hard problem.
- To compute upper bound first note that:

$$\begin{aligned}\mu_{\Delta}^{-1}(M) &= \min\{\|\Delta\| : \Delta \in \Delta, \det(I - M\Delta) = 0\} \\ &= \min\{\|\Delta\| : \Delta \in \Delta, \det(M^{-1} - \Delta) = 0\} \\ &\geq \min\{\|\Delta\| : \det(M^{-1} - \Delta) = 0\} \\ &= \|M\|^{-1}\end{aligned}$$

so that $\mu_{\Delta}(M) \leq \|M\|$.

- To tighten bound define:

$$\mathbf{D} = \{D = D^* > 0 : D\Delta = \Delta D \forall \Delta \in \Delta\}$$

Then for any $D \in \mathbf{D}$, $\Delta \in \Delta$: $D^{1/2}\Delta = \Delta D^{1/2}$.

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- Hence:

$$\begin{aligned}\mu_{\Delta}^{-1}(M) &= \min\{\|\Delta\| : \Delta \in \Delta, \det(I - M\Delta) = 0\} \\ &= \min\{\|\Delta\| : \Delta \in \Delta, \det(I - MD^{-1/2}\Delta D^{1/2}) = 0\} \\ &= \min\{\|\Delta\| : \Delta \in \Delta, \det(I - D^{1/2}MD^{-1/2}\Delta) = 0\} \\ &= \mu_{\Delta}^{-1}(D^{1/2}MD^{-1/2}) \geq \|D^{1/2}MD^{-1/2}\|^{-1}\end{aligned}$$

- We then have relaxation bound:

$$\mu_{\Delta}(M) \leq \min_{D \in \mathbf{D}} \|D^{1/2}MD^{-1/2}\| := \bar{\mu}_{\Delta}(M)$$

Since:

$$\|D^{1/2}MD^{-1/2}\| \leq \gamma \Leftrightarrow \gamma^2 D - M^*DM \geq 0$$

the computation of $\bar{\mu}_{\Delta}(M)$ now reduces to a convex programming problem.

Reducing the duality gap: $\bar{\mu}_\Delta - \mu_\Delta$ (1)

- Solve: $\mu_0 = \min_{D \in \mathbf{D}} \|D^{1/2} M D^{-1/2}\|$ and let D_o be a minimiser.
- Re-define: $M \leftarrow \mu_0^{-1} D_o^{1/2} M D_o^{-1/2}$ so that: $\bar{\mu}_\Delta(M) = \|M\| = 1$.
- If multiplicity of largest s.v. of M is $m = 1$, $\bar{\mu}_\Delta(M) = \mu_\Delta(M)$.
Therefore assume that $m > 1$.
- Let $M \in \mathcal{C}^{n \times n}$ have (ordered) SVD:

$$M = U \Sigma V^* = \begin{pmatrix} U_1 & U_2 \end{pmatrix} \begin{pmatrix} I_m & 0 \\ 0 & \Sigma_2 \end{pmatrix} \begin{pmatrix} V_1^* \\ V_2^* \end{pmatrix}$$

$$\Sigma_2 := \text{diag}(\sigma_{m+1}, \dots, \sigma_n), \quad \sigma_{m+1} < 1.$$

Reducing the duality gap: $\bar{\mu}_\Delta - \mu_\Delta$ (2)

- Define $A = \Sigma^{-1} = \text{diag}(I_m, A_2)$ where $A_2 = \text{diag}(a_{m+1}, \dots, a_n)$, $1 < a_{m+1} \leq \dots \leq a_n$.
- Consider perturbation matrix $\Delta \in \mathcal{C}^{n \times n}$:

$$\Delta = \begin{pmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{21} & \Delta_{22} \end{pmatrix}, \quad \Delta_{11} \in \mathcal{C}^{m \times m}$$

- Solve, or at least calculate upper bounds:

$$\phi_1 := \max\{\rho(V_1^* \Delta U_1) : \Delta \in \mathcal{B}\Delta\} \leq \bar{\phi}_1 \leq 1$$

$$\phi_2 := \max\{\|V_1^* \Delta U_1\| : \Delta \in \mathcal{B}\Delta\} \leq \bar{\phi}_2 \leq 1$$

Reducing the duality gap: $\bar{\mu}_\Delta - \mu_\Delta$ (3)

Then:

$$\begin{aligned} r_\Delta &= \mu_\Delta^{-1}(M) \\ &= \min\{\|\Delta\| : \Delta \in \Delta, \det(I - \Delta M) = 0\} \\ &= \min\{\|\Delta\| : \Delta \in \Delta, \det(I - \Delta U \Sigma V^*) = 0\} \\ &= \min\{\|\Delta\| : \Delta \in \Delta, \det(I - V^* \Delta U \Sigma) = 0\} \\ &= \min\{\|\Delta\| : \Delta \in \Delta, \det(A - V^* \Delta U) = 0\} \\ &\geq \min\{\|\Delta\| : \rho(\Delta_{11}) \leq \bar{\phi}_1 \|\Delta\|, \|\Delta_{11}\| \leq \bar{\phi}_2 \|\Delta\|, \det(A - \Delta) = 0\} \\ &:= \hat{r}_\Delta(M) := \hat{\mu}_\Delta^{-1}(M) \end{aligned}$$

Reducing the duality gap: $\bar{\mu}_\Delta - \mu_\Delta$ (4)

- $\hat{\mu}_\Delta(M)$ can be obtained analytically and $\hat{\mu}_\Delta(M) \leq \bar{\mu}_\Delta(M)$.
- Calculation of ϕ_1 ($\bar{\phi}_1$): Since

$$\begin{aligned}\phi_1 &= \max_{\Delta \in \mathcal{B}\Delta} \rho(V_1^* \Delta U_1) = \max_{\Delta \in \mathcal{B}\Delta} \rho(\Delta U_1 V_1^*) \\ &:= \max_{\Delta \in \mathcal{B}\Delta} \rho(\Delta M_0) \\ &= \mu_\Delta(M_0)\end{aligned}$$

this is a ssv problem *of reduced rank*.

- Calculation of ϕ_2 ($\bar{\phi}_2$): Use convex relaxation (LMI):

$$\phi_2^2 = \max_{\Delta \in \mathcal{B}\Delta} \|V_1^* \Delta U_1\|^2 \leq \min_{D \in \mathbf{D}, D - V_1 V_1^* \geq 0, \gamma I - U_1' D U_1 \geq 0} \gamma = \bar{\phi}_2^2$$

- Lemma: $\phi_2 = \bar{\phi}_2$; $\inf_{D \in \mathbf{D}} \|D^{1/2} M D^{-1/2}\| = 1 \Rightarrow \phi_2 = \bar{\phi}_2 = 1$.

Sequence of relaxation problems

General problem:

$$\gamma_{\Delta_{11}} = \min\{\|\Delta\|, \det(A - \Delta) = 0, \Delta_{11} \in \Delta_{11}\}$$

where:

- $\Delta_{11} = \{\delta \in \mathcal{C} : |\delta| \leq \phi\}, 0 \leq \phi \leq 1 \ (m = 1)$
- $\Delta_{11} = \{0_{m \times m}\}$
- $\Delta_{11} = \{\Delta_{11}\}, \|\Delta_{11}\| \leq 1, 1 \neq \lambda(\Delta_{11})$
- $\Delta_{11} = \{\Delta_{11} : \|\Delta_{11}\| \leq 1, 1 \notin \lambda(\Delta_{11})\}$
- $\Delta_{11} = \{\Delta_{11} : \rho(\Delta_{11}) \leq \phi_1, \|\Delta_{11}\| \leq \phi_2\}$

Breaching the convex bound

Theorem: Let $0 \leq \bar{\phi}_1 \leq \bar{\phi}_2 \leq 1$. Then

$$\begin{aligned}\hat{\mu}_{\Delta}^{-1} &:= \min\{\|\Delta\| : \det(A - \Delta) = 0, \rho(\Delta_{11}) \leq \bar{\phi}_1\|\Delta\|, \|\Delta_{11}\| \leq \bar{\phi}_2\|\Delta\|\} \\ &= \min\{\gamma > 1 : \|f(\gamma, \Psi)\| = a_{m+1}\}\end{aligned}$$

where: $f(\gamma, \Psi) = (\gamma I - \Psi)(\gamma^{-1}I - \Psi)^{-1}$ and

$$\Psi_{ij} = \begin{cases} 0, & j < i \\ \phi_1, & j = i \\ -\left(\frac{\phi_1}{\phi_2}\right)^{j-i-1} \frac{\phi_2^2 - \phi_1^2}{\phi_2}, & j > i \end{cases}$$

Also: (i) $\bar{\phi}_1 < 1 \Rightarrow \mu_{\Delta}(M) < 1$ and $\hat{\mu}_{\Delta}(M) < 1$ (ii) $\mu_{\Delta}(M) = \hat{\mu}_{\Delta}(M)$ iff $\exists \Delta \in \Delta$ s.t.

$$V_1^* \Delta U_1 = \hat{\mu}_{\Delta}(M) W^* \Psi W,$$

for unitary W . □

Note: Ψ is Toeplitz upper triangular. For example, if $m = 4$,

$$\Psi = \begin{bmatrix} \phi_1 & \alpha & \beta & \gamma \\ 0 & \phi_1 & \alpha & \beta \\ 0 & 0 & \phi_1 & \alpha \\ 0 & 0 & 0 & \phi_1 \end{bmatrix}$$

with

$$\alpha = \frac{\phi_2^2 - \phi_1^2}{\phi_2}, \beta = -\frac{\phi_1}{\phi_2} \frac{\phi_2^2 - \phi_1^2}{\phi_2}, \gamma = \left(\frac{\phi_1}{\phi_2}\right)^2 \frac{\phi_2^2 - \phi_1^2}{\phi_2}$$

Note: The calculation of $\hat{\mu}_{\Delta}(M)$ is a $4m \times 4m$ e-value problem. Set $a = a_{m+1}$, and $\Psi_s = \Psi + \Psi^*$. Then $a = \|(\gamma I - \Psi)(\gamma^{-1}I - \Psi)^{-1}\| \Rightarrow$

$$\det \left(\gamma I - \begin{bmatrix} 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ a^2 I - a^2 \Psi_s & (a^2 - 1) \Psi^* \Psi & \Psi_s \end{bmatrix} \right) = 0$$

Hence $\hat{\mu}_{\Delta}^{-1}$ is the smallest real e-value larger than one of a $4m \times 4m$ matrix.

P2: Quadratic Integer Programming problem

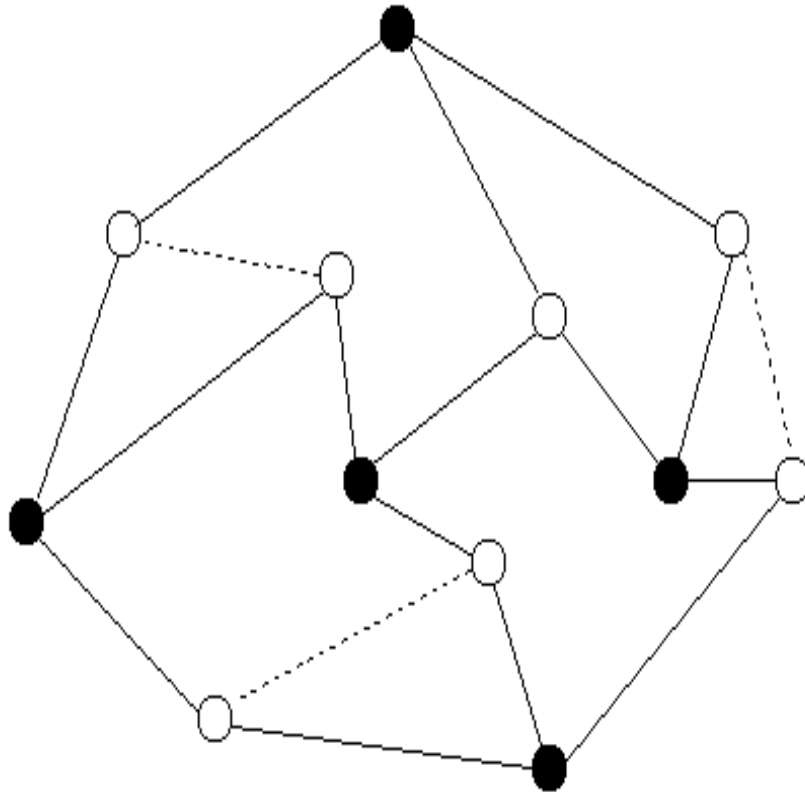
- QIP Problem has the form:

$$\gamma := \max\{x'Qx : x \in \{-1, 1\}^n\}$$

where $Q = Q'$ of mixed inertia in general.

- This needs 2^n (or 2^{n-1}) function evaluations which grows exponentially in n .
- Classical NP-hard problem with many applications in Control, Statistics, Graph Theory (“max-cut”), etc.

Example: Max-cut problem 1



Given a graph defined by:

- Vertex set $V = \{1, 2, \dots, n\}$
- Edge set $E = \{(i, j) : i, j \in V, i \neq j\}$.
- For $S \subseteq V$ capacity is number of edges connecting a node in S to a node not in S .
- Max-Cut problem: Find $S \subseteq V$ with maximum capacity

Example: Max-cut problem 2

- Define adjacency matrix:

$$\begin{aligned} Q_{ij} &= 1 && \text{if } (i, j) \in E \\ &= 0 && \text{otherwise} \end{aligned}$$

- Specify cut by vector $x \in \mathcal{R}^n$:

$$\begin{aligned} x_i &= 1 && \text{if } i \in S \\ &= -1 && \text{otherwise} \end{aligned}$$

- If $(i, j) \in E$, $1 - x_i x_j = 2$. Hence capacity:

$$c(x) = \frac{1}{4} \sum_{i=1}^n \sum_{j=1}^n (1 - x_i x_j) Q_{ij}$$

- Thus max-cut can be formulated as: $\min\{x'Qx : x \in \{-1, 1\}^n\}$.

QIP: Convex relaxation - 1

- Let $D = \text{diag}(D)$, $D - Q \geq 0$. Then for all $x \in \{-1, 1\}^n$:

$$\begin{aligned} x'Qx &= \underbrace{x'Dx - \text{trace}(D)}_{= -x'(D - Q)x + \text{trace}(D)} - x'(D - Q)x + \text{trace}(D) \\ &= -x'(D - Q)x + \text{trace}(D) \quad (x_i^2 = 1) \\ &\leq \text{trace}(D) \quad (D - Q \geq 0) \end{aligned}$$

- This results in the convex relaxation:

$$\begin{aligned} \gamma &= \max\{x'Qx : x \in \{-1, 1\}^n, Q = Q'\} \\ &\leq \min\{\text{trace}(D) : D = \text{diag}(D), D - Q \geq 0\} \\ &:= \bar{\gamma} \end{aligned}$$

- Let D_o be the (unique) optimal solution and define V such that $R(V) = \text{Ker}(D_o - Q)$, $V^*V = I_r$. If $r = 1$ duality gap is zero.

QIP: Convex relaxation - 2

- Solve RRQIP problem:

$$\gamma_r = \frac{1}{n} \max\{x'VV'x : x \in \{-1, 1\}^n\}$$

Then: $\gamma = \bar{\gamma} \Leftrightarrow \gamma_r = 1$.

- Provided $\gamma_r < 1$,

$$\gamma < \bar{\gamma} - n(1 - \gamma_r)\lambda_+(D_o - Q) < \bar{\gamma}$$

where

$$\lambda_+(D_o - Q) := \min_{\{i: \lambda_i(D_o - Q) > 0\}} \lambda_i(D_o - Q)$$

- Note that $\text{Rank}(VV') = r < n$ (typically $r \ll n$).
- Reduced-rank problem has complexity $O(n^r)$ and can be solved efficiently using theory of zonotopes, hyperplane arrangements and linear programming [Avis, Fukuda].

General characteristics of method - 1

- μ -problem:

$$\mu^{-1} = \min_{\Delta \in \Delta, \det(I - \Delta M) = 0} \|\Delta\|$$

is NP-hard.

- Formulate and solve dual:
Get (μ_0, D_o) s.t.

$$\min_{D \in \mathbf{D}} \|D^{1/2} M D^{-1/2}\|$$

$$M \leftarrow \mu_0^{-1} D_o^{1/2} M D_o^{-1/2}$$

- QIP problem:

$$\gamma := \max_{x \in \{-1, 1\}^n} x' Q x$$

is NP-hard

- Formulate and solve dual:
Get $(\bar{\gamma}, D_o)$ s.t.

$$\bar{\gamma} = \min_{D = \text{diag}(D), D - Q \geq 0} \text{trace}(D)$$

$$\text{s.t. } D = \text{diag}(D), D - Q \geq 0$$

General characteristics of method - 2

- Sufficient condition for zero duality gap: $\mu = 1$ if $m := \text{null}(I - MM^*) = 1$.
- Duality gap can be breached if we can solve a rank- m μ problem (or get a bound less than 1):
$$\max_{\Delta \in \mathcal{B}} \rho(\Delta U_1 V_1^*)$$
- Improved bound obtained via the solution of an eigenvalue problem of dimension $4m \times 4m$.
- Sufficient condition for zero duality gap: $\gamma = \bar{\gamma}$ if $r := \text{null}(D_o - Q) = 1$.
- Duality gap can be breached if we can solve a rank- r QIP problem of the form
$$\gamma_r = \frac{1}{n} \max_{x \in \{-1, 1\}^n} x' V V' x.$$
- Improved bound can be obtained by calculating the eigenvalues of a symmetric matrix.

Conclusions and Open Issues

- Development of efficient algorithms for low-rank μ problems.
- Exploit similarities in solutions. Can method be generalized to a large class of problems involving convex relaxations (e.g. new class of primal/dual algorithms)?
- Can approach be combined with probabilistic relaxation methods?