A linear algebraic method for pricing temporary life annuities

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Outline

- Introduction
- Cash flow valuation problems in actuarial science

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  1. A linear algebraic approach to random cash flow pricing
  2. Approximate solutions

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- Applications

Introduction: pricing an annuity

- With no mortality risk, the present value of cash flow $\mathbf{f}$ (e.g. purchased temporary life annuity) payable at a future times t_i, at time $t_k, 0 \leq k < i < N$ is

$$p_k = \sum_{i=k+1}^{N-1} \frac{f_i}{\prod_{j=k+1}^{i} (1 + r_j)}$$

- r_i is one period interest rate in $[t_{i-1}, t_i)$.

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Example: $f = 1$, $N = 1$,

$$p_0 = \frac{1}{1+r_1}.$$

Pricing an annuity (cont'd)

- With no interest rate risk but with force of mortality risk, the same present value is

$$p_k = \sum_{i=k+1}^{N-1} \frac{f_i}{\prod_{j=k+1}^{i} (1 + \lambda_j)}.$$

- λ_i is force of mortality in $[t_{i-1}, t_i)$.

$\frac{1}{1+\lambda_i}$ = probability that annuitant alive at t_{i-1} survives till t_i.

Pricing an annuity (cont'd)

- With both interest rate and mortality risk,

$$p_k = \sum_{i=k+1}^{N-1} \frac{f_i}{\prod_{j=k+1}^{i} (1+\lambda_j)(1+r_j)},$$

- $\mathbb{E}(p_k) = ?$
- Conventional approaches:
  1. Use affine term structure models for both interest rate and force of mortality;
  2. More complex models, with simulation-based pricing.

Pricing an annuity (cont'd)

- Our approach:
  1. Expand p_k into a polynomial in λ_j, r_j; base an approximation of $\mathbb{E}(p_k)$ on the moments of λ_j, r_j.
  2. $\mathbb{E}(r_j)$, $\mathbb{E}(r_i r_j)$ are available (forward curve data - no parametric model needed). Forward mortality information *may* be available.

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  3. Strong results on convergence of approximation + very good simulation results.
  4. Extends to various contracts (mortgage insurance, annuities, f contingent on something else).

A linear algebraic approach

- With discounting factor $\theta_i = r_i + \lambda_i + r_i\lambda_i$, the relationship

$$
\begin{aligned}
p_1 &= \frac{f_2}{1+\theta_2}, \ \textit{running} \text{ present value at } t_1 \\
p_0 &= \frac{f_1}{1+\theta_1} + \frac{f_2}{(1+\theta_2)(1+\theta_1)}, \\
&= \frac{f_1 + p_1}{1+\theta_1},
\end{aligned}
$$

  may also be written as

$$
\begin{aligned}
(1+\theta_2)\,p_1 \;+\; 0\,p_0 &= f_2, \\
(-1)\,p_1 \;+\; (1+\theta_1)p_0 &= f_1.
\end{aligned}
$$

A linear algebraic approach (cont'd)

1. Thus $\mathbf{p} = \begin{bmatrix} p_1 & p_0 \end{bmatrix}^\top$ solves a system of linear equations

$$\left(\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} + \begin{bmatrix} \theta_2 & 0 \\ 0 & \theta_1 \end{bmatrix} \right) \mathbf{p} = \mathbf{f}.$$

2. If $\mathbf{f}$ are temporary life annuity payments, $\mathbb{E}(\mathbf{p}_0)$ is the lump sum purchase price.
3. More involved formulation possible for insurance contracts; leads to random terms of both sides.
4. For future θ_i unknown and random, we have a problem of inverting a *structured* matrix with random entries.

Random matrix inversion problem

- For a series of cash flows $f_i, 0 \leq i \leq N$, we can write

$$\mathbf{f} = (Q + \Theta)\mathbf{p}$$

- $\mathbf{p}$ is $N-$vector of running present values, $\mathbf{f}$ is $N-$vector of cash flows.
-

$$\begin{aligned} [Q]_{ij} &= 1 \text{ if } i = j \\ &= -1 \text{ if } i = j + 1 \\ &= 0 \text{ otherwise} \\ [\Theta]_{ij} &= \theta_{N-i+1} \text{ if } i = j \\ &= 0 \text{ otherwise,} \end{aligned}$$

  with $\theta_i = r_i + \lambda_i + r_i\lambda_i$.
- This means the value of the series of cash flows at time t_k is $[(Q + \Theta)^{-1}\mathbf{f}]_k$.

Approximate inversion: scalar case

- For $\theta_i = \theta < 1$,

$$\frac{f}{1+\theta} = \left(1 - \theta + \theta^2 - \cdots\right) f$$
$$\approx \sum_{i=0}^{M} (-\theta)^i f.$$

- A similar result holds if $\|\theta\|$ is bounded by unity, but random.
- Does an analogous expansion hold if $\theta_i \neq \theta_j, i \neq j$?

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Convergent approximation to the solution: vector case

Theorem

Suppose that θ_i *satisfy* $\mathbb{P}(\|Q^{-1}\Theta\| < 1) = 1$, f_i *satisfy* $\max_i |f_i| < \gamma$, $\gamma < \infty$ *and the inverse of* $(Q + \Theta)$ *exists with probability* 1*. Define*

$$\mathbf{p} = (Q + \Theta)^{-1}\mathbf{f}, \ \mathbf{p}^M = \sum_{i=0}^{M} (-Q^{-1}\Theta)^i Q^{-1}\mathbf{f} \tag{1}$$

with $(-Q^{-1}\Theta)^0 = I$*. Then*

(i) $\mathbf{p}^M \to \mathbf{p}$ *with probability* 1*.*

(ii)

$$\lim_{M\to\infty} \mathbb{E}\left(\|\mathbf{p}^M - \mathbf{p}\|_2^2\right) = 0.$$

What does $\mathbf{p}^M$ look like?

- It may be shown that

$$\begin{aligned}[Q^{-1}\Theta]_{ij} &= \theta_{N-j+1} \text{ if } i \geq j, \\ &= 0 \text{ otherwise,}\end{aligned}$$

$$[Q^{-1}\mathbf{f}]_j = \sum_{i=1}^{j} f_{N-i}.$$

- *e.g.* for $N = 3$,

$$Q^{-1}\Theta = \begin{bmatrix} \theta_3 & 0 & 0 \\ \theta_3 & \theta_2 & 0 \\ \theta_3 & \theta_2 & \theta_1 \end{bmatrix}$$

$$Q^{-1}\mathbf{f} = \begin{bmatrix} f_3 \\ f_3 + f_2 \\ f_3 + f_2 + f_1 \end{bmatrix}$$

How good is the approximation?

- With probability 1,

$$\|\mathbf{p}^M - \mathbf{p}\| \le \frac{\|Q^{-1}\Theta\|^{M+1}}{1 - \|Q^{-1}\Theta\|} \|Q^{-1}\mathbf{f}\|$$

  holds for any induced norm $\|\cdot\|$, provided $\|Q^{-1}\Theta\| < 1$.

- If $\theta_i < \theta_{max}$ w.p.1,

$$\|Q^{-1}\Theta\|_2 \le \theta_{max}\sqrt{\frac{N(N+1)}{2}}.$$

  Further, given any $\epsilon > 0$, $\exists$ a matrix norm $\|\cdot\|$ s.t. $\|Q^{-1}\Theta\| < \theta_{max} + \epsilon$.

- Non-conservative bounds are difficult; but- the algorithm seems to work well in practice.

Putting it all together

Finally, $\cdots$

- Given moments $\mathbb{E}(r_i), \mathbb{E}(\lambda_i), \mathbb{E}(r_i r_j), \mathbb{E}(\lambda_i \lambda_j)$, we can construct a second order approximation for

$$\underbrace{\mathbb{E}\left(\sum_{i=1}^{N} \frac{f_i}{\prod_{j=1}^{i}(1+\lambda_j)(1+r_j)}\right)}_{\approx \text{ an element from vector } \mathbb{E}(\sum_{i=0}^{2}(-Q^{-1}\Theta)^i(Q^{-1}\mathbf{f}))},$$

in closed-form.

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  in closed-form.

- $\mathbb{E}(r_i)$ (forward curve), $\mathbb{E}(r_i r_j)$ (forward curve correlations) may be known, or found from affine term structure model.
- $\mathbb{E}(\lambda_i), \mathbb{E}(\lambda_i \lambda_j)$ may be found from an affine term structure model for force of mortality.

Simulation Experiments

- Extensive numerical simulation experiments conducted for annuity valuation, insurance premium (Date et al, 2010), general cash flows (Date et al 2007).
- The method is computationally very cheap and shown to be quite accurate.
- This requires joint moments of interest rates and mortality up to M^{th} order.
- These moments may be known (forward curve and historic correlations - **no interest rate model needed for** $M = 2$**.**)

  ...
- or they may be inferred from a given affine term structure model (*e.g.* for mortality).

Main References

- **P. Date, R. Mamon, L. Jalen and I.C. Wang**, A linear algebraic method for pricing temporary life annuities and insurance policies, *Insurance: Mathematics and Economics*, vol. 47, pages 98–104, 2010.
- P. Gaillardetz, Valuation of life insurance products under stochastic interest rates, *Insurance: Mathematics and Economics*, vol. 42, pages 212–226, 2008.
- **P. Date, R. Mamon and I.C. Wang**, Valuation of cash flows under random rates of interest: a linear algebraic approach, *Insurance: Mathematics and Economics*, vol. 41, pages 84–95, 2007.
- L. Ballotta and S. Haberman, The fair valuation problem of guaranteed annuity options: the stochastic mortality environment case, *Insurance: Mathematics and Economics*, vol. 38, pages 195–214, 2006.