Recent Results on the Inverse Real Symmetric Quadratic Eigenvalue Problem

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The Inverse Quadratic Eigenvalue Problem

Design a procedure to construct real symmetric quadratic matrix polynomials (with some definiteness constraints on the coefficients, possibly) with a prescribed set of spectral data: real and complex eigenvalues with their partial multiplicities and sign characteristics for the real spectrum.

For $L(\lambda) = L_2\lambda + L_1\lambda + L_0$, $\lambda_0 \in \mathbb{C}$ is an eigenvalues of $L(\lambda)$ with associated eigenvector x_0 if

$$L(\lambda_0)x_0 = 0$$

See the book by Chu and Golub[2]for motivation, applications and previous results

[2]M. T. Chu, G. H. Golub: *Inverse Eigenvalue Problems*. Oxford University Press, 2005

Another precedent

In a Technical Report [3] necessary and sufficient conditions were given for the existence of a selfadjoint complex $n \times n$ matrix polynomials with prescribed spectral data. The degree of such a matrix, however, was not prescribed.

[3] I. Gohberg, P. Lancaster, L. Rodman: Spectral analysis of selfadjoint matrix polynomials. *Research paper 419* (1979) Dept. Mathematics and Statistics, University of Calgary

Standard triples

$\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$

(X, T, Y) standard triple $= (X, T, Y)$ irreducible control system: minimal realization of the inverse of some matrix polynomial:

$$X(\lambda I - T)^{-1} Y = L(\lambda)^{-1} \text{ for some matrix polynomial } L(\lambda)$$

(X, T, Y)= standard triple of $L(\lambda) = L_\ell \lambda^\ell + L_{\ell-1}\lambda^{\ell-1} + \cdots + L_0$.

Equivalently ($\ell = 2$ and $\det L_2 \neq 0$)

- (X, T) right standard pair: $\operatorname{rank} \begin{bmatrix} X \\ XT \end{bmatrix} = 2n$ (Observable)
- (T, Y) left standrad pair: $\operatorname{rank} \begin{bmatrix} Y & TY \end{bmatrix} = 2n$ (Controllable)
- $XY = 0$ and XTY invertible ($I = L(\lambda)X(\lambda I - T)^{-1}Y = (L_2\lambda^2 + L_1\lambda + L_0)(XY\lambda^{-1} + XTY\lambda^{-2} + \cdots)$)

Example

$$X = \begin{bmatrix} I_n & 0 \end{bmatrix}, \quad C_R = \begin{bmatrix} 0 & I_n \\ -M^{-1}K & -M^{-1}D \end{bmatrix}, \quad Y = \begin{bmatrix} 0 \\ M^{-1} \end{bmatrix}$$

Jordan triples

(X, J, Y) is a *Jordan* triple if it is a standard triple and J is a Jordan matrix.

If $L(\lambda)^{-1} = X(\lambda I - J)^{-1}Y$ then

- J: Jordan form (over $\mathbb{R}$ or $\mathbb{C}$) of $L(\lambda)$
- X: matrix of right Jordan chains of $L(\lambda)$
- Y: matrix of left Jordan chains of $L(\lambda)$

(X, J, Y) completely determines $L(\lambda)$

1. $\begin{bmatrix} X \\ XJ \end{bmatrix} Y = \begin{bmatrix} 0 \\ M^{-1} \end{bmatrix}$
2. $\begin{bmatrix} X \\ XJ \end{bmatrix} J \begin{bmatrix} X \\ XJ \end{bmatrix}^{-1} = C \Leftrightarrow \begin{bmatrix} X \\ XJ \end{bmatrix} J = C \begin{bmatrix} X \\ XJ \end{bmatrix} \Rightarrow C = \begin{bmatrix} 0 & I_n \\ -A_0 & -A_1 \end{bmatrix}$

$$\boxed{L(\lambda) = M\lambda^2 + MA_1\lambda + MA_0}$$

Selfadjoint standard triples

$L(\lambda) = L(\lambda)^T \Rightarrow (X, T, Y) \overset{s}{\sim} (Y^T, T^T, X^T) \Leftrightarrow (Y^T, T^T, X^T) = (XP^{-1}, PTP^{-1}, PY)$, P invertible

Definition

A real standard triple (X, T, Y) is *real selfadjoint* if there is a real nonsingular *symmetric* matrix H such that

$$Y^T = XH^{-1} (\text{ or } X^T = HY) \text{ and } T^T = HTH^{-1} \tag{1}$$

Complex case: substitute

real selfadjoint	$\rightarrow$	selfadajoint
symmetric	$\rightarrow$	Hermitian
T	$\rightarrow$	*

Remarks

- If $T = J$ (real Jordan form): Real Selfadjoint Jordan Triple
- If (X, T, Y) real selfadjoint there is a **unique** H in (1)

Example: If $M\lambda^2 + D\lambda + K = M^T\lambda^2 + D^T\lambda + K^T$

$$X = \begin{bmatrix} I_n & 0 \end{bmatrix}, \quad C_R = \begin{bmatrix} 0 & I_n \\ -M^{-1}K & -M^{-1}D \end{bmatrix}, \quad Y = \begin{bmatrix} 0 \\ M^{-1} \end{bmatrix}, H = \begin{bmatrix} D & M \\ M & 0 \end{bmatrix}$$

A real Selfadjoint Jordan Triple

Any real symmetric matrix polynomial (of any degree) admits a selfadjoint Jordan triple (X, J, PX^T) with

$$J = \bigoplus_{j=1}^{q} \underbrace{\begin{bmatrix} \alpha_j & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & \alpha_j \end{bmatrix}}_{\ell_j \times \ell_j} \oplus \bigoplus_{j=1}^{s} \underbrace{\begin{bmatrix} U_j & & & \\ I_2 & \ddots & & \\ & \ddots & \ddots & \\ & & I_2 & U_j \end{bmatrix}}_{2m_j \times 2m_j}, \quad \begin{matrix} \beta_j = \mu_j \pm i\nu_j \\ U_j = \begin{bmatrix} \mu_j & -\nu_j \\ \nu_j & \mu_j \end{bmatrix} \end{matrix}$$

$$P = \bigoplus_{j=1}^{q} \epsilon_j F_{\ell_j} \oplus \bigoplus_{j=1}^{s} F_{2m_j}, \qquad F_k = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & \cdots & & 1 & 0 \\ & & \iddots & & \\ 0 & 1 & & & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix} \quad (k \times k)$$

$\epsilon_i = \pm 1$: sign characteristic

In the quadratic case: For odd ℓ_j, $\sum_{j=1}^{q} \epsilon_j = 0$

Spectral data

Spectral data of $L(\lambda)$: eigenvalues (α_j, β_j) with partial multiplicities (ℓ_j, m_j) and sign characteristic:

$$(J, P)$$

Inverse Real Symmetric Quadratic Eigenvalue Problem

1. Characterize the spectral data (J, P) admissible for some $n \times n$ $L(\lambda) = L_2\lambda^2 + L_1\lambda + L_0$ real and symmetric.
2. Construct real symmetric $n \times n$ matrices $L(\lambda) = L_2\lambda^2 + L_1\lambda + L_0$ with prescribed admissible spectral data (J, P).

Real symmetric matrix polynomials and selfadjoint triples

Theorem

(a) *If $L(\lambda)$ is real and symmetric then all its real standard triples are selfadjoint.*

(b) *If $L(\lambda)$ admits a real selfadjoint standard triple then $L(\lambda)$ is real and symmetric.*

Conclusions (quadratic case)

1. (J, P) admissible if and only if (X, J, PX^T) selfadjoint Jordan triple for some $X \in \mathbb{R}^{n\times 2n}$:
   1. $\operatorname{rank} X = n$
   2. $XPX^T = 0$
   3. $XJPX^T$ invertible
2. If (J, P) is admissible design a procedure to obtain matrices X such that (X, J, PX^T) selfadjoint Jordan triple. For each X there is a real symmetric $L(\lambda)$

The semisimple case

$\ell_j = 1,\ m_j = 1$

$$J = \mathrm{Diag}\,(r_1, \dots, r_{2q}, U_1, \dots, U_s)$$

$$r_j \in \mathbb{R},\ U_j = \begin{bmatrix} \mu_j & -\nu_j \\ \nu_j & \mu_j \end{bmatrix},\ \beta_j = \mu_j \pm i\nu_j$$

$$\sum_{j=1}^{2q} \epsilon_j = 0 \Rightarrow \begin{cases} \epsilon_j = +1, & 1 \le j \le q \\ \epsilon_j = -1, & q+1 \le j \le 2q \end{cases}$$

$$P = \mathrm{Diag}\,(I_q, -I_q, F_1, \dots, F_s),\ F_k = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$X = \begin{bmatrix} X_+ & X_- & \sqrt{2}v_1 & \sqrt{2}u_1 & \cdots & \sqrt{2}v_s & \sqrt{2}u_s \end{bmatrix}$$

Remark: $U_k^T F_k U_k = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

A convenient selfadjoint Jordan triple

$$J = \mathrm{Diag}\left(R_+, R_-, \begin{bmatrix} M & -N \\ N & M \end{bmatrix}\right) \in \mathbb{R}^{2n\times 2n}$$

with

$$R_+ = \mathrm{Diag}(r_1, \dots, r_q), \quad R_- = \mathrm{Diag}(r_{q+1}, \dots, r_{2q})$$

$$M = \mathrm{Diag}(\mu_1, \mu_2, \dots, \mu_s), \quad N = \mathrm{Diag}(\nu_1, \nu_2, \dots, \nu_s) > 0$$

$$P = \mathrm{Diag}\left(I_q, -I_q, -I_s, I_s\right) \in \mathbb{R}^{2n\times 2n}$$

$$X = \begin{bmatrix} X_+ & X_- & V & U \end{bmatrix} \in \mathbb{R}^{n\times 2n}$$

$X_+ \in \mathbb{R}^{n\times q}$, $X_- \in \mathbb{R}^{n\times q}$

$$V = \sqrt{2}\begin{bmatrix} v_1 & \cdots & v_s \end{bmatrix} \in \mathbb{R}^{n\times s}, \qquad U = \sqrt{2}\begin{bmatrix} u_1 & \cdots & u_s \end{bmatrix} \in \mathbb{R}^{n\times s}$$

The orthogonality property of eigenvectors

$$XPX^T = 0 \quad \Leftrightarrow \quad \begin{bmatrix} X_+ & X_- & V & U \end{bmatrix} \begin{bmatrix} I_q & & & \\ & -I_q & & \\ & & -I_s & \\ & & & I_s \end{bmatrix} \begin{bmatrix} X_+^T \\ X_-^T \\ V^T \\ U^T \end{bmatrix} = 0$$

$$\Leftrightarrow \quad \begin{bmatrix} X_+ & U \end{bmatrix} \begin{bmatrix} X_+^T \\ U^T \end{bmatrix} = \begin{bmatrix} X_- & V \end{bmatrix} \begin{bmatrix} X_-^T \\ V^T \end{bmatrix}$$

$$\Leftrightarrow \quad \boxed{\begin{bmatrix} X_- & V \end{bmatrix} = \begin{bmatrix} X_+ & U \end{bmatrix} \Theta}$$

$\Theta \in \mathbb{R}^{n\times n}$ **orthogonal** and

$$\mathrm{rank}\, X = n \Leftrightarrow \det \begin{bmatrix} X_+ & U \end{bmatrix} \neq 0, \det \begin{bmatrix} X_- & V \end{bmatrix} \neq 0$$

Property $\det(XJPX^T) \neq 0$

$$\begin{bmatrix} X_- & V \end{bmatrix} = \begin{bmatrix} X_+ & U \end{bmatrix} \Theta$$

$$XJPX^T =$$

$$\begin{bmatrix} X_+ & X_- & V & U \end{bmatrix} \begin{bmatrix} R_+ & & & \\ & R_- & & \\ & & M & -N \\ & & N & M \end{bmatrix} \begin{bmatrix} I_q & & & \\ & -I_q & & \\ & & -I_s & \\ & & & I_s \end{bmatrix} \begin{bmatrix} X_+^T \\ X_-^T \\ V^T \\ U^T \end{bmatrix} =$$

$$\begin{bmatrix} X_+ & U & X_- & V \end{bmatrix} \begin{bmatrix} R_+ & & & \\ & M & & -N \\ & & -R_- & \\ & -N & & -M \end{bmatrix} \begin{bmatrix} X_+^T \\ U^T \\ X_-^T \\ V^T \end{bmatrix} =$$

$$\begin{bmatrix} X_+ & U \end{bmatrix} \underbrace{\begin{bmatrix} I_n & \Theta \end{bmatrix} \begin{bmatrix} R_+ & & & \\ & M & & -N \\ & & -R_- & \\ & -N & & -M \end{bmatrix} \begin{bmatrix} I_n \\ \Theta^T \end{bmatrix}}_{H(\Theta)} \begin{bmatrix} X_+^T \\ U^T \end{bmatrix}$$

Consequences for the IRSQEP

Given the spectral data (J, P)

1. (J, P) is admissible if and only if $\det H(\Theta) \neq 0$ for some $n \times n$ orthogonal Θ.
2. If (J, P) is admissible then the following procedure yields real symmetric quadratic matrix polynomial with spectral data (J, P):
    1. Take any nonsingular $X_1 \in \mathbb{R}^{n\times n}$ and write $X_1 = \begin{bmatrix} X_+ & U \end{bmatrix}$
    2. Take any orthogonal $\Theta \in \mathbb{R}^{n\times n}$ such that $\det H(\Theta) \neq 0$
    3. Construct $\begin{bmatrix} X_- & V \end{bmatrix} = X_1\Theta$ and $X = \begin{bmatrix} X_+ & X_- & V & U \end{bmatrix}$
    4. (X, J, PX^T) is a selfadjoint Jordan triple. Construct a real symmetric $L(\lambda)= L_2\lambda^2 + L_1\lambda + L_1$. In particular

$$L_2^{-1} = XJPX^T = \begin{bmatrix} X_+ & U \end{bmatrix} H(\Theta) \begin{bmatrix} X_+^T \\ U^T \end{bmatrix}$$

3. $L_2 > 0$ if and only if $H(\Theta) > 0$

Remark

$$H(\Theta) = \begin{bmatrix} I_n & \Theta \end{bmatrix} \overbrace{\begin{bmatrix} R_+ & & & \\ & M & & -N \\ & & -R_- & \\ & -N & & -M \end{bmatrix}}^{G} \begin{bmatrix} I_n \\ \Theta^T \end{bmatrix}$$

$$\Omega = \frac{1}{\sqrt{2}} \begin{bmatrix} I_n & \Theta \\ -\Theta^T & I_n \end{bmatrix} \quad \text{orthogonal}$$

$H(\Theta)$ is a principal submatrix of $\frac{1}{2}\Omega G \Omega^T$:

$$\lambda_i(G) \geq \lambda_i(H(\Theta)) \geq \lambda_{i+n}(G)$$

$$\Lambda(G) = (r_1, \ldots, r_q, -r_{q+1}, \ldots, -r_{2q}, \pm|\beta_1|, \ldots, \pm|\beta_s|)$$

Example

$J = \mathrm{Diag}(3, 2, 1, -1, -2, -3)$, $P = \mathrm{Diag}(+1, -1, -1, +1, +1, -1)$, $\Lambda(G) = (3, -1, -2, -2, -1, 3)$. (J, P) may be admissible but $\nexists L(\lambda) = L_2\lambda^2 + L_1\lambda + L_0$ with $L_2 > 0$.

Admissible spectral data

(J, P) admissible $\Rightarrow$ J= linearization of $L(\lambda)$ $\overset{\text{Theorem 1.7[4]}}{\Longrightarrow}$ if p=largest geometric multiplicity of eigenvalues of $L(\lambda)$ ($\dim \mathrm{Ker}\, L(\lambda_0)$) then $p \leq n$.
Since $n = q + s$

$$(J, P) \text{ admissible } \Rightarrow s \geq p - q$$

Theorem

(J, P) admissible spectral data if and only if $s \geq p - q$

Two possible cases

- $q \geq p$. $L(\lambda)$ can be taken diagonal.
- $p > q$. No diagonal $L(\lambda)$ with (J, P) as spectral data.

[4] I. Gohberg, P. Lancaster, L. Rodman: *Matrix Polynomials*. SIAM, 2009

Example I

$$J = \mathrm{Diag}\left(-2, -1, \begin{bmatrix} -2 & -1 \\ 1 & -2 \end{bmatrix}\right) \qquad P = \mathrm{Diag}(1, -1, -1, 1)$$

$$L(\lambda) = \begin{bmatrix} -(\lambda+2)(\lambda+1) & 0 \\ 0 & \lambda^2 + 4\lambda + 5 \end{bmatrix}$$

$$L_2 = \mathrm{Diag}(-1, 1) = L_2^{-1} = X_1 H(\Theta) X_1^T$$

$$\begin{aligned} H(\Theta) &= \begin{bmatrix} I_n & \Theta \end{bmatrix} \begin{bmatrix} R_+ & & & \\ & M & & -N \\ & & -R_- & \\ & -N & & -M \end{bmatrix} \begin{bmatrix} I_n \\ \Theta^T \end{bmatrix} \\ &= \begin{bmatrix} R_+ & 0 \\ 0 & M \end{bmatrix} - \Theta \begin{bmatrix} 0 & 0 \\ 0 & N \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & N \end{bmatrix} \Theta^T - \Theta \begin{bmatrix} R_- & 0 \\ 0 & M \end{bmatrix} \Theta^T \\ &= \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} - \Theta \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \Theta^T - \Theta \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \Theta^T \end{aligned}$$

$$\Theta = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad X_1 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Example I (cont.)

With the same Θ and $X_1 = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$

```
>> Th=sym([1 0; 0 -1]); X1=sym([2 1;0 2]);
>> G=sym([-2 0 0 0;0 -2 0 -1;0 0 1 0;0 -1 0 2]);
>> L2=inv(X1*[eye(2) Th]*G*[eye(2) Th]'*X1')
L2 =
[ -1/4,  1/8]
[  1/8, 1/16]
>> J=sym(blkdiag(diag([-2 -1]),[-2 -1;1 -2]));
>> X=X1*[eye(2) Th]; X=X(:,[1 3 4 2]);T=[X;X*J]; C=T*J*T^(-1)
C =
[  0,    0,  1,    0]
[  0,    0,  0,    1]
[ -2, -3/2, -3, -1/2]
[  0,   -5,  0,   -4]
>> L0=L2*C([3 4],[1 2]), L1=L2*C([3,4],[3,4])
L0 =                              L1=
[  1/2, -1/4]                       [   3/4, -3/8]
[ -1/4, -1/2]                       [ -3/8, -5/16]
```

Example II

$$J = \mathrm{Diag}\left(0,0,\begin{bmatrix}0&0&-1&0\\0&0&0&-1\\1&0&0&0\\0&1&0&0\end{bmatrix}\right) \qquad P = \mathrm{Diag}(1,-1,-1,-1,1,1)$$

$s = 2 > p - q = 1$

No diagonalizable quadratic with this spectral data

$$H(\Theta) = \begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix} - \Theta\begin{bmatrix}0&0&0\\0&1&0\\0&0&1\end{bmatrix} - \begin{bmatrix}0&0&0\\0&1&0\\0&0&1\end{bmatrix}\Theta^T - \Theta\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}\Theta^T$$

With $\Theta = \begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}$

$$H(\Theta) = \begin{bmatrix}0&-1&0\\-1&0&0\\0&0&-2\end{bmatrix}$$

Positive definite leading coefficient

The semisimple case

Theorem

Any semisimple real symmetric quadratic matrix polynomial with positive definite leading coefficient is diagonalizable over $\mathbb{R}$

Consequence: There is no real symmetric quadratic matrix polynomial with positive definite leading coefficient and, for example, λ, λ, λ^2+1 and λ^2+1 as elementary divisors.

Advantage

- Sign characteristic easy to compute from diagonal matrix polynomials

Drawback

- Nonconstructive proof.

Eigenfunctions[5]

$$\det(\mu I_n - L(\lambda)) = 0$$

implicitly defines n real functions $\mu_1(\lambda), \ldots, \mu_n(\lambda)$ which are analytic for real λ.

- $L(\lambda_0) = 0$ if and only if $\mu_j(\lambda_0) = 0$ for some j.
- $\dim \operatorname{Ker} L(\lambda_0) = \#\{j : \mu_j(\lambda_0) = 0\}$

Semisimple case: For each $\lambda_k \in \Lambda(L)$ there is j such that

$$\mu_j(\lambda) = (\lambda - \lambda_k)\nu_j(\lambda), \quad \nu_j(\lambda_k) \neq 0$$

$$\operatorname{sgn}(\nu_j(\lambda_k)) = \text{sign characteristic of } L(\lambda) \text{ at } \lambda_k$$

$$\nu(\lambda_k) = \mu_j'(\lambda_k)$$

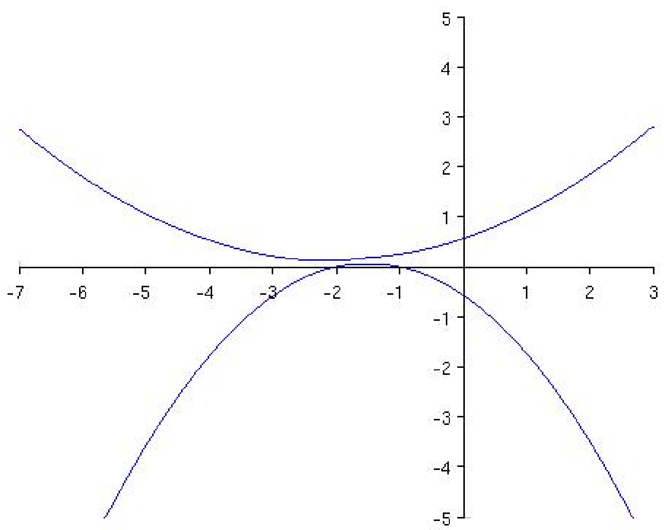
[5] I. Gohberg, P. Lancaster, L. Rodman: Spectral analysis of selfadjoint matrix polynomials. *Research paper 419* (1979) Dept. Mathematics and Statistics, University of Calgary

Example

$$L(\lambda) = \begin{bmatrix} -1/4 & 1/8 \\ 1/8 & 1/16 \end{bmatrix}\lambda^2 + \begin{bmatrix} 3/4 & -3/8 \\ -3/8 & -5/16 \end{bmatrix}\lambda + \begin{bmatrix} 1/2 & -1/4 \\ -1/4 & -1/2 \end{bmatrix}$$

has spectral data $J = \mathrm{Diag}\left(-2, -1, \begin{bmatrix} -2 & -1 \\ 1 & -2 \end{bmatrix}\right)$, $P = \mathrm{Diag}(1, -1, -1, 1)$

$$\det(\mu I - L(\lambda)) = 0$$



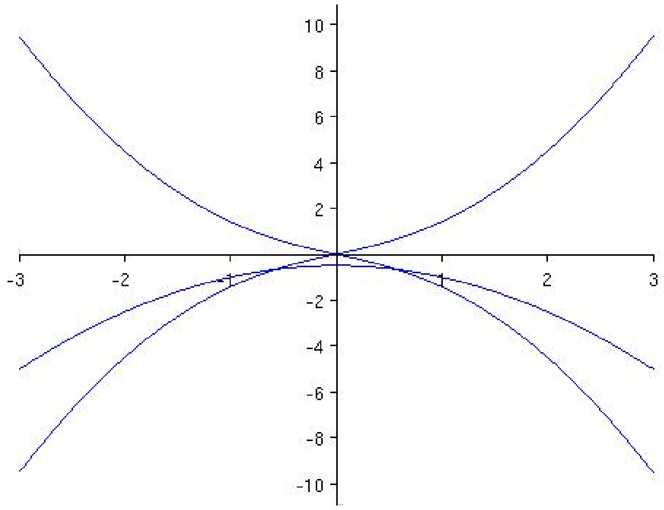
$\Lambda(L_2) = \{-0.2938\ldots, 0.10634\ldots\}$

Example

$$L(\lambda) = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1/2 \end{bmatrix} \lambda^2 + \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \lambda + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$$

Spectral data: λ , λ , λ^2+1, λ^2+1. Sign characteristic: $+1$ and -1.

$$\det(\mu I - L(\lambda)) = 0$$



$\Lambda(L_2) = \{-1, -1/2, 1\}$

Asymptotic behaviour of the eigenfunctions

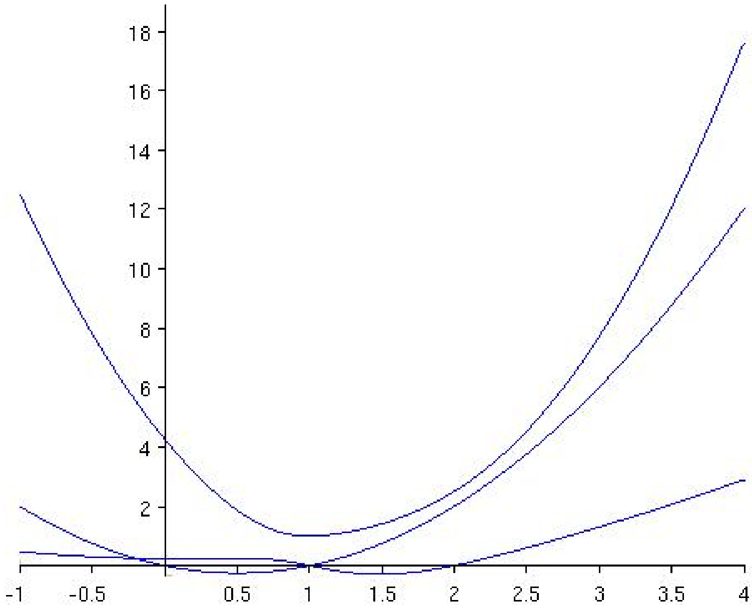
Lemma

Let $L(\lambda) = L_\ell \lambda^\ell + L_{\ell-1}\lambda^{\ell-1} + \cdots + L_0$ be a selfadjoint matrix polynomial with $\det L_\ell \neq 0$. Let $\mu_1(\lambda)$, ..., $\mu_n(\lambda)$ be the eigenfunctions of $L(\lambda)$ and let $(\pi, n-\pi, 0)$ the inertia of L_ℓ. Let $\lambda_{\max}$ be the highest real eigenvalue of $L(\lambda)$. Then there are π indices $\{i_1, \ldots, i_\pi\} \subseteq \{1, \ldots, n\}$ such that for $\lambda > \lambda_{\max}$ $\mu_j(\lambda) > 0$ if $j \in \{i_1, \ldots, i_\pi\}$ and $\mu_j(\lambda) < 0$ if $j \notin \{i_1, \ldots, i_\pi\}$.

Positive leading coefficient. Example

$$L(\lambda) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & \frac{3}{2} \end{bmatrix} \lambda^2 + \begin{bmatrix} 3 & 0 & -3 \\ 0 & 1 & 0 \\ -3 & 0 & 3 \end{bmatrix} \lambda + \begin{bmatrix} -2 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & -\frac{5}{2} \end{bmatrix}$$

Spectral data: $\lambda - 1$, $\lambda - 1$, $\lambda - 2$, λ and $\lambda^2 + 1$ and the following sign characteristic: $+1$, -1, $+1$ and -1



$\Lambda(L) = \{1, 0.2192\ldots, 2.2807\ldots\}$

The Sign Characteristic of MP with PD leading coeff.

Theorem

Let $L(\lambda)$ be a semisimple symmetric matrix polynomial with $L_\ell > 0$ and a maximal and minimal real eigenvalue λ_{max} and λ_{min}, respectively. For any $\alpha \leq \lambda_{\text{max}}$, let $p(\alpha)$ denote the number of real eigenvalues (counting multiplicities) of $L(\lambda)$ of positive type in $(\alpha, \lambda_{\text{max}}]$ and $n(\alpha)$ the number of real eigenvalues (counting multiplicites) of $L(\lambda)$ of negative type in $[\alpha, \lambda_{\text{max}}]$. Then

$$n(\alpha) \leq p(\alpha)$$

for all $\alpha \in [\lambda_{\text{min}}, \lambda_{\text{max}}]$.

The IRSQEP with positive definite leading coefficient

Theorem

There exists a semisimple real symmetric quadratic matrix polynomial $L(\lambda)$ with positive leading coefficient and (J, P) as spectral data if and only if

$$n(\alpha) \leq p(\alpha)$$

for all $\alpha \in [\lambda_{\min}, \lambda_{\max}]$, $\lambda_{\max}$ and $\lambda_{\min}$ the maximal and minimal prescribed real eigenvalues. Moreover, under this condition the matrix polynomial can be constructed diagonal.

Example

Eigenvalues	2	1	1	0	$\pm i$
Sign Characteristic	$+1$	$+1$	-1	-1	

$$D(\lambda) = \begin{bmatrix} (\lambda-2)(\lambda-1) & & \\ & (\lambda-1)\lambda & \\ & & \lambda^2+1 \end{bmatrix}$$

$$H(\Theta) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \Theta \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Theta^T - \Theta \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Theta^T$$

Taken $\Theta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ and $X_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/\sqrt{2} \end{bmatrix}$, $X_1 H(\Theta) X_1^T = I_3$

With this Θ, choose any non-singular X_1 and use the technique of page 14 to produce many nondiagonal $L(\lambda)$ with the prescribed spectral data.

Changing the orthogonal matrix

- Nonsingular leading coefficient: $\det H(\Theta) \neq 0$

$$\mathcal{S} = \{\Theta \in \mathcal{O}_n | \det H(\Theta) \neq 0\}$$

  $\mathcal{O}_n$ set of $n \times n$ orthogonal matrices. $\mathcal{S} \neq \emptyset$ if and only if $s > p - q$ and then its open and dense in $\mathcal{O}_n$.

- Positive definite leading coefficient: $H(\Theta) > 0$

$$\mathcal{P} = \{\Theta \in \mathcal{O}_n | H(\Theta) \text{ is positive definite}\}$$

  $\mathcal{P}$ is open in $\mathcal{O}_n$.

Thank you very much