Enclosures for the eigenvalues of operators in electromagnetism

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The Maxwell eigenvalue problem

$\Omega \subset \mathbb{R}^n$ - compact connected $\quad n = 2, 3$ and poly(gon/hedr)al boundary

μ - magnetic permeability and ϵ - electric permittivity, both L^∞ and

$$\exists c > 1 \qquad \begin{array}{c} c^{-1} \leq \epsilon(x) \leq c \\ c^{-1} \leq \mu(x) \leq c \end{array} \qquad \text{for almost all } x \in \Omega$$

Eigenfrequencies $\omega \in \mathbb{R}$
Electromagnetic fields $\mathbf{E}$ and $\mathbf{H}$ $(n{=}3)$ or H $(n{=}2)$

$$n{=}3 \left\{ \begin{array}{l} \operatorname{curl} \mathbf{E} = i\omega\mu\, \mathbf{H} \\ \operatorname{curl} \mathbf{H} = -i\omega\epsilon\, \mathbf{E} \\ \mathbf{E} \times \mathbf{n} \restriction_{\partial\Omega} = \mathbf{0} \end{array} \right.$$

$$\left. \begin{array}{l} \operatorname{curl} \mathbf{E} = i\omega\mu\, H \\ \begin{pmatrix} \partial_y H \\ -\partial_x H \end{pmatrix} = -i\omega\epsilon\, \mathbf{E} \\ \mathbf{E} \times \mathbf{n} \restriction_{\partial\Omega} = \mathbf{0} \end{array} \right\} n{=}2$$

$\mathbf{n}$ - outer normal vector to $\partial\Omega$.

Basic context

Problem. Given a subspace $\mathcal{L}$ on Ω generated by "standard" finite elements, extract from $\mathcal{L}$ "certified" information about $(\mathbf{E}, \omega)$.

In matrix form, writing $(\tilde{\mathbf{E}}, \tilde{\mathbf{H}}) = (\sqrt{\epsilon}\mathbf{E}, \sqrt{\mu}\mathbf{H})$ for $n = 3$, we get

$$\underbrace{\begin{pmatrix} \epsilon^{-1/2} & 0 \\ 0 & \mu^{-1/2} \end{pmatrix} \begin{pmatrix} 0 & i\,\mathrm{curl} \\ -i\,\mathrm{curl} & 0 \end{pmatrix} \begin{pmatrix} \epsilon^{-1/2} & 0 \\ 0 & \mu^{-1/2} \end{pmatrix}}_{M = \text{underlying self-adjoint operator } : \mathrm{D}(M) \longrightarrow L^2(\Omega; \mathbb{C}^{3n-3})} \begin{pmatrix} \tilde{\mathbf{E}} \\ \tilde{\mathbf{H}} \end{pmatrix} = \omega \begin{pmatrix} \tilde{\mathbf{E}} \\ \tilde{\mathbf{H}} \end{pmatrix}$$

Some remarks.

(a) $M = M^*$ is strongly indefinite (spectrum symmetric w.r.t. 0).

(b) $(\mathbf{E}, \mathbf{H}) = \mathrm{grad}\,\phi \Rightarrow (\mathbf{E}, \mathbf{H}) \in \ker(M)$ - infinite-dimensional.

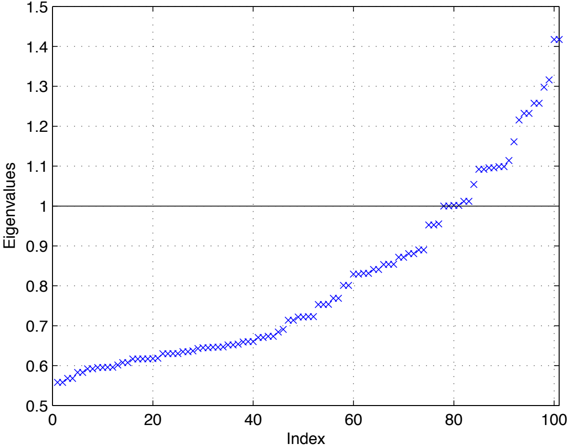
(c) Vector elements and mixed formulation for e.v.p. See [Boffi. *Acta Numerica* 2010] & [Arnold, Falke & Winter. *Bull AMS* 2010]. No "one-sided" bounds for eigenvalues. Not certified?

(d) What about nodal elements?

(e) Maybe use widely available computer packages, commercial or otherwise, and get certified information up to machine precision.

Nodal elements can fail dramatically

Galerkin method with Lagrange elements order 5 in $\Omega = [0, \pi]^2$. Calculations here and elsewhere fixing $\mu = \epsilon = 1$. Unstructured mesh with 3092 dof. The true $\sigma(M) = \left\{ \pm\sqrt{j^2 + k^2} \right\}_{jk \in \mathbb{N}}$.



The 100 eigenvalues of the reduced problem $M\restriction_{\mathcal{L}} \underline{v} = \lambda \underline{v}$ near $\omega = 1$

A general view of the strategy

Philosophy.
Certain matrix polynomials, where the coefficients are obtained from the action of M on $\mathcal{L}$, give certified information about $\sigma(M)$. Proofs usually involve a spectral mapping argument.

Concrete aim.
Compute enclosure $\omega \in (\omega^-, \omega^+)$ and $u \in \mathcal{L}$ such that, for $Mv = \omega v$,

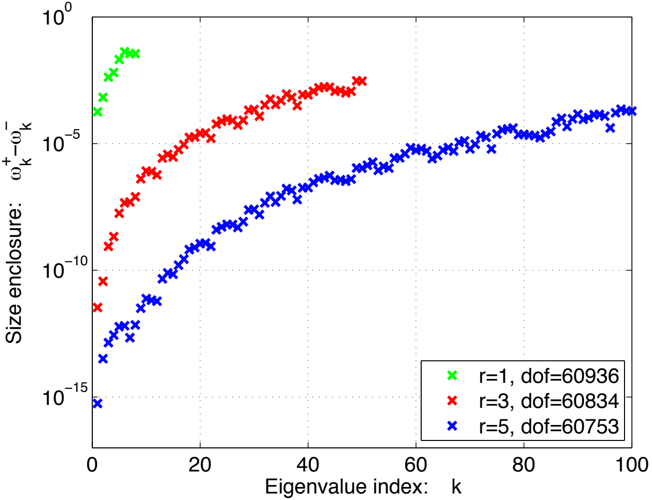
$$\frac{\|u - v\|}{\|u\|} \leq c(\omega^+ - \omega^-)^p$$

Actual strategy. Assuming we know nothing about $\sigma(M)$

1. Find a (big) interval $[a, b] \cap \sigma(M) = \{\omega\}$ from the second order spectrum of M relative to $\mathcal{L}$. [Boulton & Strauss, *Proc. Royal. Soc. A* 2011] and references therein.
2. Find $\omega^\pm$ and u from an extension of the Temple-Lehmann-Goerisch method [Zimmermann & Mertins, *Z. Anal. Anwendungen* 1995] and [Davies & Plum, *IMA J Numer. Anal.* 2004]. Results below are ongoing research with Boussaid & Barrenechea: we found $p = 1/2$.

Test region $\Omega = [0, \pi]^2$

Enclosure for the first 100 positive eigenvalues (not counting multiplicity)



Unstructured mesh on Lagrange elements order r.

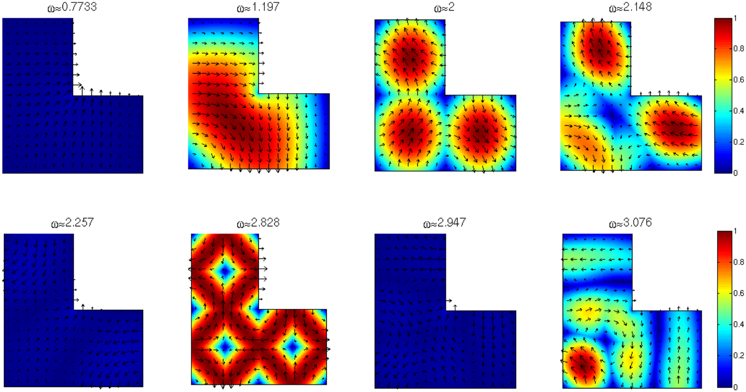
Test region $\Omega = [0,\pi]^2 \setminus [\pi/2,\pi]^2$

k	[Boffi *etal*, 1999] $\approx \omega_k$	$(\omega_k)^+_-$ [1.]	$(\omega_k)^+_-$ [2.]	(a_k, b_k)	dof
1	0.768192684	0.7^{81}_{54}	0.77333^{504}_{476}	$(0.316, 1.14)$	98733
2	1.196779010	1.19^{71}_{63}	1.1967827557^{625}_{339}	$(1, 1.73)$	69213
3	1.999784988	$^{2.00006}_{1.99994}$	$^{2.0000000000018}_{1.9999999999816}$	$(1.73, 2.02)$	8253
4	2.148306309	2.14^{89}_{79}	2.148483682^{711}_{572}	$(2.02, 2.21)$	69213
5	2.252760528	$2.^{28}_{23}$	2.257^{300}_{298}	$(2.17, 2.79)$	81018
6	2.828075317	2.82^{88}_{80}	2.82842712^{479}_{308}	$(2.24, 2.85)$	8253
7	2.938491109	2.9^{81}_{08}	2.9467^{180}_{083}	$(2.85, 3.07)$	81018
8	3.075901493	3.07^{71}_{45}	3.07589297478^{9}_{5}	$(3, 3.36)$	81018
9	3.390427701	$3.^{43}_{35}$	3.39807^{377}_{027}	$(3.08, 3.74)$	81018

We fixed order $r = 5$ in this table

Eigenfunctions on test region

For the previous table, the "**E**" component u_E of u are as follows.

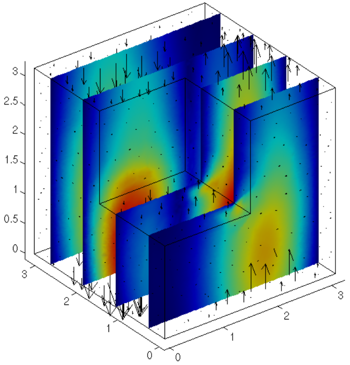


Colours are $|u_E|$ with $\|u_E\|_\infty = 1$ and the arrows point towards u_E.

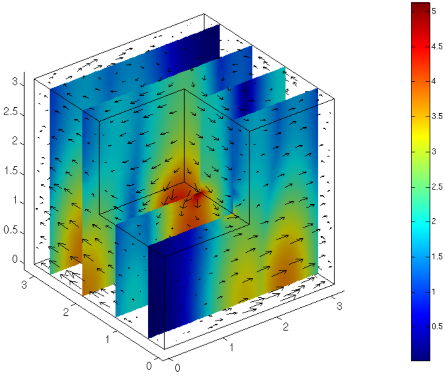
Test region $\Omega = [0,\pi]^3 \setminus \left([0,\pi/2]^2 \times [\pi/2,\pi]\right)$

Lagrange order $r = 3$, unstructured mesh maximum element size $h = \pi/8$

$$\omega = 2.^{50}_{18}$$



Electric field u_E



Magnetic field u_H

The *second order spectrum* of M relative to $\mathcal{L} \subset \mathrm{D}(M)$

$$\lambda \in \sigma_2(M,\mathcal{L}) \quad \overset{def}{\Longleftrightarrow} \quad \begin{array}{l} \text{for } \lambda \in \mathbb{C} \quad \exists 0 \neq u \in \mathcal{L} \\ \langle (M-\lambda)u, (M-\overline{\lambda})v \rangle = 0 \qquad \forall v \in \mathcal{L} \end{array}$$

$$\Longleftrightarrow \quad \begin{array}{l} \exists 0 \neq \underline{u} \in \mathbb{C}^d \qquad (K_{\mathcal{L}} - 2\lambda L_{\mathcal{L}} + \lambda^2 B_{\mathcal{L}})\underline{u} = 0 \\ K_{\mathcal{L}} = [\langle Mb_j, Mb_k \rangle] \quad L_{\mathcal{L}} = [\langle Mb_j, b_k \rangle] \quad B_{\mathcal{L}} = [\langle b_j, b_k \rangle] \end{array}$$

Theorem A. [Shargorodsky 2001]

$$\lambda, \overline{\lambda} \in \sigma_2(M,\mathcal{L}) \qquad \Longrightarrow \qquad \big[\mathrm{Re}\lambda - |\mathrm{Im}\lambda|, \mathrm{Re}\lambda + |\mathrm{Im}\lambda|\big] \cap \sigma(M) \neq \varnothing$$

Theorem B.
Let $\mathcal{L}_h \subset \mathrm{D}(M)$ be an $h > 0$ dependant family of finite element spaces. Denote by $v_h \in \mathcal{L}_h$ the finite element interpolant of $v \in \mathrm{D}(M)$. If

$$\|u_h - u\|_{\mathrm{D}(M^2)} < c\|u\|h^p \qquad \text{for} \qquad Mu = \omega u,$$

then there exists $\omega_h, \overline{\omega}_h \in \sigma_2(M,\mathcal{L}_h)$ such that $|\omega_h - \omega| = \mathcal{O}(h^{p/2})$. Moreover the algebraic multiplicities match.

The *Goerisch-Zimmermann-Mertins enclosure*

For $\rho \in \mathbb{R}$ consider the following eigenvalue problem

$$\text{find } u \in \mathcal{L} \setminus \{0\} \text{ and } \tau \in \mathbb{R} \quad \text{such that}$$
$$\tau \langle (M-\rho)u, (M-\rho)v \rangle = \langle (M-\rho)u, v \rangle \qquad \forall v \in \mathcal{L} \tag{P_ρ}$$

Theorem C.

Let $a < b$ such that $a, b \notin \sigma(M)$. Suppose that (P_a) has at least m_a positive eigenvalues $0 < \tau_+^{m_a} \leq \ldots \leq \tau_+^1$ and that (P_b) has at least m_b negative eigenvalues $\tau_-^1 \leq \ldots \leq \tau_-^{m_b} < 0$. Then M has

- at least j eigenvalues in $[a, a + \left(\tau_+^j\right)^{-1}]$ for all $j \in \{1, \ldots, m_a\}$
- at least k eigenvalues in $[b + \left(\tau_-^k\right)^{-1}, b]$ for all $k \in \{1, \ldots, m_b\}$

Corollary D.

Suppose that $[a, b] \cap \sigma(M) = \{\omega\}$. If (P_a) has a positive eigenvalue and (P_b) has a negative eigenvalue, then

$$\underbrace{b + \left(\tau_-^1\right)^{-1}}_{\omega^-} < \omega < \underbrace{a + \left(\tau_+^1\right)^{-1}}_{\omega^+}$$

Approximation of eigenvectors

$$\begin{gathered}\text{find } u \in \mathcal{L} \setminus \{0\} \text{ and } \tau \in \mathbb{R} \quad \text{such that} \\ \tau \langle (M-\rho)u, (M-\rho)v \rangle = \langle (M-\rho)u, v \rangle \qquad \forall v \in \mathcal{L}\end{gathered} \tag{P_ρ}$$

Theorem E.

Suppose that $[a, b] \cap \sigma(M) = \{\omega\}$ and let $\Pi = \int_a^b \mathrm{d}E_\lambda$ be the spectral projection associated to $[a, b]$. There exists a constant $c > 0$ only dependant on a, b and M but independent of $\mathcal{L}$ such that, if (P_a) has a positive eigenvalue with associated eigenvector u^+ and (P_b) has a negative eigenvalue with associated eigenvector u^-, then

$$\|(I - \Pi)u^{\pm}\| \leq c(\omega^+ - \omega^-)^{1/2} \|u^{\pm}\|.$$

Convergence for Lagrange elements

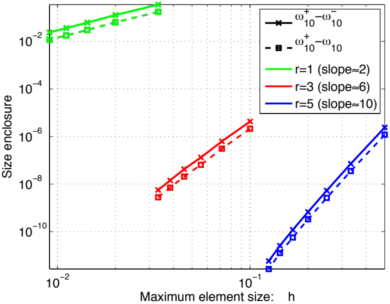
Lagrange finite element spaces $r \geq 1$

$$\begin{aligned}
\mathbf{V}_h^r(k) &= \{v_h \in [C^0(\overline{\Omega})]^k : v_h \restriction_K \in [\mathbb{P}_r(K)]^n \ \forall K \in \mathcal{T}_h\} \\
\mathbf{V}_{h,0}^r(n) &= \{v_h \in \mathbf{V}_h^r : v_h \times \mathbf{n} = 0 \text{ on } \partial\Omega\} \\
\tilde{\mathcal{L}}_h &= \quad \mathbf{V}_{h,0}^r(n) \times \mathbf{V}_h^r(2n-3) \quad \subset H_0^1(\mathrm{curl};\Omega) \times H^1(\Omega) \\
\mathcal{L}_h &= \quad \sqrt{\epsilon}\,\mathbf{V}_{h,0}^r(n) \times \sqrt{\mu}\,\mathbf{V}_h^r(2n-3) \quad \subset \mathrm{D}(M)
\end{aligned}$$

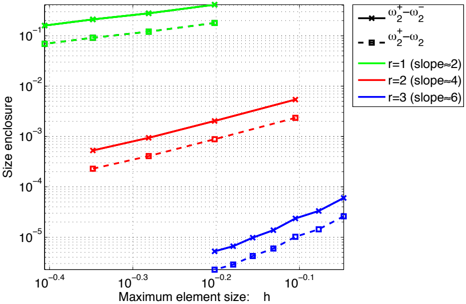
Theorem F.
Suppose that $[a,b] \cap \sigma(M) = \{\omega\}$. Let $\omega \in (\omega_h^-, \omega_h^+)$ and $u_h^{\pm}$ be the enclosure and normalised eigenvector found by applying Corollary D and Theorem E above with $\mathcal{L} = \mathcal{L}_h$. If an associated eigenfunction of ω is $H^{r+1}(\Omega)$, then

$$\omega^+ - \omega^- = \mathcal{O}(h^{2r}) \qquad \text{and} \qquad \frac{\|(I-\Pi)u_h^{\pm}\|}{\|u_h^{\pm}\|} = \mathcal{O}(h)$$

How sharp is Theorem F



$\Omega = [0, 2\pi]^2$



$\Omega = [0, 2\pi]^3$

The approximated spectral distance

An important ingredient in the proof of Theorems A, B, C and Corollary D is the close examination of the pseudospectrum of the quadratic matrix polynomial

$$Q(\lambda) = (K_{\mathcal{L}} - 2\lambda L_{\mathcal{L}} + \lambda^2 B_{\mathcal{L}})$$

[Higham & Tisseur, *SIAM J. Matrix Anal. Appl.* 2001] and also [Boulton, Lancaster & Psarrakos, Math. Comp. 2007].

$$F(t) = \min_{u \in \mathcal{L}} \frac{\|(M - t)u\|}{\|u\|} \qquad G(z) = \min_{u \in \mathcal{L}} \frac{\|Q(z)u\|}{\|u\|}$$

$$F^2(t) = G(t) \qquad t \in \mathbb{R}$$