Computing a Nearest Correlation Matrix with Factor Structure

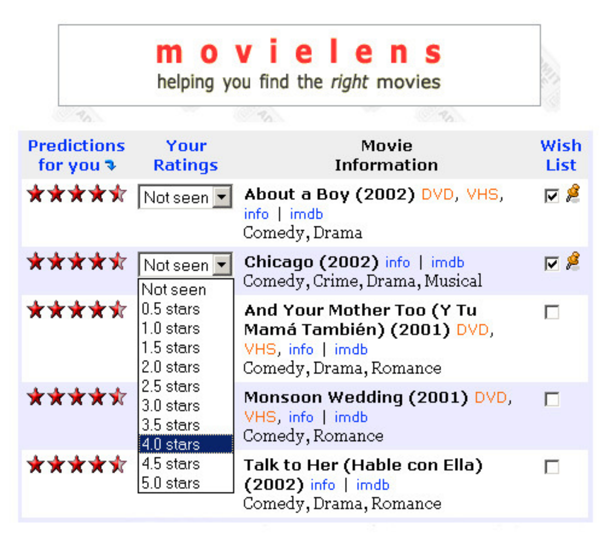
Nick Higham
School of Mathematics
The University of Manchester

higham@ma.man.ac.uk
http://www.ma.man.ac.uk/~higham/

Joint work with
Rüdiger Borsdorf, Marcos Raydan

Linear Algebra and its Applications to Financial Engineering—in honour of Prof. Peter Lancaster.
January 2012

http://www.movielens.org



http://www.netflixprize.com/

Correlation Matrix

$n \times n$ symmetric positive semidefinite matrix A with $a_{ii} \equiv 1$.

- symmetric,
- 1s on the diagonal,
- eigenvalues nonnegative *or*
  all principal minors nonnegative.

Properties:

- off-diagonal elements between -1 and 1,
- convex set.

Quiz

Is this a correlation matrix?

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Quiz

Is this a correlation matrix?

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}. \qquad \text{Spectrum: } -0.4142,\ 1.0000,\ 2.4142.$$

Quiz

Is this a correlation matrix?

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}. \qquad \text{Spectrum: } -0.4142,\ 1.0000,\ 2.4142.$$

For what w is this a correlation matrix?

$$\begin{bmatrix} 1 & w & w \\ w & 1 & w \\ w & w & 1 \end{bmatrix}.$$

Quiz

Is this a correlation matrix?

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}. \qquad \text{Spectrum: } -0.4142,\ 1.0000,\ 2.4142.$$

For what *w* is this a correlation matrix?

$$\begin{bmatrix} 1 & w & w \\ w & 1 & w \\ w & w & 1 \end{bmatrix}. \qquad \frac{-1}{n-1} \le w \le 1.$$

Structured Correlation Matrices

- Nonnegative:

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & 1 & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & 1 \end{bmatrix}.$$

Structured Correlation Matrices

- Nonnegative:

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & 1 & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & 1 \end{bmatrix}.$$

- Low rank:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Structured Correlation Matrices

- Nonnegative:

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & 1 & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & 1 \end{bmatrix}.$$

- Low rank:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

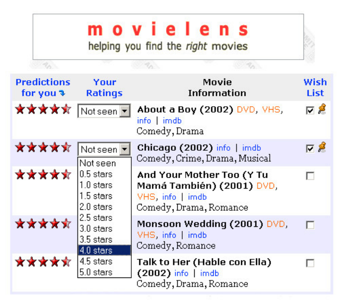
- Factor structure:

$$\begin{bmatrix} 1 & x_1x_2 & x_1x_3 \\ x_1x_2 & 1 & x_2x_3 \\ x_1x_3 & x_2x_3 & 1 \end{bmatrix}.$$

Approximate Correlation Matrices

Empirical correlation matrices often not true correlation matrices, due to

- asynchronous data
- missing data
- limited precision
- stress testing



Nearest Correlation Matrix

Find X achieving

$$\min\{\, \|A - X\|_F : X \text{ is a correlation matrix}\,\},$$

where $\|A\|_F^2 = \sum_{i,j} a_{ij}^2$.

- ★ Constraint set is a closed, convex set, so unique minimizer.
- ★ Nonlinear optimization problem.

Questions From Finance Practitioners

"Given a real symmetric matrix A which is almost a correlation matrix what is the best approximating (in Frobenius norm?) correlation matrix?"

"I am researching ways to make our company's correlation matrix positive semi-definite."

"Currently, I am trying to implement some real options multivariate models in a simulation framework. Therefore, I estimate correlation matrices from inconsistent data set which eventually are non psd."

Google scholar correlation matrix Search Advanced Scholar Search My Citations

Scholar Articles and patents | anytime | include citations | Create email alert | Results **1** - **10** of about **3,450,000**. (**0.11** sec)

Tests for comparing elements of a **correlation matrix**. — [PDF] from wisc.edu · Find It via JRUL

JH Steiger - Psychological Bulletin, 1980 - psycnet.apa.org

Abstract 1. In psychological research, it is desirable to be able to make statistical comparisons between **correlation** coefficients measured on the same individuals. For example, an experimenter (E) may wish to assess whether 2 predictors correlate equally ...

Cited by 1326 - Related articles - All 9 versions

Correlation matrix memories — Find It via JRUL

T Kohonen - Computers, IEEE Transactions on, 1972 - ieeexplore.ieee.org

Abstract A new model for associative memory, based on a **correlation matrix**, is suggested. In this model information is accumulated on memory elements as products of component data. Denoting a key vector by q (p), and the data associated with it by another vector x (p) ...

Cited by 662 - Related articles - All 7 versions

[BOOK] Principles and procedures of statistics: a biometrical approach

RGD Steel... - 1980 - orton.catie.ac.cr

... distribution%comparisons involving two sample means%principles of experimental design%analysis of variance I: the one-way classification%multiple comparisons%analysis of variance II: multiway classifications%linear regression%linear **correlation%matrix** notation%linear ...

Cited by 36886 - Related articles - Cached - Find It via JRUL - Library Search - All 10 versions

---

Longitudinal data analysis using generalized linear models — [PDF] from jstor.org · Find It via JRUL

KY Liang, SL Zeger - Biometrika, 1986 - Biometrika Trust

... Section 3 introduces and presents asymptotic theory for the 'generalized' estimating equation in which we borrow strength across subjects to estimate a 'working' **correlation matrix** and hence explicitly account for the time dependence to achieve greater asymptotic efficiency. ...

Cited by 8734 - Related articles - All 22 versions

Computing the nearest **correlation matrix**—a problem from finance — [PDF] from oxfordjournals.org · Find It via JRUL

NJ Higham - IMA Journal of Numerical Analysis, 2002 - imajna.oxfordjournals.org

Abstract Given a symmetric **matrix**, what is the nearest **correlation matrix**—that is, the nearest symmetric positive semidefinite **matrix** with unit diagonal? This problem arises in the finance industry, where the correlations are between stocks. For distance measured in two ...

Cited by 184 - Related articles - BL Direct - All 18 versions

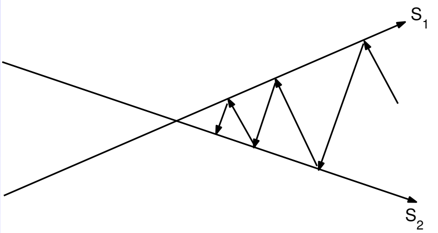
Create email alert

Goooooooooogle ▶

Result Page: 1 2 3 4 5 6 7 8 9 10 **Next**

Alternating Projections Method

H (2002): repeatedly **project** onto the positive semidefinite matrices then the unit diagonal matrices.



- Easy to implement.
- Guaranteed convergence, at a linear rate.
- Can add further constraints/projections, e.g., fixed elements (Lucas, 2001).

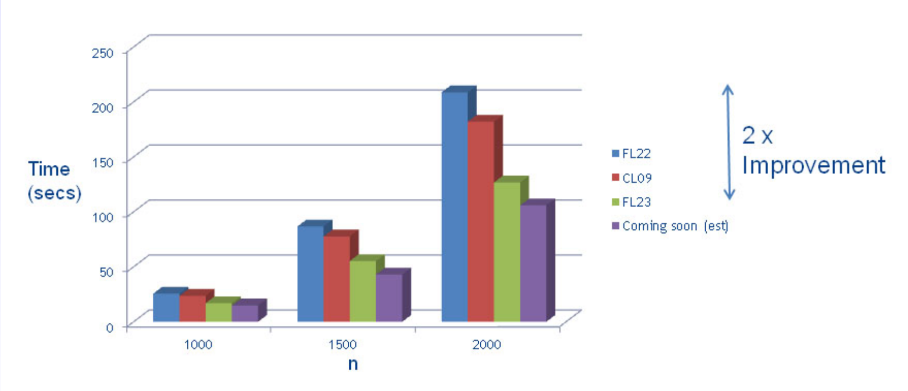
Newton Method

Qi & Sun (2006): **Newton method** based on theory of strongly semismooth matrix functions.

- Applies Newton to **dual** (unconstrained) of $\min \frac{1}{2}\|A - X\|_F^2$ problem.
- **Globally** and **quadratically** convergent.
- H & Borsdorf (2010) improve efficiency and reliability:
  - use minres with preconditioning,
  - reliability improved by line search tweaks,
  - extra scaling step to ensure unit diagonal.

NAG Library

Implemented in NAG codes **g02aaf** (**g02aac**) and **g02abf** (weights, lower bound on ei'vals—Mark 23).



Factor Model

$$\xi = \underbrace{X}_{n\times k}\underbrace{\eta}_{k\times 1} + \underbrace{F}_{n\times n}\underbrace{\varepsilon}_{n\times 1}, \qquad \eta_i, \varepsilon_i \in N(0,1),$$

where $F = \mathrm{diag}(f_{ii})$. Since $E(\xi) = 0$,

$$\mathrm{cov}(\xi) = E(\xi\xi^T) = XX^T + F^2.$$

Assume $\mathrm{var}(\xi_i) \equiv 1$. Then $\sum_{j=1}^k x_{ij}^2 + f_{ii}^2 = 1$, so

$$\sum_{j=1}^k x_{ij}^2 \le 1, \qquad i = 1\colon n.$$

- Collateralized debt obligations (CDOs),
- multifactor normal copula model.

Structured Correlation Matrix

Yields correlation matrix of form

$$C(X) = D + XX^T = D + \sum_{j=1}^{k} x_j x_j^T,$$

$$D = \mathrm{diag}(I - XX^T), \qquad X = [x_1, \dots, x_k].$$

$C(X)$ has k **factor correlation matrix structure**.

Structured Correlation Matrix

Yields correlation matrix of form

$$C(X) = D + XX^T = D + \sum_{j=1}^{k} x_j x_j^T,$$

$$D = \mathrm{diag}(I - XX^T), \qquad X = [x_1, \ldots, x_k].$$

$C(X)$ has k **factor correlation matrix structure**.

$$C(X) = \begin{bmatrix} 1 & y_1^T y_2 & \ldots & y_1^T y_n \\ y_1^T y_2 & 1 & \ldots & \vdots \\ \vdots & & \ddots & y_{n-1}^T y_n \\ y_1^T y_n & \ldots & y_{n-1}^T y_n & 1 \end{bmatrix}, \quad y_i \in \mathbb{R}^k.$$

Aims

For k factor correlation matrices, investigate

- mathematical properties,
- nearness problem.

1-Parameter Correlation Matrix

$$X(w) = \begin{bmatrix} 1 & w & w \\ w & 1 & w \\ w & w & 1 \end{bmatrix}, \qquad w \in \mathbb{R}.$$

Theorem

$\min\{\, \|A - X(w)\|_F : X(w)$ *a corr. matrix* $\}$ *has unique solution the projection of*

$$w = \frac{e^T A e - \mathrm{trace}(A)}{n^2 - n},$$

onto $[-1/(n-1), 1]$.

Block Structured Correlation Matrix

$$\left[\begin{array}{cc|cc} 1 & \gamma_{11} & \gamma_{12} & \gamma_{12} \\ \gamma_{11} & 1 & \gamma_{12} & \gamma_{12} \\ \hline \gamma_{12} & \gamma_{12} & 1 & \gamma_{22} \\ \gamma_{12} & \gamma_{12} & \gamma_{22} & 1 \end{array}\right], \qquad C_{ij} = \begin{cases} C(\gamma_{ii}) \in \mathbb{R}^{n_i \times n_i}, & i = j, \\ \gamma_{ij} ee^T \in \mathbb{R}^{n_i \times n_j}, & i \neq j. \end{cases}$$

Objective function:

$$f(\Gamma) = \|A - C(\Gamma)\|_F^2 = \sum_{i=1}^{m} \|A_{ii} - C(\gamma_{ii})\|_F^2 + \sum_{i \neq j} \|A_{ij} - \gamma_{ij} ee^T\|_F^2.$$

- Convex constraint set $\Rightarrow$ unique minimizer.
- Alternating projections converges (use prev. theorem for projection onto pattern).

1-Factor Correlation Matrix

$$C(x) = \mathrm{diag}(1 - x_i^2) + xx^T, \qquad x \in \mathbb{R}^n$$

i.e., $c_{ij} = x_i x_j$, $i \neq j$.

Lemma

$$\det(C(x)) = \prod_{i=1}^{n} (1 - x_i^2) + \sum_{i=1}^{n} x_i^2 \prod_{\substack{j=1 \\ j \neq i}}^{n} (1 - x_j^2).$$

Corollary

If $|x| \leq e$ with $x_i = 1$ for at most one i then $C(x)$ is nonsingular. $C(x)$ is singular if $x_i = x_j = 1$ for some $i \neq j$.

Rank Result

$$C(x) = \mathrm{diag}(1 - x_i^2) + xx^T,$$

Theorem

Let $x \in \mathbb{R}^n$ with $|x| \le e$. Then $\mathrm{rank}(C(x)) = \min(p+1, n)$, *where p is the number of x_i for which $|x_i| < 1$.*

$$x = [1\ 1\ 1\ x_4\ x_5] \quad \Rightarrow \quad C(x) = \begin{bmatrix} 1 & 1 & 1 & x_4 & x_5 \\ 1 & 1 & 1 & x_4 & x_5 \\ 1 & 1 & 1 & x_4 & x_5 \\ x_4 & x_4 & x_4 & 1 & x_4x_5 \\ x_5 & x_5 & x_5 & x_4x_5 & 1 \end{bmatrix}.$$

All your hedges in one basket

Leif Andersen, Jakob Sidenius and Susanta Basu present new techniques for single-tranche CDO sensitivity and hedge ratio calculations. Using factorisation of the copula correlation matrix, discretisation of the conditional loss distribution followed by a recursion-based probability calculation, and derivation of analytical formulas for deltas, they demonstrate a significant improvement in computational speeds

In a traditional synthetic collateralised debt obligation (CDO), the arranger tranches out credit losses on a pool of credit default swaps (CDSs) and passes them through to different investors. Assuming that investors for all tranches can be identified, the arranger is typically left with fairly moderate market exposure. For various reasons, placing the entire pool capital structure with investors has become increasingly difficult, and many recent credit basket derivatives expose the dealer to significant market risk. For instance, the recent 'single-tranche' CDO (STCDO) product involves the sale of a single CDO tranche to a single customer, leaving it to the arranger to manage the risk of the remaining capital structure. As STCDOs and similar 'custom' products offer significant customer benefits and are much less difficult to originate than traditional CDOs, such products are likely to increase in importance. This is especially true for managed trades where the customer has certain rights to alter the composition of the reference portfolio over time.

A basic prerequisite for active management of the risk of a credit basket derivative is the ability to accurately calculate the sensitivity of the security with respect to market and model parameters, most prominently the par CDS spreads of the underlying reference pool. The numbers of such sensitivities can be very large – many thousands – and can put considerable strain on computing resources. Moreover, the calculation of each of

where Q is the risk-neutral probability measure and λ_k is a (forward) default hazard rate function. The functions $p_k(T)$, $k = 1, \dots, N$ can be bootstrapped by standard means from the quoted CDS spreads and are assumed known for all T.

Equation (1) fully establishes the risk-neutral marginal distribution of each default time τ_k. To construct the joint distribution of all default times, we here choose[2] to employ a Student-*t* copula, which we quickly define for reference. Defining vectors $\boldsymbol{\tau} = (\tau_1, \dots, \tau_N)^T$ and $\mathbf{T} = (T_1, \dots, T_N)^T$, the joint default time distribution in the Student-*t* copula, becomes:

$$Q(\boldsymbol{\tau} \le \mathbf{T}) = t_{N,\nu}\left(t_{1,\nu}^{-1}\left(p_1(T_1)\right),\dots,t_{1,\nu}^{-1}\left(p_N(T_N)\right)\right) \qquad (2)$$

where $t_{1,\nu}$ and $t_{N,\nu}$ are the one- and N-dimensional cumulative Student-*t* distribution functions with ν degrees of freedom, respectively. Recall that the density $\eta_{N,\nu}$ of an N-dimensional Student-*t* distribution with correlation matrix Σ is:

$$\eta_{N,\nu}(\mathbf{z}) = C_{N,\nu}\left(1+\nu^{-1}\mathbf{z}^T\Sigma^{-1}\mathbf{z}\right)^{-\frac{\nu+N}{2}}, \quad C_{N,\nu} \equiv \frac{\Gamma\left(\frac{\nu+N}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{|\Sigma|(\nu\pi)^N}} \qquad (3)$$

where Γ is the gamma function. For high degrees of freedom, (3) approaches

One-Factor Problem

$$\min_{x\in\mathbb{R}^n} f(x) := \|A - C(x)\|_F^2$$
$$\text{subject to} \quad -e \le x \le e.$$

- Objective function is nonconvex.
- The constraint implies $C(x)$ is a correlation matrix.

One-Factor Problem: Derivatives

- **Objective**:

  $$f(x) = \langle A - I, A - I \rangle_F - 2x^T(A - I)x + (x^Tx)^2 - \sum_{i=1}^n x_i^4.$$

- **Gradient**:

  $$\nabla f(x) = 4((x^Tx)x - (A - I)x - \mathrm{diag}(x_i^2)x).$$

- **Hessian**:

  $$\nabla^2 f(x) = 4(2xx^T + (x^Tx + 1)I - A - 3\mathrm{diag}(x_i^2)).$$

- $\nabla f(x)$, $\nabla^2 f(x)$ cheap.
- $f(x)$ has a saddle point at $x = 0$.

k Factor Problem

$$C(X) := I - \mathrm{diag}(XX^T) + XX^T \quad \text{with } X \in \mathbb{R}^{n\times k}.$$

Representation not unique!

$$\sum_{j=1}^{k} x_{ij}^2 \le 1 \quad \Longrightarrow \quad C(X) \text{ is a correlation matrix.}$$

The k factor problem is

$$\min_{X\in\mathbb{R}^{n\times k}} f(x) := \|A - C(X)\|_F^2 \quad \text{subject to} \quad \sum_{j=1}^{k} x_{ij}^2 \le 1.$$

k Factor Problem: Derivatives

- Gradient

$$\nabla f(X) = 4(X(X^T X) - AX + X - \mathrm{diag}(XX^T)X)$$

- Hessian given implicitly, can be viewed as a matrix representation of the Fréchet derivative of $\nabla f(X)$.

Choice of Optimization Method

- Derivatives available.
- Ignore the constraints?
- Starting matrix, convergence test?

Choice of Optimization Method

- Derivatives available.
- Ignore the constraints?
- Starting matrix, convergence test?

- Rich set of solvers in NAG Library, Mark 22:
  - E04 - Minimizing or Maximizing a Function
  - E05 - Global Optimization of a Function
- MATLAB Optimization toolbox.

Alternating Directions

$$f(x_{ij}) = \text{const.} + 2\sum_{q\neq i}\left(a_{iq} - \sum_{s=1}^{k} x_{is}x_{qs}\right)^2.$$

Hence $f'(x_{ij}) = 0$ if

$$x_{ij} = \frac{\sum_{q\neq i} x_{qj}\left(a_{iq} - \sum_{s\neq j} x_{is}x_{qs}\right)}{\sum_{q\neq i} x_{qj}^2}.$$

Project x_{ij} onto $[-1, 1]$.

- Convergence to stationary point of f guaranteed.
- Limit may not be feasible for $k > 1$.

Principal Factors Method

Anderson, Sidenius & Basu (2003): with
$F(X) = I - \mathrm{diag}(XX^T)$ [so $C(X) = F(X) + XX^T$]

$$X_i = \mathrm{argmin}_{X \in \mathbb{R}^{n\times k}} \|A - F(X_{i-1}) - XX^T\|_F.$$

Min obtained by eigendecomposition of $A - F(X_{i-1})$.
Equivalent to **alternating projections method** for

$$\mathcal{U} := \{W \in \mathbb{R}^{n\times n} : w_{ij} = a_{ij} \text{ for } i \neq j\}$$ **convex,**

$$\mathcal{S} := \{W \in \mathbb{R}^{n\times n} : W = XX^T \text{ for } X \in \mathbb{R}^{n\times k}\}$$ **nonconvex!**

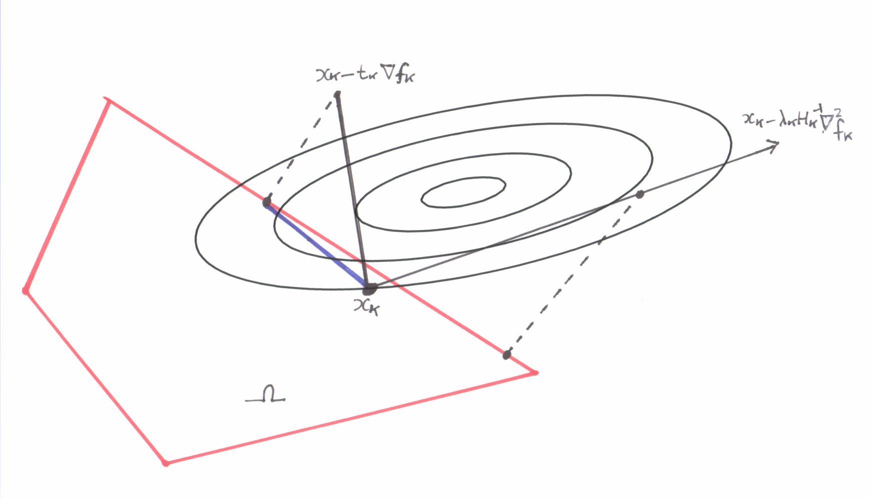
- Alt proj theory says no guarantee of convergence!
- Constraints ignored, so project final iterate onto them.

Spectral Projected Gradient Method

Birgin, Martínez & Raydan (2000).
To minimize $f : \mathbb{R}^n \to \mathbb{R}$ over convex set Ω:

$$x_{k+1} = x_k + \alpha_k d_k.$$

- $d_k = P_\Omega(x_k - t_k \nabla f(x_k)) - x_k$ is descent direction,
- $\alpha_k \in [-1, 1]$ chosen through **nonmonotone line search** strategy.
- Promising since $P_\Omega(\cdot)$ is cheap to compute.



- Fortran Code in ACM TOMS (Alg 813).
- R package by Varadhan & Gilbert (2009).

Test Examples

- **corr**: `gallery('randcorr',n)`
- **nrand**: $\frac{1}{2}(B + B^T) + \mathrm{diag}(I - B)$ with $B \in [-1, 1]^{n \times n}$ such that $\lambda_{\min}(B) < 0$.

Results averaged over 10 instances.

- **AD**: alternating directions.
- **PFM**: principal factors method.
- **Nwt**: `e04lb` of NAG Toolbox for MATLAB (modified Newton), bound constraints.
- **SPGM**: spectral projected gradient method.

Comparison: $k = 1$, $n = 2000$

	tol$= 10^{-3}$			tol$= 10^{-6}$		
	t(sec.)	**#its**	$\sqrt{f(X^*)}$	**t(sec.)**	**#its**	$\sqrt{f(X^*)}$
corr, $f(X_0) = 26.0$						
AD	3.3	5.2	26.0	3938	7282	26.0
PFM	68	1.1	26.0	827	18	26.0
Nwt	23	1.8	26.0	36	5.0	26.0
SPGM	9.8	5.2	26.0	638	760	26.0
nrand $f(X_0) = 825.13$						
AD	3.8	7.2	815.79	3.4	10.0	815.79
PFM	22	3.0	815.81	19.0	4.0	815.81
Nwt	4167	1222	815.79	4312	1229	815.79
SPGM	9.4	7.2	815.79	11	9.6	815.79

Comparison: $k = 6, n = 1000$

	tol$= 10^{-3}$			**tol**$= 10^{-6}$		
	t(sec.)	**#its**	$\sqrt{f(X^*)}$	**t(sec.)**	**#its**	$\sqrt{f(X^*)}$
corr, $f(X_0) = 18.5$						
AD	704	836	18.38	5060	5955	18.38
PFM	10	4.1	18.38	95	28.1	18.38
Nwt	167	52	18.38	280	68.2	18.38
SPGM	24	235	18.38	108	892	18.38
nrand , $f(X_0) = 415$						
AD	8694	9816	421	1.13e4	1.28e4	414
PFM	10.1	6.0	421	9.8	10	420
Nwt	146	40.8	421	109	56	420
SPGM	122	1263	407	276	2925	407

Overall Conclusions

- Nearest *k* factor correlation matrix problem relevant in many applications.
- **Principal factors method** has no convergence theory and can converge to an incorrect answer!
- Important to use methods that respect the constraints and converge to a feasible stationary point.
- **Spectral projected gradient** method is best choice: exploits convexity of constraints.
  Implemented in NAG routine **`g02aef`** (Mark 23, 2011).

R. Borsdorf, N. J. Higham and M. Raydan
Computing a Nearest Correlation Matrix with Factor Structure, SIMAX, 31(5): 2603–2622, 2010.

References I

- L. Anderson, J. Sidenius, and S. Basu.
  All your hedges in one basket.
  Risk, pages 67–72, Nov. 2003.
  |www.risk.net|.

- J. Barzilai and J. M. Borwein.
  Two-point step size gradient methods.
  IMA J. Numer. Anal., 8:141–148, 1988.

- E. G. Birgin, J. M. Martínez, and M. Raydan.
  Nonmonotone spectral projected gradient methods on convex sets.
  SIAM J. Optim., 10(4):1196–1211, 2000.

References II

E. G. Birgin, J. M. Martínez, and M. Raydan.
Algorithm 813: SPG—Software for convex-constrained optimization.
ACM Trans. Math. Software, 27(3):340–349, 2001.

E. G. Birgin, J. M. Martínez, and M. Raydan.
Spectral projected gradient methods.
In C. A. Floudas and P. M. Pardalos, editors, *Encyclopedia of Optimization*, pages 3652–3659.
Springer-Verlag, Berlin, second edition, 2009.

R. Borsdorf, N. J. Higham, and M. Raydan.
Computing a nearest correlation matrix with factor structure.
SIAM J. Matrix Anal. Appl., 31(5):2603–2622, 2010.

References III

P. Glasserman and S. Suchintabandid.
Correlation expansions for CDO pricing.
Journal of Banking & Finance, 31:1375–1398, 2007.

N. J. Higham.
Computing the nearest correlation matrix—A problem from finance.
IMA J. Numer. Anal., 22(3):329–343, 2002.

C. Lucas.
Computing nearest covariance and correlation matrices.

M.Sc. Thesis, University of Manchester, Manchester, England, Oct. 2001.
68 pp.

References IV

H.-D. Qi and D. Sun.
A quadratically convergent Newton method for computing the nearest correlation matrix.
SIAM J. Matrix Anal. Appl., 28(2):360–385, 2006.

P. Sonneveld, J. J. I. M. van Kan, X. Huang, and C. W. Oosterlee.
Nonnegative matrix factorization of a correlation matrix.
Linear Algebra Appl., 431:334–349, 2009.

A. Vandendorpe, N.-D. Ho, S. Vanduffel, and P. Van Dooren.
On the parameterization of the CreditRisk$^+$ model for estimating credit portfolio risk.
Insurance Math. Econom., 42(2):736–745, 2008.

References V

R. Varadhan and P. D. Gilbert.
BB: An R package for solving a large system of nonlinear equations and for optimizing a high-dimensional nonlinear objective function.
J. Statist. Software, 32(4):1–26, 2009.