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Linear Descriptor Systems for Modelling Insurance Portfolios: A Matrix Pencil Approach

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Introduction

We develop our analysis in **three** stages:

- We specify a model to describe the **premium rating process** associated with the sharing claim experience for each product in the pool.
- We formulate and examine the **interaction** of the surpluses among the insurance products in the pool.
- In the special case, where a target for zero surplus is required for some of the products (e.g. due to a regulatory constraint), a linear **descriptor** model is derived where the solution is more complex and the notion of causality does not exist.

Two main objectives:

(i) To provide a comprehensive, convenient and practical actuarial model for the management of a portfolio of different insurance products using the standard tools of control theory:

- claims may be regarded as the input ($\underline{u}$),
- the accumulated surplus as the state ($\underline{x}$) and
- the gross premiums as the output ($\underline{y}$) vector-variable.

(ii) To introduce the mathematical framework for manipulating and solving systems (1) by presenting some preliminary concepts and definitions from *matrix pencil theory*:

Note also that for a given pair of constant matrices E and $A \in \mathbb{R}^{m \times n}$, with $\det E = 0$ which uniquely determine the matrix pencil $sE - A$ of system (1) is defined.

$$\begin{cases} E\underline{x}_k = A\underline{x}_{k-1} + B\underline{u}_k \\ \\ \underline{y}_k = C\underline{x}_k + D\underline{u}_k \end{cases} \tag{1}$$

Notations and Model Framework

Consider:

m : The total **number** of different products, participating in the portfolio of the insurance company.

e_i : The **expense** factor for the i^{th}-insurance product, i.e. $(1-e_i)\times$Gross Premium is the margin for expenses.

r_i : The **annual rate of investment return** for the i^{th}-insurance product.

λ_{ij} : The **interaction factor** $i, j = 1, 2, \ldots, m$, is the proportion of accumulated surplus of the i^{th}-product transferred to the j^{th}-insurance product.

ε_i : The **profit sharing factor** (feedback factor) for the i-insurance product, which includes premium repayments and determines the percentage of accumulated surplus repaid to the policyholders.

$\left\{C_{i,k}\right\}_{k\in\mathbb{N}}$: The **actual total amount incurred claims** sequence for the i-company in year k, i.e. $(k-1, k]$.

$\left\{\hat{C}_{i,k}\right\}_{k\in\mathbb{N}}$: The **estimated total expected annual incurred claims** sequence in year k for the i^{th}- insurance product. Obviously, there is always a small (or lar-ger) delay period of d_i years in updating information.

In that case and using the latest information of the two available years, we ob-tain:

$$\hat{C}_{i,k} = f\begin{pmatrix}\text{previous}\\ \text{years}\end{pmatrix} = w_i C_{i,k-d_i-1} + \left(1 - w_i\right) C_{i,k-d_i-2}. \tag{2}$$

w_i : The **weighted factor** for the average claims (over the two recent years) for the i^{th}-insurance product.

d_i : The **length** of time delay (measured in years) for the i^{th}-insurance product. Thus, it takes about d_i years for incurred claims to be fully reported, processed and settled. Obviously, the available claim information at the beginning of the k year (or at the end of $k-1$) refers to the experience of the years $k-d_i-1$, $k-d_i-2$, $k-d_i-3$, . . ., 2, 1, 0, i.e. years prior to and inclusive of years $k-d_i-1$.

$\left\{P_{i,k}\right\}_{k\in\mathbb{N}}$: The **gross annual premium** (**GAP**) sequence paid at the end of the k^{th} year for the i^{th}-insurance product. The GAP is determined as an expense-adjusted premium $P_{i,k}^{(e)}$ less the surplus adjustment, see also Zimbidis and Haberman (2001a), where

$$P_{i,k}^{(e)} = \hat{C}_{i,k} + \left(1-e_i\right) P_{i,k}^{(e)} = \frac{\hat{C}_{i,k}}{e_i}.$$

Thus, it follows that

$$P_{i,k} = P_{i,k}^{(e)} - \sum_{j=1}^{m} \varepsilon_j \lambda_{ij} \left(S_{j,k} - S_{j,k-d_j-1}\right) = \frac{\hat{C}_{i,k}}{e_i} - \sum_{j=1}^{m} \varepsilon_j \lambda_{ij} \left(S_{j,k} - S_{j,k-d_j-1}\right),$$

for $i = 1, 2, \ldots, m.$ (3)

Equation (3) is the decision function.

Note that, GAP is calculated annually at the beginning of each year according to a base premium and a profit sharing scheme. The last one mandates an extra modification of the base premium through a refund (charge) to the policyholder a certain percentage of the benefit scheme's total accumulated surplus (deficit). Obviously, the manager of the portfolio should firstly consider the difference between the real and the target-surplus; see expression (3).

$\{S_{i,k}\}_{k\in\mathbb{N}}$: The **accumulated surplus** (**AS**) sequence at the end of the k year for the i^{th} -insurance product, where

$$S_{i,k} = (1+r_i)\sum_{j=1}^{m}\lambda_{ij}S_{j,k-1} + e_i P_{i,k} - C_{i,k} \quad \text{for } i = 1, 2, \ldots, m. \tag{4}$$

The Model and System of Equations

The system, which is a classical MIMO (multi input-multi output) starts from an initial point for the first year's premium, then claim data provide the input background for the development of the surplus level etc.

For the i^{th}-insurance product, the k^{th}-year's premium and surplus proceedings are determined according to the following equations,

$$P_{i,k} = \frac{1}{e_i}\left(w_i C_{i,k-d_i-1} + \left(1-w_i\right) C_{i,k-d_i-2}\right) - \sum_{j=1}^{m} \varepsilon_j \lambda_{ij} \left(S_{j,k} - S_{j,k-d_j-1}\right), \qquad (5)$$

$$S_{i,k}=\left(1+r_i\right)\sum_{j=1}^{m}\lambda_{ij}S_{j,k-1}+w_iC_{i,k-d_i-1}+\left(1-w_i\right)C_{i,k-d_i-2}$$
$$-e_i\sum_{j=1}^{m}\varepsilon_j\lambda_{ij}\left(S_{j,k}-S_{j,k-d_j-1}\right)-C_{i,k} \quad . \tag{6}$$

Each of the m insurance products generates its own system of equations.

These systems cannot be solved independently since the interaction factors λ_{ij} exists in them. Thus, considering expressions (5) and (6), the following systems (**S1** and **S2**) of $2m$ delay difference equations that describe the premium rating, the surplus process and the interaction within the portfolio of insurance product are derived respectively.

$$
\left.
\begin{array}{l}
P_{1,k} = \dfrac{1}{e_1}\left(w_1 C_{1,k-d_1-1} + (1-w_1) C_{1,k-d_1-2}\right) - \displaystyle\sum_{j=1}^{m} \varepsilon_j \lambda_{1j}\left(S_{j,k} - S_{j,k-d_j-1}\right) \\
\vdots \\
P_{i,k} = \dfrac{1}{e_i}\left(w_i C_{i,k-d_i-1} + (1-w_i) C_{i,k-d_i-2}\right) - \displaystyle\sum_{j=1}^{m} \varepsilon_j \lambda_{ij}\left(S_{j,k} - S_{j,k-d_j-1}\right) \\
\vdots \\
P_{m,k} = \dfrac{1}{e_m}\left(w_m C_{m,k-d_m-1} + (1-w_m) C_{m,k-d_m-2}\right) - \displaystyle\sum_{j=1}^{m} \varepsilon_j \lambda_{mj}\left(S_{j,k} - S_{j,k-d_j-1}\right)
\end{array}
\right\}
\tag{S1}
$$

$$
\left.\begin{aligned}
&\left(1+e_1\varepsilon_1\lambda_{11}\right)S_{1,k}+e_1\sum_{j=2}^{m}\varepsilon_j\lambda_{1j}S_{j,k}=\left(1+r_1\right)\sum_{j=1}^{m}\lambda_{1j}S_{j,k-1}+e_1\sum_{j=1}^{m}\varepsilon_j\lambda_{1j}S_{j,k-d_j-1}\\
&\qquad\qquad\qquad\qquad -C_{1,k}+w_1C_{1,k-d_1-1}+\left(1-w_1\right)C_{1,k-d_1-2}\\
&\vdots\\
&e_i\sum_{j=1}^{i-1}\varepsilon_j\lambda_{ij}S_{j,k}+\left(1+e_i\varepsilon_i\lambda_{ii}\right)S_{i,k}+e_i\sum_{j=i+1}^{m}\varepsilon_j\lambda_{ij}S_{j,k}=\left(1+r_i\right)\sum_{j=1}^{m}\lambda_{ij}S_{j,k-1}+e_i\sum_{j=1}^{m}\varepsilon_j\lambda_{ij}S_{j,k-d_j-1}\\
&\qquad\qquad\qquad\qquad -C_{i,k}+w_iC_{i,k-d_i-1}+\left(1-w_i\right)C_{i,k-d_i-2}\\
&\vdots\\
&e_m\sum_{j=2}^{m-1}\varepsilon_j\lambda_{mj}S_{j,k}+\left(1+e_m\varepsilon_m\lambda_{mm}\right)S_{m,k}=\left(1+r_m\right)\sum_{j=1}^{m}\lambda_{mj}S_{j,k-1}+e_m\sum_{j=1}^{m}\varepsilon_j\lambda_{mj}S_{j,k-d_j-1}\\
&\qquad\qquad\qquad\qquad -C_{m,k}+w_mC_{m,k-d_m-1}+\left(1-w_m\right)C_{m,k-d_m-2}
\end{aligned}\right\}\quad \text{(S2)}
$$

Obviously, working with systems (S1) and (S2) is not an easy task.

Thus, the matrix-vector reformulation is more appropriate. So, we may denote

$$
\underline{S} = \begin{bmatrix} S_{1,k} \\ S_{1,k-1} \\ \vdots \\ S_{1,k-d_1} \\ --- \\ S_{2,k} \\ S_{2,k-1} \\ \vdots \\ S_{2,k-d_2} \\ --- \\ \vdots \\ --- \\ S_{m,k} \\ S_{m,k-1} \\ \vdots \\ S_{m,k-d_m} \end{bmatrix} \in \mathbb{R}^{\sum_{i=1}^{m}(1+d_i)}, \quad \underline{u} = \begin{bmatrix} C_{1,k} \\ C_{1,k-1} \\ \vdots \\ C_{1,k-d_1-2} \\ --- \\ C_{2,k} \\ C_{2,k-1} \\ \vdots \\ C_{2,k-d_2-2} \\ --- \\ \vdots \\ --- \\ C_{m,k} \\ C_{m,k-1} \\ \vdots \\ C_{m,k-d_m-2} \end{bmatrix} \in \mathbb{R}^{\sum_{i=1}^{m}(3+d_i)} \quad \underline{P} = \begin{bmatrix} P_{1,k} \\ P_{2,k} \\ \vdots \\ P_{m,k} \end{bmatrix} \in \mathbb{R}^{m}.
$$

It should be mentioned that the input vector is determined by considering the actual $C_{i,k-d_i}$'s when they are available or the expectation $\hat{C}_{i,k-d_i}$'s otherwise. In other words, we obtain

$$C_{i,k-j} = \begin{cases} \text{is replaced by } \hat{C}_{i,k-j}, & \text{for } j = 0,1,2,\ldots,d_i \\ \text{remain unchanged,} & \text{for } j = d_i+1, d_i+2,\ldots \end{cases}.$$

For the use of systems S1 and S2, five super-matrices, **C**, **D** and **A, B**, **E** are respectively introduced. We start with (S2):

$$\mathbf{E} = \begin{bmatrix} \mathbf{E_{11}} & \mathbf{E_{12}} & \cdots & \mathbf{E_{1m}} \\ \mathbf{E_{21}} & \mathbf{E_{22}} & \cdots & \mathbf{E_{2m}} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{E_{m1}} & \mathbf{E_{m2}} & \cdots & \mathbf{E_{mm}} \end{bmatrix} \in \mathbb{R}^{\sum_{i=1}^{m}(1+d_i)\times\sum_{i=1}^{m}(1+d_i)},$$

where

$$\mathbf{E_{ii}} = \begin{bmatrix} 1+e_i\varepsilon_i\lambda_{ii} & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \in \mathbb{R}^{(1+d_i)\times(1+d_i)} \text{ and } \mathbf{E_{ij}} = \begin{bmatrix} e_i\varepsilon_j\lambda_{ij} & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{(1+d_i)\times(1+d_j)}.$$

The matrix
$$\mathbf{A}=\begin{bmatrix}\mathbf{A_{11}} & \mathbf{A_{12}} & \cdots & \mathbf{A_{1m}}\\ \mathbf{A_{21}} & \mathbf{A_{22}} & \cdots & \mathbf{A_{2m}}\\ \vdots & \vdots & \ddots & \vdots\\ \mathbf{A_{m1}} & \mathbf{A_{m2}} & \cdots & \mathbf{A_{mm}}\end{bmatrix}\in\mathbb{R}^{\sum_{i=1}^{m}(1+d_i)\times\sum_{i=1}^{m}(1+d_i)},$$

where its elements are
$$\mathbf{A_{ii}}=\begin{bmatrix}(1+r_i)\lambda_{ii} & 0 & 0 & \cdots & 0 & e_i\varepsilon_i\lambda_{ii}\\ 1 & 0 & 0 & \cdots & 0 & 0\\ 0 & 1 & 0 & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \cdots & \vdots & \vdots\\ 0 & 0 & 0 & \ddots & 0 & 0\\ 0 & 0 & 0 & \cdots & 1 & 0\end{bmatrix}\in\mathbb{R}^{(1+d_i)\times(1+d_i)}$$

and
$$\mathbf{A_{ij}}=\begin{bmatrix}(1+r_i)\lambda_{ij} & 0 & 0 & \cdots & 0 & e_i\varepsilon_j\lambda_{ij}\\ 0 & 0 & 0 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 0 & 0\end{bmatrix}\in\mathbb{R}^{(1+d_i)\times(1+d_j)}.$$

The matrix
$$\mathbf{B}=\begin{bmatrix}\mathbf{B_{11}} & \mathbf{B_{12}} & \cdots & \mathbf{B_{1m}}\\ \mathbf{B_{21}} & \mathbf{B_{22}} & \cdots & \mathbf{B_{2m}}\\ \vdots & \vdots & \ddots & \vdots\\ \mathbf{B_{m1}} & \mathbf{B_{m2}} & \cdots & \mathbf{B_{mm}}\end{bmatrix}\in\mathbb{R}^{\sum_{i=1}^{m}(1+d_i)\times\sum_{i=1}^{m}(3+d_i)},$$

where its elements are

$$\mathbf{B_{ii}}=\begin{bmatrix}-1 & 0 & \cdots & 0 & w_i & 1-w_i\\ 0 & 0 & \cdots & 0 & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ 0 & 0 & \cdots & 0 & 0 & 0\end{bmatrix}\in\mathbb{R}^{(1+d_i)\times(3+d_i)} \text{ and } \mathbf{B_{ij}}=\mathbb{O}\in\mathbb{R}^{(1+d_i)\times(3+d_j)} \text{ for } i\neq j.$$

Finally, for the system (S1), we define

$$\mathbf{C}=\begin{bmatrix}\mathbf{C_1} & \mathbf{C_2} & \cdots & \mathbf{C_m}\end{bmatrix}\in\mathbb{R}^{m\times\sum_{i=1}^{m}(1+d_i)},$$

where $$\mathbf{C_i} = \begin{bmatrix} -\varepsilon_{i1}\lambda_{i1} & 0 & 0 & \cdots & 0 & \varepsilon_{i1}\lambda_{i1} \\ -\varepsilon_{i2}\lambda_{i2} & 0 & 0 & \cdots & 0 & \varepsilon_{i2}\lambda_{i2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -\varepsilon_{im}\lambda_{im} & 0 & 0 & \cdots & 0 & \varepsilon_{im}\lambda_{im} \end{bmatrix} \in \mathbb{R}^{m\times(1+d_i)},$$

and $$\mathbf{D} = \begin{bmatrix} \mathbf{D_1} & \mathbf{D_2} & \cdots & \mathbf{D_m} \end{bmatrix} \in \mathbb{R}^{m\times\sum_{i=1}^{m}(3+d_i)},$$

where $$\mathbf{D_i} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & w_i & 1-w_i \\ 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix} \in \mathbb{R}^{m\times\sum_{i=1}^{m}(3+d_i)}.$$

Thus, the complex input-output systems (S1) and (S2) can be combined and expressed as a linear generalized difference system of type (1), i.e.

$$\begin{cases} \mathbf{E}\underline{S}_k = \mathbf{A}\underline{S}_{k-1} + \mathbf{B}\underline{u}_k \\ \\ \underline{P}_k = \mathbf{C}\underline{S}_k + \mathbf{D}\underline{u}_k \end{cases} \tag{7}$$

It is clear that system (7) is a generalized difference system of first order, since the (super) matrix $\mathbf{E}$ holds. Following now, the classical theory of difference equations, see for instance Bellmann and Cooke (1963), the analytical solution to equation (7) is given by:

$$\underline{S}_k = \mathbf{E}^{-1}\mathbf{A}\underline{S}_{k-1} + \mathbf{E}^{-1}\mathbf{B}\underline{u}_k$$

$$\Rightarrow \underline{S}_k = \left(\mathbf{E}^{-1}\mathbf{A}\right)^k \underline{S}_o + \sum_{j=0}^{k-1}\left(\mathbf{E}^{-1}\mathbf{B}\right)^{k-1-j} \underline{u}_j, \tag{8}$$

and

$$\underline{P}_k = \mathbf{C}\left[\left(\mathbf{E}^{-1}\mathbf{A}\right)^k \underline{S}_o + \sum_{j=0}^{k-1}\left(\mathbf{E}^{-1}\mathbf{B}\right)^{k-1-j} \underline{u}_j\right] + \mathbf{D}\underline{u}_k. \tag{9}$$

However, the matrix $\mathbf{E}$ can easily be singular (or $\det\mathbf{E} \to 0$) e.g. assuming that some of the different insurance products do not accumulate a surplus due to a forced regulatory constraint i.e.

$$
\begin{aligned}
0 = &\left(1+r_i\right)\sum_{j=1}^{m}\lambda_{ij}S_{j,k-1} \\
&+e_i\sum_{j=1}^{i-1}\varepsilon_j\lambda_{ij}\left(0-S_{j,k-d_j-1}\right)-e_i\varepsilon_i\lambda_{ii}S_{i,k-d_i-1}+e_i\sum_{j=i+1}^{m}\varepsilon_j\lambda_{ij}\left(0-S_{j,k-d_j-1}\right) \\
&-C_{i,k}+w_iC_{i,k-d_1-1}+\left(1-w_i\right)C_{i,k-d_i-2}
\end{aligned}
$$

or equivalently,

$$
0=\left(1+r_i\right)\sum_{j=1}^{m}\lambda_{ij}S_{j,k-1}+e_i\sum_{j=1}^{m}\varepsilon_j\lambda_{ij}S_{j,k-d_j-1}-C_{i,k}+w_iC_{i,k-d_i-1}+\left(1-w_i\right)C_{i,k-d_i-2}.
$$

If we consider the strategy above, we manipulate our system as follows

$$\mathbf{E_{ii}} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \in \mathbb{R}^{(1+d_i)\times(1+d_i)}, \ \mathbf{E_{ij}} = \mathbb{O} \in \mathbb{R}^{(1+d_i)\times(1+d_j)}$$

and, obviously the $\det \mathbf{E} = 0$, so the system (7) becomes a descriptor.

Now, by considering the regular matrix pencil approach, we decompose system (7) in two subsystems, whose solutions are provided below

$$\underline{S}_k = Q_{n\times p}\left[A_p^k \underline{\psi}_{p,o} + \sum_{j=0}^{k-1} A_p^{k-1-j}\left[PB\right]_{p\times n}\underline{u}_j\right] - Q_{n\times q}\left[\sum_{j=0}^{q-1} H_q^j \left[PB\right]_{q\times n}\underline{u}_{k+j}\right] \tag{10}$$

where the consistent initial condition satisfies the following important expression

$$\underline{S}_o \in C_o = \left\{ \underline{S}_o \in \mathbb{F} : \underline{S}_o = Q_{n\times p}\underline{\psi}_p - Q_{n\times q}\left[\sum_{j=0}^{q-1} H_q^j [PB]_{q\times n} \underline{u}_j \right] \right\},$$

and

$$\underline{\psi}_o = Q^{-1}\underline{S}_o = \begin{bmatrix} \underline{\psi}_{p,o} \\ \underline{\psi}_{q,o} \end{bmatrix}.$$

$$\underline{P}_k = CQ_{n\times p}\left[A_p^k \underline{\psi}_{p,o} + \sum_{j=0}^{k-1} A_p^{k-1-j} [PB]_{p\times n} \underline{u}_j \right] + D\underline{u}_k - CQ_{n\times q}\left[\sum_{j=0}^{q-1} H_q^j [PB]_{q\times n} \underline{u}_{k+j} \right] \qquad (11)$$

Non-causality occurs in many physical phenomena and certainly non-causal systems are by no means useless.

For instance, one can consider several cases of data processing that have been recorded, such as

- speech,
- meteorological data,
- demographic data,
- stock market fluctuations etc,

where their collection is not constrained causally.

Granger's causality

We should notice that the causality in economic systems is defined in Granger's (1969) sense. Characteristically, one of the most popular example for such analyses is the money-income relationship, see for instance Sims (1972), Barth and Bennett (1974), Williams, Gogdhart and Gowland (1976), Ciccolo (1978), Feige and Pearce (1979), Hsiao (1979, 1981) etc. However, it should be mentioned that whenever the money's models are not caused by income in Granger's sense, their forecast ability cannot be improved by using only the information in past income data (this is an essential definition of Granger causality), see Lütkepohl (1982).

A numerical application for the special case of two insurance products

In this numerical application, we consider a simple situation with two insurance products, $m = 2$, with $d_1 = 2$ and $d_2 = 3$ years delay, respectively. Before we go further, it is important to determine the values of the variables, which are taken into consideration in section 2.

- Firstly, we assume that the *expense* factor for the first insurance product is $e_1 = 80\%$ and for the second $e_2 = 90\%$.
- The *annual rate of investment returns* for both the first and the second insurance product is the same, i.e. $r_1 = r_2 = 4\%$.

- Moreover, we suppose the *interaction factors* $\lambda_{11} = 90\%$, $\lambda_{12} = 10\%$ and $\lambda_{22} = 95\%$, $\lambda_{21} = 5\%$. That means a greater proportion of accumulated surplus of the first (more profitable product) is transferred to the second.
- In order to obtain a faster response, we assume that the *profit sharing factor* (feedback factor) for the first product is $\varepsilon_1 = 0.3$, and for the second is almost the same, $\varepsilon_2 = 0.35$.
- the weighted factor for the average claims (over the two recent years) both for the first and the second insurance product is equal to 1/2.

In this numerical application, we are going to investigate the special case, where the *second* product accumulates zero surpluses.

As already mentioned, this strategy can be realistic, since the product managers of the insurance company may hope a further success and development in the other, most profitable, product.

Now, we are going to examine the behaviour of the system with respect to the *spike signal*. This kind of input corresponds to the appearance of an unexpected claim into the system. Moreover, we suppose that a spike signal appears as the input of the first subsystem while the second subsystem has a zero input, i.e.

$$C_{1,k} = \begin{cases} 1, & \text{for } k = 0 \\ 0, & \text{for } k = 1, 2, \ldots \end{cases}, \text{ and } C_{2,k} = 0 \text{ for } k = 0, 1, 2, \ldots .$$

a) The spike signal appears only to the 1st product.

The input vectors are the following

$$\underline{u}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \underline{u}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \underline{u}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \underline{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \underline{u}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and $\underline{u}_k = 0$ for $k = 5, 6, \ldots.$

So, according to (10) and (11) expressions we obtain (12) and (13), respectively. Moreover, we assume that $\underline{\psi}_o = Q^{-1}\underline{S}_o = \underline{0} \Rightarrow \underline{S}_o = \underline{0}$.

Consequently,

$$\begin{aligned}\underline{S}_k = Q_{7\times 6}\Big[& A_6^{k-1}[PB]_{6\times 11}\,\underline{u}_o + A_6^{k-2}[PB]_{6\times 11}\,\underline{u}_1 + A_6^{k-3}[PB]_{6\times 11}\,\underline{u}_2 \\ & + A_6^{k-4}[PB]_{6\times 11}\,\underline{u}_3 + A_6^{k-5}[PB]_{6\times 11}\,\underline{u}_4\Big] - Q_{7\times 1}[PB]_{1\times 11}\,\underline{u}_k\end{aligned} \tag{12}$$

and

$$\begin{aligned}\underline{P}_k = CQ_{7\times 6}\Big[& A_6^{k-1}[PB]_{6\times 11}\,\underline{u}_o + A_6^{k-2}[PB]_{6\times 11}\,\underline{u}_1 + A_6^{k-3}[PB]_{6\times 11}\,\underline{u}_2 \\ & + A_6^{k-4}[PB]_{6\times 11}\,\underline{u}_3 + A_6^{k-5}[PB]_{6\times 11}\,\underline{u}_4\Big] + \Big[D - CQ_{7\times 1}[PB]_{1\times 11}\Big]\underline{u}_k\end{aligned} \tag{13}$$

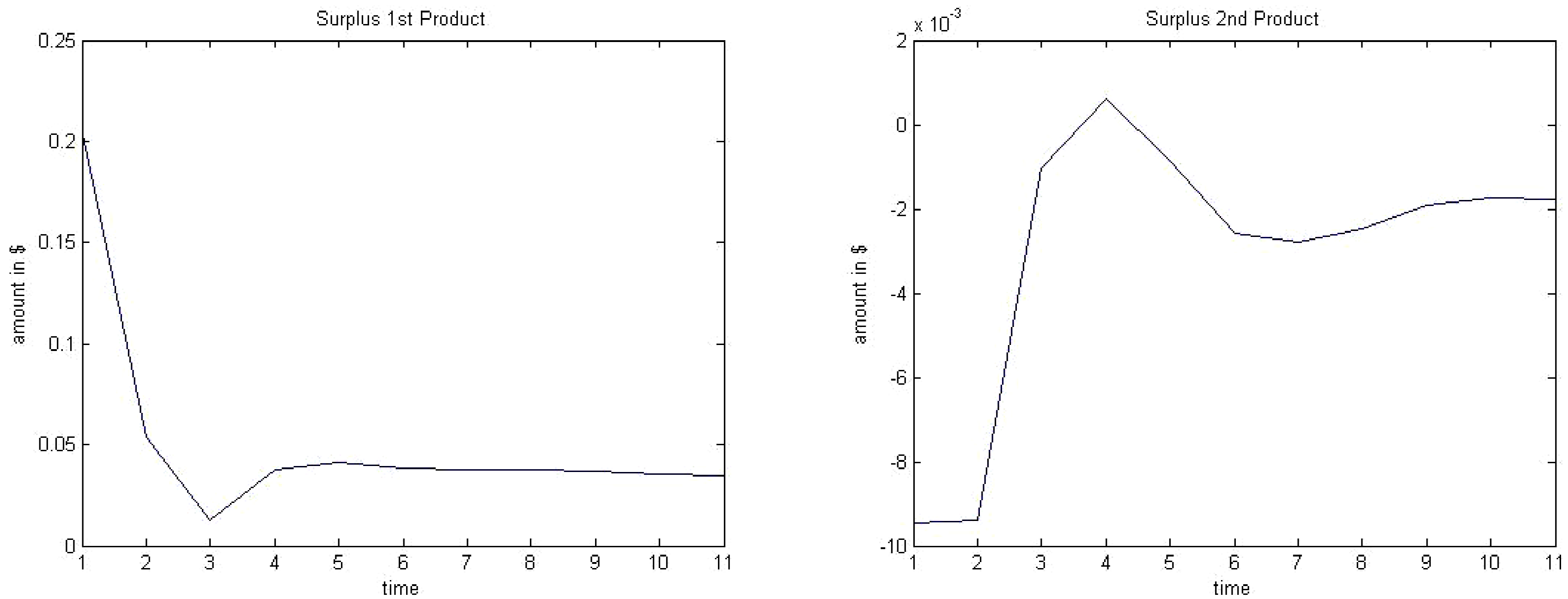


figure 1 (a), (b): The surplus for the 1st and the 2nd insurance product, respectively.

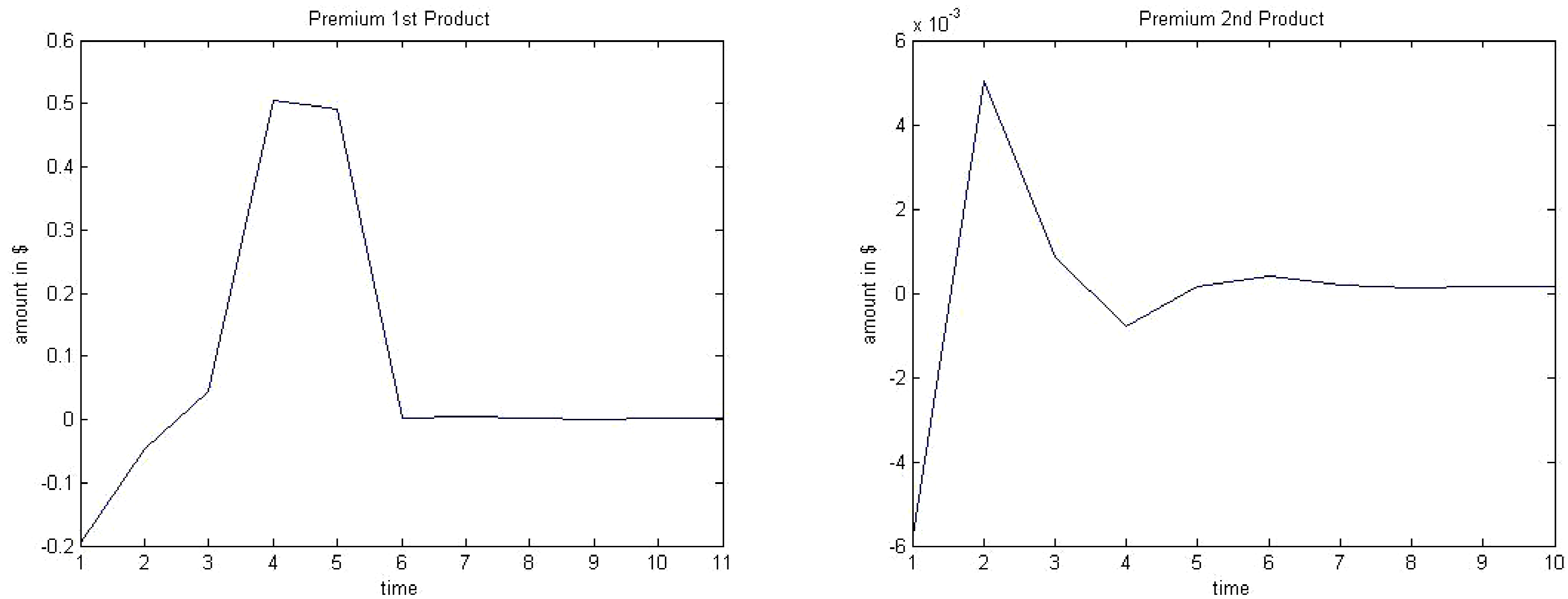


figure 2 (a), (b): The premium for the 1st and the 2nd insurance product, respectively.

b) The spike signal appears only to the 2nd product.

$$\underline{u}_o = \begin{bmatrix} 0\\0\\0\\0\\0\\1\\0\\0\\0\\0\\0 \end{bmatrix}, \quad \underline{u}_1 = \begin{bmatrix} 0\\0\\0\\0\\0\\0\\1\\0\\0\\0\\0 \end{bmatrix}, \quad \underline{u}_2 = \begin{bmatrix} 0\\0\\0\\0\\0\\0\\0\\1\\0\\0\\0 \end{bmatrix}, \quad \underline{u}_3 = \begin{bmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\1\\0\\0 \end{bmatrix}, \quad \underline{u}_4 = \begin{bmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\0\\1\\0 \end{bmatrix} \quad \underline{u}_5 = \begin{bmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\1 \end{bmatrix}$$

and $\underline{u}_k = 0$ for $k = 6, 7, \ldots$.

Again, according to (10) and (11) we obtain (14) and (15), respectively. Note that we also assume that $\underline{\psi}_o = Q^{-1}\underline{S}_o = \underline{0} \Rightarrow \underline{S}_o = \underline{0}$,

So,

$$\begin{aligned}\underline{S}_k = Q_{7\times 6}\Big[& A_6^{k-1}[PB]_{6\times 11}\underline{u}_o + A_6^{k-2}[PB]_{6\times 11}\underline{u}_1 + A_6^{k-3}[PB]_{6\times 11}\underline{u}_2 \\ & + A_6^{k-4}[PB]_{6\times 11}\underline{u}_3 + A_6^{k-5}[PB]_{6\times 11}\underline{u}_4 + A_6^{k-6}[PB]_{6\times 11}\underline{u}_5\Big] - Q_{7\times 1}[PB]_{1\times 11}\underline{u}_k\end{aligned}, \quad (14)$$

$$\begin{aligned}\underline{P}_k = CQ_{7\times 6}\Big[& A_6^{k-1}[PB]_{6\times 11}\underline{u}_o + A_6^{k-2}[PB]_{6\times 11}\underline{u}_1 + A_6^{k-3}[PB]_{6\times 11}\underline{u}_2 \\ & + A_6^{k-4}[PB]_{6\times 11}\underline{u}_3 + A_6^{k-5}[PB]_{6\times 11}\underline{u}_4 + A_6^{k-6}[PB]_{6\times 11}\underline{u}_5\Big] + \Big[D - CQ_{7\times 1}[PB]_{1\times 11}\Big]\underline{u}_k\end{aligned}$$

(15)

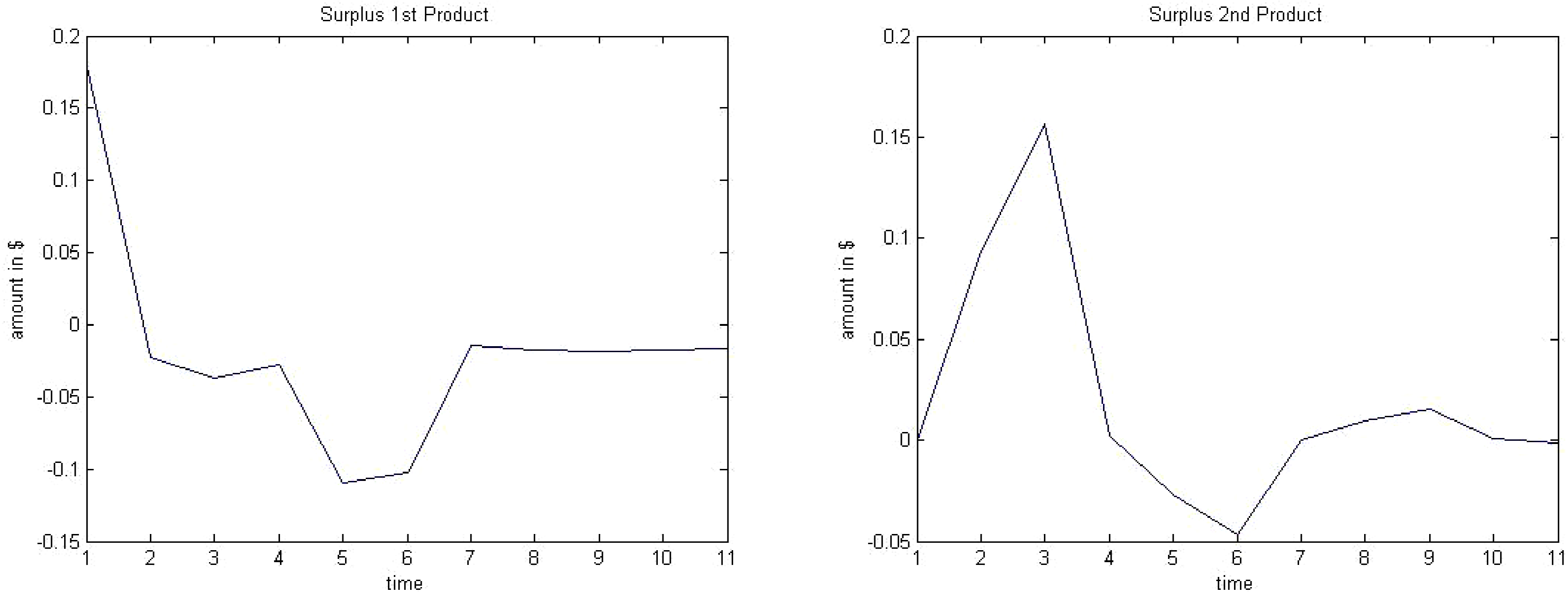
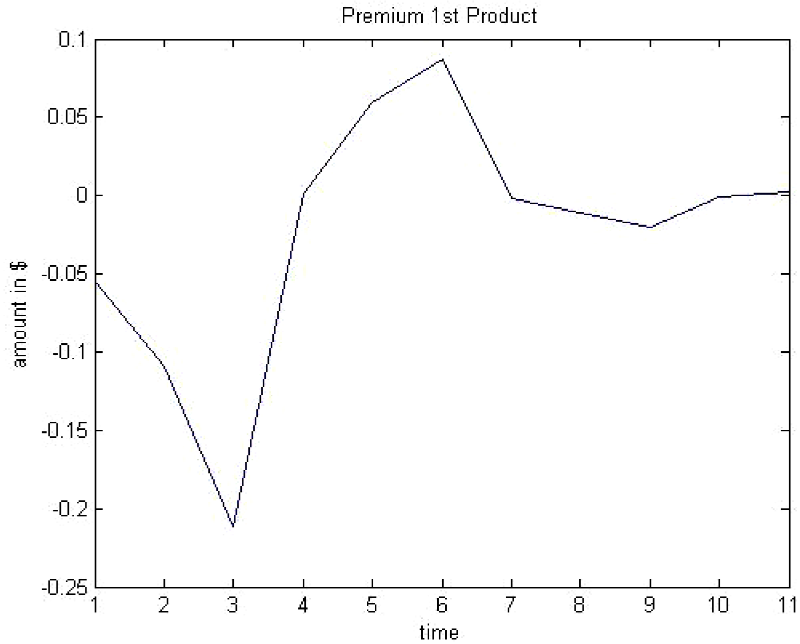


figure 3 (a), (b): The surplus for the 1st and the 2nd insurance product, respectively.



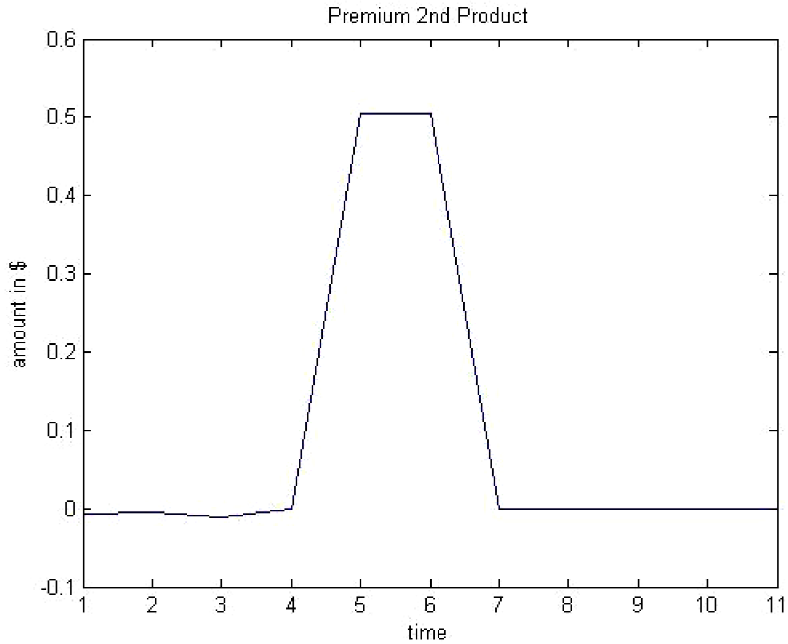


figure 4 (a), (b): The surplus for the 1st and the 2nd insurance product, respectively.

Conclusions

The use of matrix pencil theory appears to be unavoidable in order to manipulate the singularities of a multiple input-output system.

In the numerical application, the diagrams of the surplus and premium response with respect to the spike input signal are quite interesting.

On the other hand, we could follow the opposite direction in our analysis. Define the pattern for the surplus or premium response and go back to the optimal choices for the basic controlled parameters, as the loading or interaction factors.

The research in this area is being continued.

Thank you very much for your attention

(SELECTIVE) REFERENCES

- Pantelous, A.A., Zimbidis, A.A., and Kalogeropoulos, G.I. (2010), Linear Generalized Stochastic Systems for Insurance Portfolio, *Stochastic Analysis and Applications*, Vol. 28, No. 6, pp. 946-971.
- Grispos, E., Kalogeropoulos, G.I., (1998), The causality for generalized linear discrete time dynamical systems. *Journal of Institute of Mathematics and Computer Science (Mathematical Series)* 11 (3), 145-154.
- Kendrick, D.A., (2005), Stochastic control for economic models: past, present and the paths ahead. Journal of Economic Dynamics and Control 29, 3-30.
- Livesey, D.A., (1973), The singularity problem in the dynamic input-output model, International Journal of System Science 4, 437-440.
- Zimbidis, A.A., Haberman, S., (2001), Controlling the solvency interaction among group of insurance companies. Journal of Actuarial Practice 9, 151-187.