Regularity of the American Option Value Function and Hedging Error of the American Put Option in Jump-Diffusion Process

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Introduction

This work is devoted to the regularity properties of the American option value function as well as the discrete time hedging error of the American put option, when there are brusque variations in prices of assets. We assume that there are finite number of jumps in each finite time interval and the asset price jumps in the proportions which are independent and identically distributed such that all the moments of the proportions are finite.

Formulation of the problem

Consider a probability space $(\Omega, \mathcal{F}, P)$ on which we define a standard Wiener process $W = (W_t)_{0 \leq t \leq T}$, a Poisson process $N = (N_t)_{0 \leq t \leq T}$ with intensity λ and a sequence $(U_j)_{j \geq 1}$ of independent, identically distributed random variables taking values in $(-1, \infty)$. We will assume that the time horizon T is finite and the $\sigma-$algebras generated respectively by $(W_t)_{0 \leq t \leq T}$, $(N_t)_{0 \leq t \leq T}$, $(U_j)_{j \geq 1}$ are independent.

Formulation of the problem

We denote by $(\mathcal{F}_t)_{0\leq t\leq T}$ the $P-$completion of the natural filtration of (W_t), (N_t) and $(U_j)I_{j\leq N_t}$, $j \geq 1, 0 \leq t \leq T$. On a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)_{0\leq t\leq T}$ we consider a financial market with two assets $m_t, 0 \leq t \leq T$, the price of a unit of a money market account at time t, and $S_t, 0 \leq t \leq T$, the value at time t of the share of a stock whose price jumps in the proportions $U_1, U_2, ...,$ at some times $\tau_1, \tau_2, ...$ We assume that the τ_j,s corresponds to the jump times of a Poisson process.

Formulation of the problem

The evolution of the assets m_t and S_t obeys the following ordinary and stochastic differential equations differential equation

$$dm_t = r(t)m_t dt, \;\; m_0 = 1, \;\;\; 0 \leq t \leq T, \tag{1}$$

$$dS_t = S_{t-}\left(b(t)dt + \sigma(t)dW_t + d\left(\sum_{j=1}^{N_t} U_j\right)\right). \tag{2}$$

We assume that $(b(t), \mathcal{F}_t)_{0\leq t\leq T}$ is certain progressively measurable process and the deterministic time-varying interest rate $r(t)$ and the volatility $\sigma(t)$ are continuously differentiable functions of time which satisfy the requirements

$$0 \leq r(t) \leq \bar{r}, \;\;\; 0 < \underline{\sigma} \leq \sigma(t) \leq \bar{\sigma}, \tag{3}$$

$$|r(t) - r(s)| + |\sigma(t) - \sigma(s)| \leq K|t - s|, \tag{4}$$

where $s, t \in [0, T]$ and $\bar{r}$, $\underline{\sigma}$, $\bar{\sigma}$ and K are some positive constants.

Formulation of the problem

From the above stochastic differential equation we find that the dynamics of S_t can be described as:

$$S_t = S_0 \left(\prod_{j=1}^{N_t} (1 + U_j) \right) \exp \left[\int_0^t \left(b(u) - \frac{\sigma^2(u)}{2} \right) du + \int_0^t \sigma(u) dW_u \right]. \tag{5}$$

The discounted stock price $\tilde{S}_t = e^{-\int_0^t r(u)du} S_t$ is a martingale (see, for example, Lamberton and Lapeyre [6]) if and only if

$$\int_0^t b(u)du = \int_0^t r(u)du - \lambda t E(U_1). \tag{6}$$

In this work we investigate the regularity properties of the American option value function with a nonnegative non-increasing convex payoff function $g(x), x \geq 0$. We assume that $g(0) = g(0+)$. The typical example is American put option with payoffs $g(x) = (L - x)^+$ where L is the exercise price.

Formulation of the problem

Recall the fundamental fact that the American option value function $v(t,x), x \geq 0, 0 \leq t \leq T$, can be considered as the value function of the corresponding optimal stopping problem (see, for example, Karatzas and Shereve, Section 2.5 [7]), in particular

$$v(t,x) = \sup_{\tau \in \mathcal{T}_{t,T}} E\left[\exp\left(-\int_t^\tau r(v)dv\right) g(S_\tau(t,x))\right], x \geq 0, 0 \leq t \leq T, \quad (7)$$

where $\mathcal{T}_{t,T}$ denotes the set of all stopping times τ such that $t \leq \tau \leq T$, and the stochastic process $S_u(t,x), t \leq u \leq T$ satisfies the same stochastic differential equation

$$dS_u(t,x) = S_{u-}(t,x)\left(b(u)du + \sigma(u)dW_u + d\left(\sum_{j=1+N_t}^{N_u} U_j\right)\right), \ t \leq u \leq \quad (8)$$

with the initial condition $S_t(t,x) = x, x \geq 0$.

Formulation of the problem

The unique solution $(S_u(t,x), \mathcal{F}_u)_{t\leq u\leq T}$ of this equation is given by the exponential

$$S_u(t,x) = x\left(\prod_{j=1+N_t}^{N_u}(1+U_j)\right)\exp\left[\int_t^u\left(b(u)-\frac{\sigma^2(u)}{2}\right)du + \int_t^u \sigma(u)dW_u\right].$$

Condition (6) leads to

$$S_u(t,x) = x\left(\prod_{j=1+N_t}^{N_u}(1+U_j)\right)\exp\left[\int_t^u\left(r(u)-\lambda E(U_1)-\frac{\sigma^2(u)}{2}\right) + \int_t^u \sigma(u)dW_u\right]$$

Introduce the new stochastic

$$S_u(t,x) = \exp\left[\ln x + \int_t^u \left(r(u) - \lambda E(U_1) - \frac{\sigma^2(u)}{2}\right) du + \int_t^u \sigma(u) dW_u\right.$$

process $(X_u(t,x), \mathcal{F}_u)_{t \leq u \leq T}$

$$X_u(t,y) = y + \int_t^u \left(r(u) - \lambda E(U_1) - \frac{\sigma^2(v)}{2}\right) dv + \int_t^u \sigma(v) dW_v + \sum_{j=N_t+1}^{N_u} \ln(1+U_j),$$

$t \leq u \leq T, -\infty < y < \infty, U_j \in (-1,\infty), j = 1,2,....$
It is easy to show that

$$S_u(t,x) = \exp\left[X_u(t,\ln x)\right], \ t \leq u \leq T \ x > 0, , \tag{9}$$

and for arbitrary stopping time τ, $t \leq \tau \leq T$, we have

$$g(S_\tau(t,x)) = \psi(X_\tau(t,\ln x)),$$

Variational Inequalities

where $\psi(y) = g(e^y), -\infty < y < \infty$, is the new payoff function. Let us define the corresponding optimal stopping problem

$$u(t,y) = \sup_{\tau \in \mathcal{T}_{t,T}} E\left[\exp\left(-\int_t^\tau r(v)dv\right)\psi(X_\tau(t,y))\right], \qquad (10)$$

with $0 \le t \le T$ and $-\infty < y < \infty$, then we find

$$v(t,x) = u(t, \ln x),\ x > 0,\ 0 \le t \le T. \qquad (11)$$

We will use the following lemma [2]:

Theorem

Let $g(x), x \ge 0$ be a nonnegative, non-increasing convex function. Then the new payoff function defined by $\psi(y) = g(e^y), -\infty < y < \infty$ is Lipschitz continuous, that is,

$$|\psi(y_2) - \psi(y_1)| \le g(0)|y_2 - y_1|,\ y_1, y_2 \in \mathbb{R}. \qquad (12)$$

By the scaling property of the Brownian motion we can express the value function $u(t, y)$ of the optimal stopping problem (10) as follows (see Jaillet, Lamberton and Lapeyre [5])

$$\begin{aligned} u(t,y) &= \sup_{\tau \in \mathcal{T}_{0,1}} E\Bigg[\exp\left(- \int_t^{t+\tau(T-t)} r(v)dv \right) \\ &\quad \psi\Bigg(y + \int_t^{t+\tau(T-t)} \left(r(v) - \lambda E(U_1) - \frac{\sigma^2(v)}{2} \right) dv \\ &\quad + \int_0^{\tau} \sqrt{T-t}\, \sigma(t + v(T-t)) dW_v + \sum_{j=1}^{N_{t+\tau(T-t)}} \ln(1+U_j) \Bigg) \Bigg] \end{aligned} \quad (13)$$

where $\mathcal{T}_{0,1}$ denotes the set of all stopping times τ with respect to the filtration $(\mathcal{F}_u)_{0 \le u \le 1}$ taking values in $[0, 1]$.
Let us come to the following result:

Theorem

The value function $u(t, y), 0 \leq t \leq T, -\infty < y < \infty$ *of the optimal stopping problem (10) is Lipschitz continuous in the argument* y *and locally Lipschitz continuous in* t *i.e.*

$$|u(t, y) - u(t, z)| \leq g(0)\ |y - z|,\ \ y, z \in \mathbb{R}, 0 \leq t \leq T, \qquad (14)$$

$$|u(t, y) - u(s, y)| \leq \frac{A}{\sqrt{T - t}} |t - s|, \qquad (15)$$

where A *is some nonnegative constant depending on parameters* $\bar{r}$, $\bar{\sigma}$, $g(0)$, λ, $E(U_1)$, K *and* T.

Let $\tilde{S}_t = e^{-\int_0^t r(u)du} S_t$ is the discounted stock price, then the discounted price function

$$\tilde{v}(t,x) = e^{-\int_0^t r(u)du} v(t, xe^{\int_0^t r(u)du}), \ 0 \le t \le T, x > 0 \qquad (16)$$

of the option at time t is C^2 on $[0,T) \times \mathbb{R}^+$ (see, Lamberton and Lapeyre [6]) and between the jump times, satisfies

$$\begin{aligned} \tilde{v}(t,\tilde{S}_t) &= v(0,S_0) + \int_0^t \frac{\partial \tilde{v}}{\partial u}(u,\tilde{S}_u)du \\ &+ \int_0^t \frac{\partial \tilde{v}}{\partial x}(u,\tilde{S}_u)\tilde{S}_u\Big(-\lambda E(U_1)du + \sigma(u)dW_u\Big) \\ &+ \frac{1}{2}\int_0^t \frac{\partial^2 \tilde{v}}{\partial x^2}(u,\tilde{S}_u)\sigma^2(u)\tilde{S}_u^2 du \\ &+ \sum_{j=1}^{N_t}\Big(\tilde{v}(\tau_j,\tilde{S}_{\tau_j}) - \tilde{v}(\tau_j,\tilde{S}_{\tau_j-})\Big). \qquad (17) \end{aligned}$$

The function $\tilde{v}(t,x)$ is Lipschitz of order 1 with respect to x and with $S_{\tau_{j-}} = S_{\tau_j}(1+U_j), j = 1, 2, \ldots$ the process

$$M_t = \sum_{j=1}^{N_t} \left(\tilde{v}(\tau_j, \tilde{S}_{\tau_j}) - \tilde{v}(\tau_j, \tilde{S}_{\tau_j -}) \right) - \lambda \int_0^t \int \left(\tilde{v}(u, \tilde{S}_u(1+z)) - \tilde{v}(u, \tilde{S}_u) \right) d\nu(z) du \quad (18)$$

is a square integrable martingale.

Combining (17) and (18) we find that

$$\begin{aligned}\tilde{v}(t,\tilde{S}_t) \quad - \quad & \int_0^t \left[\frac{\partial \tilde{v}}{\partial u}(u,\tilde{S}_u) - \lambda E U_1 \tilde{S}_u \frac{\partial \tilde{v}}{\partial x}(u,\tilde{S}_u) \right. \\ & + \frac{1}{2}\sigma^2(u)\tilde{S}_u^2 \frac{\partial^2 \tilde{v}}{\partial x^2}(u,\tilde{S}_u) \\ & \left. - \lambda \int \Big(\tilde{v}(u,\tilde{S}_u(1+z)) - \tilde{v}(u,\tilde{S}_u) \Big)\, d\nu(z) \right] du \end{aligned} \tag{19}$$

is a martingale and therefore (see, Israel and Rincon [2])

$$\frac{\partial \tilde{v}}{\partial u}(u, \tilde{S}_u) \quad - \quad \lambda E U_1 \tilde{S}_u \frac{\partial \tilde{v}}{\partial x}(u, \tilde{S}_u) + \frac{1}{2}\sigma^2(u)\tilde{S}_u^2 \frac{\partial^2 \tilde{v}}{\partial x^2}(u, \tilde{S}_u)$$
$$-\lambda \int \Big(\tilde{v}(u, \tilde{S}_u(1+z)) - \tilde{v}(u, \tilde{S}_u) \Big) \, d\nu(z) \leq 0 \tag{20}$$

a.e. in $[0, T) \times \mathbb{R}^+$.

From El Karoui, Jeanblanc-Picque and Shreve [1] we know that, if the interest rate process is deterministic and the stock volatility depends only on time and current price of the stock, then the price of the contingent claim is a convex function of the stock.

Therefore, we can write

$$\frac{\partial^2 \tilde{v}(t,x)}{\partial x^2} \geq 0 \tag{21}$$

a.e. in $[0, T) \times \mathbb{R}^+$.

Theorem

The mapping $\varsigma(t,x) = x\ v(t,x)$ is Lipschitz continuous in x and locally Lipschitz continuous in the argument of t, i.e.

$$|\varsigma(t,x) - \varsigma(t,y)| \leq 2\ g(0)|x-y|,\ 0 \leq t \leq T,\ 0 < x \leq y < \infty, \tag{22}$$

$$|\varsigma(t,x) - \varsigma(s,x)| \leq \frac{C\ x}{\sqrt{T-t}}\ |t-s|,\ 0 \leq s \leq t < T,\ x > 0, \tag{23}$$

where the constant C is the function of $\bar{r}$, $\bar{\sigma}$, $g(0)$, λ, $E(U_1)$, K and T.

Theorem

The second order weak partial derivative $\frac{\partial^2 v(t,x)}{\partial x^2}$ *of the value function (7) satisfies with respect to x*

$$x^2 \left| \frac{\partial^2 v(t,x)}{\partial x^2} \right| \leq \frac{D}{\sqrt{T-t}}, \ x > 0, \ 0 \leq t < T,$$

where D is a nonnegative constant depends on the parameters $\overline{r}$, $\overline{\sigma}$, $\underline{\sigma}$, $g(0)$, λ, $E|U_1|$, $E\left(\frac{|U_1|}{1+U_1}\right)$, K, T.

Theorem

For the function $\gamma(t,y) = y\,\frac{\partial v(t,y)}{\partial y}, 0 \le t_1 \le t_2 < T, y > 0$, *of the value function (7) we have the following bound*

$$\begin{aligned}
|\gamma(t_2,y) - \gamma(t_1,y)| \le & \frac{1}{h}\Big[\int_y^{y+h} |\gamma(t_2,y) - \gamma(t_2,z)|dz \\
+ & \int_y^{y+h} |\gamma(t_1,y) - \gamma(t_1,z)|dz \\
+ & (y+h)|v(t_2,y+h) - v(t_1,y+h)| \\
+ & y|v(t_2,y) - v(t_1 y)| \\
+ & \int_y^{y+h} |v(t_2,z) - v(t_1,z)|dz\Big],
\end{aligned}$$

where $h > 0$.

In the next result we shall apply Corollary 3.7, from Jailet, Lamberton and Lapeyre [5], which states that the partial derivative $\frac{\partial v(t,x)}{\partial x}$ of the value function (10) is continuous with respect to the pair of arguments $(t, y), 0 \leq t < T, x > 0$.

Theorem

The mapping $\gamma(t,x) = x\frac{\partial v(t,x)}{\partial x}$ satisfies with respect to time argument local Heldor estimate with exponent $\frac{1}{2}$, i.e.,

$$|\gamma(t,x) - \gamma(s,x)| \leq \frac{G + x\ H}{\sqrt{T - t}}|t - s|^{\frac{1}{2}}, 0 \leq s \leq t < T, x > 0, \quad (24)$$

where G and H are positive constants depend on the parameters $\bar{r}$, $\bar{\sigma}$, $\underline{\sigma}$, $g(0)$, λ, $E(U_1)$, $E\left(\frac{|U_1|}{1+U_1}\right)$, K and T.

Discrete Time Hedging Strategy

In practice, the discrete time hedging process is as the following

$$\triangle_\delta(t) = \triangle(t_{k-1}), t_{k-1} \leq t < t_k, k = 1, 2, ...n, \qquad (25)$$

where $\delta = \frac{T}{n}$, $t_k = k \cdot \delta$, the subscript δ indicates the error of approximation.

Hedging Error

Theorem

Assume $\triangle(t) = \frac{\partial v(t,x-)}{\partial x}$ *represents the continuous time hedging strategy of the American put option in jump-diffusion process and* $\triangle_\delta(t)$ *the corresponding discrete time hedge defined as (25). Then we have the estimate*

$$\widetilde{E} \int_0^T \left(\triangle(u) - \triangle_\delta(u)\right)^2 S_u^2 du \leq a \; \ln \frac{T}{\delta} \cdot \delta, \tag{26}$$

where a *is a positive function of* S_0, $g(0)$, $\bar{r}$, $\bar{\sigma}$, $\underline{\sigma}$, λ, β, $\mathcal{K}$, T *and all the moments of the random variable* U_1.

Hedging Error

Theorem

If $\Pi(t), 0 \leq t \leq T$, *denotes the continuous time portfolio and* $\Pi_\delta(t), 0 \leq t \leq T$, *the portfolio value process of the corresponding discrete time hedging process defined as (25), then we have the following hedging error*

$$E_r \sup_{0 \leq t \leq T} |\Pi(t) - \Pi_\delta(t)| \leq 8e^{\left(\left(\frac{\lambda}{\tilde{\lambda}}\right)^2 \mathcal{L} - 2\lambda + 3\tilde{\lambda} + 2\bar{r}\right)\frac{T}{2}} \left[d \cdot \delta \ln \frac{T}{\delta} + e \cdot h(\widetilde{E} U_1) \right.$$

where $h(x)$ *is function such that* $h(x) \to 0$ *as* $x \to 0$. *The non-negative constant* d *depends on the parameters* $\bar{r}$, $\bar{\sigma}$, $\underline{\sigma}$, λ, β, $g(0)$, T, $E(U_1^n), n \in \mathbb{N}$, $\mathcal{K}$, S_0 *and the constant* e *on* $\bar{r}$, $\bar{\sigma}$, $\underline{\sigma}$, λ, β, T, $g(0)$, $\widetilde{E}\left(\sum_{j=1}^{N_T} \frac{1}{T-\tau_j}\right)$, $\mathcal{L} = \int_{-\infty}^{\infty} \frac{f^2(u)}{\tilde{f}(u)} du$ *and* $E(U_1^n), n \in \mathbb{N}$.

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