On non-tradeable endowments

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June 2012

Main questions:

1. How do financial markets respond to the presence of risk which is not tradeable?
2. What happens if such risks become tradeable?

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1. How do financial markets respond to the presence of risk which is not tradeable?
2. What happens if such risks become tradeable?

- Pricing of payoffs that are not traded in a financial market
- Effect of non-tradeable endowments on asset prices
- Innovation: introduction of new secturities
- Securitisation

1. LeRoy and Werner: Principles of Financial Economics
2. Incomplete markets literature
3. CAPM literature

- Two period model
- Non-storable consumption good, serves as a numeraire
- Uncertainty: $\Omega = \{s_1, \dots, s_N\}$
- Investment possibilities
  - Risk-free asset, interest factor $R_f = 1 + r_f > 0$
  - K risky assets (stock of firms)
- $i = 1, \dots, I$ investors: μ-σ preferences represented by utility function

$$U^i : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}, \quad (\mu, \sigma) \mapsto U(\mu, \sigma)$$

  - strictly increasing in μ, strictly decreasing in σ
  - strictly quasi-concave

Assumption 1 (Financial instruments)

- *Market subspace*

$$\mathcal{M} := \mathrm{span}\{R_f, q_1, \cdots, q_K\}.$$

- *Orthogonal decomposition:* $\mathcal{C} = \mathcal{M} \oplus \mathcal{M}^\perp$.
- *Expected payoffs:* $\overline{q} = (\overline{q}_1, \ldots, \overline{q}_K) \in \mathbb{R}^K$
- *Covariance matrix* $V = (V_{kl})$ *is positive definite*

Assumption 2 (Agents and their endowments)

- *Total Initial endowment of agent i:*

$$e^i = \langle q, x_0^i \rangle \quad + \quad \underbrace{e_N^i}_{\textit{non-tradeable}} \quad \in \mathcal{C}$$

- *Orthogonal decomposition*

$$e_N^i = \underbrace{\langle q, y_0^i \rangle + R_f b_0^i}_{\textit{hedgeable}} \quad + \quad \underbrace{e_N^{i,\perp}}_{\textit{non-hedgeable}} \quad \in \mathcal{M} \oplus \mathcal{M}^\perp$$

- *Can only borrow against* $\langle p, x_0^i \rangle$

Assumption 3 (Aggregate endowment and market portfolio)

- *Market portfolio*

$$\sum_{i=1}^{I} x_0^i = x_m$$

- *Portfolio replicating aggregate non-tradeable endowment*

$$y_m = \sum_{i=1}^{I} y_0^i$$

- *Extended market portfolio:* $z_m = x_m + y_m$
- *Aggregate hedgeable endowment:* $e_m = \langle q, z_m \rangle \in \mathcal{M}$

Decision problem:

$$\max_{x \in \mathbb{R}^K} U\big(\mu_c(\pi, x), \sigma_c(x)\big). \tag{1}$$

Expected date-1 consumption

$$\mu_c(\pi, x) := \mathbb{E}[c] = \overline{e} + \langle \pi, x - x_0 \rangle.$$

Standard deviation of date-1 consumption

$$\sigma_c(x) \;:=\; \sqrt{\mathbb{V}\mathrm{ar}[c]} = \sqrt{\langle x + y_0, V(x + y_0)\rangle + \epsilon^2},$$

$\epsilon := \sqrt{\mathbb{V}\mathrm{ar}[e_N^{\perp}]}$ ... residual risk

Variance-minimising problem:

$$\min_{x \in \mathbb{R}^K} \frac{1}{2}\sigma_c(x)^2 \quad \text{s.t.} \quad \mu_c(\pi, x) = \mu$$

Solution

$$x_{\text{eff}}(\mu, \pi) := \frac{\mu - \mu_0}{\langle \pi, V^{-1}\pi \rangle} V^{-1}\pi - y_0 \,, \tag{2}$$

where $\mu_0 = \overline{e} - \langle \pi, x_0 + y_0 \rangle$ consists of

1. classical variance-minimizing portfolio $\frac{\mu - \mu_0}{\langle \pi, V^{-1}\pi \rangle} V^{-1}\pi$
2. $-y_0 \in \mathbb{R}^K$ offseting the risk of the orthogonal projection of e_N on $\mathcal{M}$.

- Standard deviation

$$\sigma_c(x_{\text{eff}}(\mu,\pi)) = \sqrt{\left(\frac{\mu-\mu_0}{\rho}\right)^2 + \epsilon^2}\,, \tag{3}$$

$\rho := \sqrt{\langle \pi, V^{-1}\pi \rangle} \ldots$ *price of risk*.
$\epsilon \ldots$ *residual risk* which cannot be hegded

- Efficient frontier

$$\mu = \mu_0 + \rho\sqrt{\sigma^2 - \epsilon^2}, \quad \sigma \geq \epsilon \tag{4}$$

If all risk is hedgeable, $\epsilon = 0$, the classical efficient frontier $\mu = \mu_0 + \rho\sigma$ obtains.

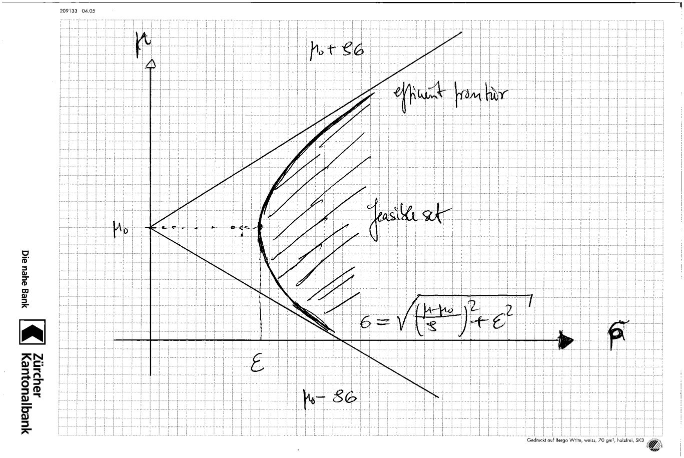

Fig. 1: Feasible portfolios

Theorem 1 (Two-fund Separation)

Under the above hyotheses, let $e \in \mathcal{C}$ *with*

$$e = \langle q, x_0 + y_0 \rangle + R_f b_0 + e_N^{\perp},$$

Then for any $0 < \sqrt{\langle \pi, V^{-1}\pi \rangle} < \rho_U(e)$, *the optimization problem (1) has a unique maximizer*

$$x_\star = \frac{\sigma_\star}{\sqrt{\langle \pi, V^{-1}\pi \rangle}} V^{-1}\pi - y_0, \tag{5}$$

where optimal risk

$$\sigma_\star = \underset{\sigma \geq 0}{\operatorname{argmax}}\, U\left(\mu_0 + \sigma\sqrt{\langle \pi, V^{-1}\pi \rangle},\ \sqrt{\sigma^2 + \epsilon^2}\right) \tag{6}$$

is finite with $\mu_0 = \overline{e} - \langle \pi, x_0 + y_0 \rangle$ *and* $\epsilon = \sqrt{\mathbb{V}\mathrm{ar}[e_N^{\perp}]}$ *the residual risk*

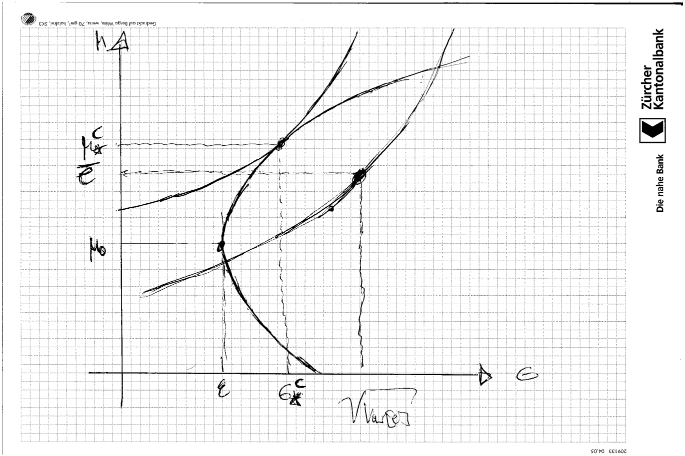

Fig. 2: Two-fund separation

Interpretation

Given expected excess return $\pi = \overline{q} - R_f p$, the investor chooses

1. optimal amount of hedgeable risk $\sigma_\star \rightarrow$ 'demand-for-risk'
2. an efficient portfolio (=classical variance minimising portfolio corrected by a portolio that hedges non-tradeable endowment)

Remarks

1. Two fund separation in terms of demand functions as in Lintner (1965)
2. Could be viewed as a three fund separation
3. Transforms a multivariate problem into a two-dimensional one
4. Demand-for-risk function $\sigma_\star = \varphi(e, \rho)$ crucial

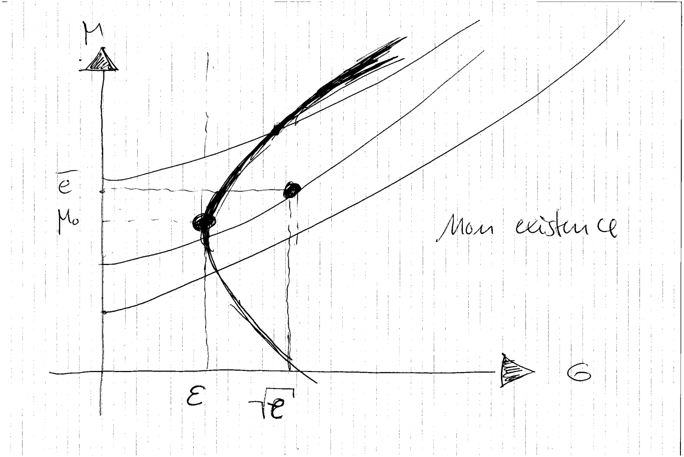

Fig. 3: Non-existence

Theorem 3 (Existence and uniqueness of CAPM equilibrium)

Let $(\overline{q}, V)$, $e^1, \ldots, e^I \in \mathcal{C}$, and $z_m \in \mathbb{R}^K$ with $0 < \sigma_m < \sigma_{\max}$ be given. Then there exists a CAPM equilibrium with market clearing prices

$$p_\star \;=\; \frac{1}{R_f}\left(\overline{q} - \frac{\rho_\star}{\sigma_m} V z_m\right), \tag{12}$$

where $\rho_\star > 0$ solves

$$\phi(\rho) := \sum_{i=1}^{I} \varphi^i(e^i, \rho) = \sigma_m.$$

The equilibrium portfolio allocation is

$$x_\star^i = \frac{\varphi^i(e^i, \rho_\star)}{\sigma_m} z_m - y_0^i, \quad i = 1, \ldots, I.$$

If, in addition **aggregate demand for risk** *ϕ is strictly monotonically increasing for all ρ with $\phi(\rho) > 0$, then the equilibrium is unique.*

Remarks

1. Existence and uniqueness reduced to a one-dimensional problem
2. Standard pricing formula, but with **extended market portfolio** $z_m = x_m + y_m$
3. Investors hold a portion of the extended market portfolio
4. Only the equilibrium price of risk $\rho_\star$ depends on preferences
5. Existence may fail to hold if aggregate risk is too high

Standard Valution of Non-traded Payoffs

Given: $e \in \mathcal{C}$
Decomposition: $e = e_M + e_M^{\perp}, \;\; e_M \in \mathcal{M}, \;\; e_M^{\perp} \in \mathcal{M}^{\perp}$
Replicating: $e_M = R_f a_e + \langle q, x_e \rangle$
Pricing:

$$\begin{aligned} \mathcal{V}(e) &= a_e + \langle p_\star, x_e \rangle \\ &= \frac{1}{R_f}\left[R_f a_e + \langle \overline{q}, x_e \rangle - \frac{\rho_\star}{\sigma_m}\langle x_e, V z_m \rangle \right] \\ &= \frac{1}{R_f}\left[\mathbb{E}[e] - \frac{\mathbb{C}\mathrm{ov}[e, R_M]}{\sigma_M^2}(\mu_M - R_f) \right] \end{aligned}$$

with market return

$$R_M = \frac{\langle q, z_m \rangle}{\langle p_\star, z_m \rangle}, \quad \mu_M = \mathbb{E}[R_M], \quad \sigma_M = \sqrt{\mathbb{V}\mathrm{ar}[R_M]}$$

Result

- Pricing can be done as 'usual' **but**, in order to be consistent with equilibrium theory, with the extended market portfolio

Innovations

- non-redundant financial instrument, newly introduced
- payoff $\mathfrak{q}$
- Replicates non-hedgeable endowment: $e_N = \mathfrak{q}\,\mathfrak{x}_m$
- New market portfolio: $x_m^+ = (x_m, \mathfrak{x}_m)$
- Expected payoffs are $\overline{q}^+ = (\overline{q}, \overline{\mathfrak{q}}) \in \mathbb{R}^{K+1}$
  Covariance matrix
  $$V_+ = \left(\begin{array}{c|c} V & v \\ \hline v^\top & \mathfrak{v} \end{array}\right),$$
  with $v_k = \mathbb{C}\mathrm{ov}[q_k, \mathfrak{q}]$
  and $\mathfrak{v} = \mathbb{V}\mathrm{ar}[\mathfrak{q}]$

Proposition 3 (Change of Prices)

With the introduction of the innovation above, one has

(i) *Equilibrium price of risk:* $\rho_\star^+ > \rho_\star$

(ii) *Equilibrium asset prices:*

$$\begin{aligned} p_{\star k}^+ &= p_{\star k} + \frac{1}{R_f}\left(\frac{\rho_\star}{\sigma_m} - \frac{\rho_\star^+}{\sigma_m^+}\right)(Vz_m)_k, \quad k = 1, \ldots, K \\ \mathfrak{p}_\star &= \overline{\mathfrak{q}} - \frac{\rho^+}{\sigma_m^+}(\langle v, x_m\rangle + \mathfrak{v}\mathfrak{x}_m) \end{aligned}$$

Corollary 1 (Change of prices)

With the introduction of the above innovation,

$$\frac{\rho_\star}{\sigma_m} \begin{matrix} > \\ = \\ < \end{matrix} \frac{\rho_\star^+}{\sigma_m^+} \quad \Longleftrightarrow \quad \text{Aggregate demand for risk is} \begin{cases} \text{strictly concave} \\ \text{linear} \\ \text{strictly convex} \end{cases}$$

Proposition 4 (Change of Valuation)

Let $e \in \mathcal{C}$ be given. Then

$$\mathcal{V}^+(e) = \mathcal{V}(e) + \underbrace{\frac{1}{R_f}\left(\frac{\rho_\star}{\sigma_m} - \frac{\rho_\star^+}{\sigma_m^+}\right)\langle x_e, Vz_m\rangle}_{\textit{preference-dependent}} - \underbrace{\frac{1}{R_f}\frac{\rho_\star^+}{\sigma_m^+}\mathbb{C}\mathrm{ov}[e, e_N]}_{\textit{orthogonal component}}$$

Results

- Innovations increase the equilibrium price of risk
- Investors are willing to accept more risk
- Individual risk may increase/decrease
- Allocation of risk is more 'efficient'
- Aggregrate risk remains the same
- Innovations may change equilibrium asset prices in **either direction**, depending on preferences and the correlation of the payoff with the market