The Omega model: From bankruptcy to occupation times in the red

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Let $(\Omega, \mathcal{F}, P)$

be a probability space

Let $(\text{Rolex}, \mathcal{F}, P)$

be a probability space

I let you choose between

this Omega watch and this box of chocolate.

What do you choose?

The meaning of

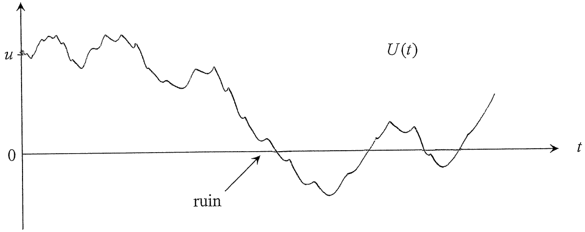
$$(\Omega, \mathcal{F}, P)$$

Probably a Fake Omega

The Brownian motion model and some notation

- $U(t) = u + \mu t + \sigma W(t)$: surplus at time t
- $\{W(t)\}$: standard Wiener process
- μ: expected increment per unit time
- σ^2: variance of the increment per unit time
- convenient notation: $D = \frac{1}{2}\sigma^2$
- u: initial surplus

Sample path of $U(t)$



Ruin versus bankruptcy

In the Omega model, there is a distinction between ruin (negative surplus) and *bankruptcy* (going out of business). It is assumed that even with a negative surplus, the company can do business as usual and continue until bankruptcy occurs. The probability for bankruptcy is quantified by a bankruptcy rate function $\omega(x)$, where x is the value of the negative surplus. The idea of distinguishing ruin from bankruptcy comes from the impression that some companies and certain industries seem to be able to continue doing business even when they are technically ruined. This may especially be true for companies that are owned by governments or other companies.

Bankruptcy rate function $\omega(x)$, $x < 0$

- If $U(t) = x < 0$:
  bankruptcy by time $t + dt$ with probability $\omega(x)dt$
- Examples:
  1). $\omega(x) = \lambda$ constant: easier calculations!

  2). $\omega(x) = -\lambda x$: more realistic!
- For convenience we set $\omega(x) = 0$ if $x > 0$.

Expected discounted penalty at bankruptcy

- $w(x)$: penalty function
- $\delta > 0$: force of interest
- $\phi(u)$, $-\infty < u < \infty$:

  expected discounted penalty at bankruptcy,

  a function of the initial surplus u.

$\phi(u)$ is the solution of

$$
\begin{aligned}
&D\phi''(u) + \mu\phi'(u) - (\omega(u) + \delta)\phi(u) + \omega(u)w(u) = 0, && u < 0,\\
&D\phi''(u) + \mu\phi'(u) - \delta\phi(u) = 0, && u > 0,
\end{aligned}
$$

with the conditions that $\phi(u)$ and $\phi'(u)$ are continuous at $u = 0$.

Immediate consequence

$$\phi(u) = \phi(0)e^{-\rho u}, \qquad u > 0,$$

where $-\rho$ is the negative solution of

$$D\xi^2 + \mu\xi - \delta = 0.$$

$\phi(u)$ for $\omega(x) = \lambda$, $x < 0$

- We need some facts from exponential stopping of Brownian motion

$$X(t) = \mu t + \sigma W(t), \quad t > 0$$

- notation: $f_{X(t)}(x)$ pdf of $X(t)$
- τ: independent exponential random variable, mean $1/\lambda$
- $X(t)$ is stopped at time τ

Discounted density of $X(\tau)$

$$f^{\delta}_{X(\tau)}(x) = E[e^{-\delta\tau} f_{X(\tau)}(x)] = \lambda \int_0^{\infty} e^{-(\lambda+\delta)t} f_{X(t)}(x) dt.$$

It is known that

$$f^{\delta}_{X(\tau)}(x) = \begin{cases} \kappa e^{-\alpha x}, & \text{if} \quad x \leq 0, \\ \kappa e^{-\beta x}, & \text{if} \quad x > 0, \end{cases}$$

where $\alpha < 0$ and $\beta > 0$ are the two solutions of the equation

$$D\xi^2 + \mu\xi - (\lambda + \delta) = 0,$$

and

$$\kappa = \frac{\lambda}{D(\beta - \alpha)}.$$

Proof with Laplace transforms

$$
\begin{aligned}
&\int_{-\infty}^{\infty} e^{-zx} f^{\delta}_{X(\tau)}(x)dx \\
=\ & \lambda \int_0^{\infty} e^{-(\lambda+\delta)t-\mu tz+Dtz^2} dt \\
=\ & \frac{\lambda}{\lambda+\delta+\mu z - Dz^2} \\
=\ & \frac{\kappa}{\beta+z} - \frac{\kappa}{\alpha+z},
\end{aligned}
$$

for $-\beta < z < -\alpha$.

Equivalent formulation of bankruptcy if $\omega(x) = \lambda$, $x < 0$

- $\tau_1, \tau_2, \cdots$, i.i.d. exponential with mean $1/\lambda$
- observation times: $T_1 = \tau_1, \; T_2 = \tau_1 + \tau_2, \cdots$
- bankruptcy is the event that the surplus is negative at one of the observation times
- Albrecher, Cheung and Thonhauser (2011, 2012)

$\phi(0)$ if $\omega(x) = \lambda$, $x < 0$

By conditioning on $U(\tau_1)$, we see that

$$\phi(0) = \kappa \int_{-\infty}^{0} w(x) e^{-\alpha x} dx + \kappa \int_{0}^{\infty} \phi(x) e^{-\beta x} dx.$$

Solving $\phi(0)$, we obtain

$$\begin{aligned} \phi(0) &= \frac{\kappa \int_{-\infty}^{0} w(x) e^{-\alpha x} dx}{1 - \kappa \int_{0}^{\infty} e^{-(\rho+\beta)x} dx} \\ &= \frac{\kappa(\rho+\beta)}{\rho+\beta-\kappa} \int_{-\infty}^{0} w(x) e^{-\alpha x} dx \\ &= -(\rho+\alpha) \int_{-\infty}^{0} w(x) e^{-\alpha x} dx. \end{aligned}$$

$\phi(u)$ if $u < 0$ and $\omega(x) = \lambda$, $x < 0$

Again conditioning on $U(\tau_1)$ yields now

$$\begin{aligned}
\phi(u) &= \kappa \int_{-\infty}^{0} w(u+y)e^{-\alpha y}dy + \kappa \int_{0}^{-u} w(u+y)e^{-\beta y}dy \\
&\quad + \kappa \int_{-u}^{\infty} \phi(u+y)e^{-\beta y}dy \\
&= \kappa \int_{-\infty}^{u} w(x)e^{\alpha(u-x)}dx + \kappa \int_{u}^{0} w(x)e^{\beta(u-x)}dx \\
&\quad + \frac{\kappa}{\rho+\beta}\phi(0)e^{\beta u},
\end{aligned}$$

$f(x|u)$: discounted density of surplus at bankruptcy

It is the coefficient of $w(x)$ in the expressions for $\phi(u)$.

$$f(x|u) = -(\rho+\alpha)e^{-\rho u-\alpha x}, \qquad \text{if } \ u \geq 0$$

and

$$f(x|u) = \begin{cases} \kappa e^{\alpha(u-x)} - \kappa \frac{\rho+\alpha}{\rho+\beta} e^{\beta u-\alpha x}, & \text{if } \ x < u < 0, \\ \kappa e^{\beta(u-x)} - \kappa \frac{\rho+\alpha}{\rho+\beta} e^{\beta u-\alpha x}, & \text{if } \ u < x < 0. \end{cases}$$

$\psi(u)$ probability of bankruptcy (assume $\mu > 0$)

- Special case of $\phi(u)$ when $\delta = 0$, $w(x) = 1$
- Important relationship
  between $\psi(u)$ and the times that $U(t) < 0$
- We are doubly motivated for calculating $\psi(u)$

Comparison for $u \geq 0$

- Probability of ruin: $e^{-\rho u}$
- Probability of bankruptcy: $\psi(0)e^{-\rho u}$
- Hence, their ratio is $\psi(0)$, independently of u.

Exposure of a sample path

$$\mathcal{E} = \int_0^\infty \omega(U(t))dt;$$

recall that $\omega(x) = 0$ if $x > 0$. For a given sample path, the conditional probability for no bankruptcy is $e^{-\mathcal{E}}$. It follows that

$$1 - \psi(u) = E[e^{-\mathcal{E}}].$$

Special case: $\omega(x) = \lambda$ constant

- Then $\mathcal{E} = \lambda L$

  where L is the total time spent in the red

- $$1 - \psi(u) = E[e^{-\lambda L}]$$

  Laplace transform of L

$\psi(u)$ for $\omega(x) = \lambda$ constant

- $$\psi(u) = 1 - \frac{\rho}{\rho+\beta} e^{\beta u}, \qquad \text{if} \quad u \leq 0,$$
- $$\psi(u) = \frac{\beta}{\rho+\beta} e^{-\rho u}, \qquad \text{if} \quad u > 0,$$
- Here $\rho = \mu/D$ and

$$\beta = \frac{-\mu + \sqrt{\mu^2 + 4\lambda D}}{2D} = \frac{\rho}{2}\left[-1 + \sqrt{1 + \lambda/b}\right],$$

with

$$b = \frac{\mu^2}{4D}.$$

Laplace transform of L with $u = 0$

$$
\begin{aligned}
E[e^{-\lambda L}] &= 1 - \psi(0) = \frac{\rho}{\rho + \beta} \\
&= \frac{2}{1 + \sqrt{1 + \lambda / b}}.
\end{aligned}
$$

How can we invert this Laplace transform?

Laplace transform of L with $u = 0$

$$
\begin{aligned}
E[e^{-\lambda L}] &= 1 - \psi(0) = \frac{\rho}{\rho + \beta} \\
&= \frac{2}{1 + \sqrt{1 + \lambda/b}} \\
&= 2\,\frac{1}{\sqrt{1 + \lambda/b}} - 2b\,\frac{1 - \frac{1}{\sqrt{1+\lambda/b}}}{\lambda} \\
&= 2\ell(\lambda) - 2b\frac{1 - \ell(\lambda)}{\lambda}
\end{aligned}
$$

Now we can invert it!

- $\ell(\lambda)$ Laplace transform of
  $g(t)$ gamma pdf
  shape parameter $1/2$ and scale parameter b
- $\frac{1-\ell(\lambda)}{\lambda}$ Laplace transform of $1 - G(t)$
- Hence, the pdf of L is

$$2g(t) - 2b[1 - G(t)], \qquad t > 0$$

if $u = 0$

$\psi(u)$ for a general $\omega(x)$

$$D\psi''(u) + \mu\psi'(u) - \omega(u)\psi(u) + \omega(u) = 0, \qquad u < 0.$$

The constant 1 is a particular solution. It follows that

$$\psi(u) = 1 + Ch(u), \qquad u < 0,$$

where $h(u)$ is a non-trivial solution of the corresponding homogeneous differential equation

$$Dh''(u) + \mu h'(u) - \omega(u)h(u) = 0, \qquad u < 0.$$

Illustration: for $\omega(x) = \lambda$, $h(u) = e^{\beta u}$, $u < 0$.

$$1-\psi(u)=\frac{h(u)}{h(0)}[1-\psi(0)], \qquad u\le 0. \tag{1}$$

Then $\psi'(0-)=\psi'(0+)$ is the condition that

$$-\frac{h'(0)}{h(0)}[1-\psi(0)]=-\rho\psi(0), \tag{2}$$

which yields

$$\psi(0)=\frac{h'(0)}{h'(0)+\rho h(0)} \tag{3}$$

and hence

$$1-\psi(u)=\frac{\rho h(u)}{h'(0)+\rho h(0)}, \qquad u\le 0. \tag{4}$$

$\psi(u)$ for $\omega(x) = -\lambda x$

$$\text{Here} \qquad \mathcal{E} = \lambda \int_0^\infty U(t)_- dt$$

$1 - \psi(u)$ Laplace transform of $\int_0^\infty U(t)_- dt$

The equation

$$Dh''(u) + \mu h'(u) + \lambda u h(u) = 0$$

can be transformed into the Airy equation.
Result:

$$h(u) = e^{-\frac{\rho u}{2}} Ai\left(\left(\frac{\lambda}{D}\right)^{1/3}\left(-u + \frac{\mu^2}{4D\lambda}\right)\right)$$

where $Ai(x)$ denotes the *Airy function of the first kind*.

Laplace transform for $\int_0^\infty U(t)_- dt$

$$\frac{2\rho Ai(z^2)}{\rho Ai(z^2) - 2(\frac{\lambda}{D})^{1/3} Ai'(z^2)},$$

where

$$z = \frac{\rho}{2}(\frac{D}{\lambda})^{1/3}.$$

From this

$$E[\int_0^\infty U(t)_- dt] = \frac{D^2}{\mu^3}.$$