Estimates for Approximations of American Put Option Price

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7th June 2012
Actuarial & Financial Mathematics 2012: Theory & Practice

American put options

A derivative contract

- Strike K_{strike}
- Expiry $T > 0$
- Underlying security with price S_t at time $t \in [0, T]$
- Can be exercised anytime between now (time 0) and expiry (time T)

For put option on one asset the payoff at exercise time T^* is

$$\bar{g}(S) := [K_{\mathrm{strike}} - S]_+.$$

Mathematical formulation leads to optimal stopping problem

$$v(t, S) := \sup_{t \leq T^* \leq T} \mathbb{E}_{t,S} \left(e^{-\int_t^{T^*} \bar{\rho}(u, S_u) du} \bar{g}(S_{T^*}) \right).$$

The model

We have to choose how to model the underlying asset(s). We take d risky assets satisfying

$$dS_u^i = S_u^i \bar{\rho}(u, S_u) du + S_u^i \sum_{j=1}^{d} \bar{\sigma}^{ij}(u, S_u) dW_u^j, \quad S_t^i = S^i$$

Remarks

- Even in the one dimensional case with $\bar{\sigma}$ and $\bar{\rho}$ constant there is no “formula” for American put option price.
- There is a “formula” for perpetual American put option.
- American call option with payoff $[S - K_{\text{strike}}]_+$ has optimal exercise at expiry and hence is equivalent to European call (convexity and Doob's optional sampling theorem).

Simple Transformation

Let

$$\bar{\beta}^i(t,S) := \bar{\rho}(t,S) - \frac{1}{2}\sum_{j=1}^{d} \bar{\sigma}^{ij}(t,S)^2$$

Change of variable to remove unbounded growth in S in the SDE

- Let $x_t^i := \ln S_t^i$
- Let $\sigma(t,x) := \bar{\sigma}(t,e^x)$, $\beta(t,x) = \bar{\beta}(t,e^x)$, $\rho(t,x) = \bar{\rho}(t,e^x)$ and $g(x) := \bar{g}(e^{x_i})$
- Itô's formula gives

$$dx_u^i = \beta^i(u,x_u^i)du + \sum_{j=1}^{d} \sigma^{ij}(u,x_u^i)dW_u^j,\ x_t^i = x^i = \ln S^i,\ u \in [t,T]$$

Optimal stopping problem

$$w(t,x) = \sup_{T^* \in \mathfrak{T}[t,T]} \mathbb{E}_{t,x}(e^{-\int_t^{T^*} \rho(u,x_u)du} g(x_{T^*}))$$

PDE Formulation

The payoff w of the optimal stopping problem

$$w(t,x) = \sup_{T^* \in \mathfrak{T}[t,T]} \mathbb{E}_{t,x}(e^{-\int_t^{T^*} \rho(u,x_u)du} g(x_{T^*}))$$

is the unique solution to

$$\max\left[w_t + \mathrm{L}\, w, g - w\right] = 0 \quad \text{on} \quad [0,T) \times \mathbb{R}^d$$
$$w(T,x) = g(x) \quad \text{for all} \quad x \in \mathbb{R}^d,$$

where L is the diffusion generator i.e. for a smooth function η

$$\mathrm{L}\,\eta := \sum_{i,j=1}^{d} \tfrac{1}{2}(\sigma\sigma^T)^{ij}\eta_{x^i x^j} + \sum_{i=1}^{d} \beta^i \eta_{x^i} - \rho\eta$$

Available pricing methods

What are the choices

- Formulae giving approximate value
- Monte Carlo simulation
- Binomial and trinomial recombining trees
- Finite element methods
- Finite difference schemes

What are the issues

- Convergence
- Convergence rates
- Artificial boundary conditions

Finite Difference Approximations

We restrict ourselves to diffusion generators L which can be written as

$$\mathsf{L}\,\eta = \sum_{k=\pm 1,\ldots,\pm d_1} \left(a_k\, \mathsf{D}^2_{\ell_k}\, \eta + b_k\, \mathsf{D}_{\ell_k}\, \eta \right) - \rho\eta$$

for the directional derivatives D_ℓ, in the direction of $\ell \in \mathbb{R}^d$.
Finite difference approximations

$$\delta^T_\tau\, \eta(t,x) := \frac{\eta(t+\tau_T(t),x) - \eta(t,x)}{\tau},$$

$$\delta_{h,\ell}\, \eta(t,x) := \frac{\eta(t,x+h\ell) - \eta(t,x)}{h},$$

$$\Delta_{h,\ell}\, \eta := -\,\delta_{h,\ell}\, \delta_{h,-\ell}\, \eta = \frac{1}{h}(\delta_{h,\ell}\, \eta + \delta_{h,-\ell}\, \eta)$$

lead to

$$\mathsf{L}_h\, \eta := \sum_{k=\pm 1,\ldots,\pm d_1} (a_k\, \Delta_{h_k,\ell_k}\, \eta + b_k\, \delta_{h_k,\ell_k}\, \eta) - \rho\eta$$

Fully Discrete Problem

Grid $\mathcal{M}_T := \bar{\mathcal{M}}_T \cap \big([0, T) \times \mathbb{R}^d\big)$, where

$$\bar{\mathcal{M}}_T := \{(t, x) \in [0, T] \times \mathbb{R}^d : (t, x) = ((t_0 + j\tau) \wedge T, \\ x_0 + h(i_1\ell_1 + \cdots + i_{d_1}\ell_{d_1})), j \in \{0\} \cup \mathbb{N}, i_k \in \mathbb{Z}, k = \pm 1, \ldots, \pm d_1\}$$

Fully discrete problem

$$\max\left[\delta_\tau^T\, w_{\tau,h} + \mathsf{L}_h\, w_{\tau,h}, g - w_{\tau,h}\right] = 0 \quad \text{on} \quad Q,$$
$$w_{\tau,h} = g \quad \text{on} \quad \bar{\mathcal{M}}_T \setminus Q$$

- Implicit scheme since we're solving backward in time
- The set Q may be finite or infinite e.g. $Q = \mathcal{M}_T$
- $w_{\tau,h}$ defined for any point (t, x) as the grid can be centred arbitrarily.

Error Estimate

Under appropriate regularity assumptions in particular that σ_k, b_k, ρ and g:

- Lipschitz continuous in space
- $1/2$-Hölder continuous in time

we get

$$|w - w_{\tau,h}| \leq C(\tau^{1/4} + h^{1/2}),$$

with C independent of τ and h.

Remarks:

- The assumptions are restrictive since they essentially mean that $S \mapsto S\bar{\sigma}(t, S)$ etc. are Lipschitz continuos
- Error estimate is optimal (we allow diffusion coefficients to degenerate)
- Computationally not directly applicable - $\mathcal{M}_T$ contains infinitely many points

Ingredients for proof

- Comparison theorem for discrete problem and continuous problem
- A priori estimate for the derivative, in any direction, of the solution to the discrete problem. Obtained by applying discrete derivative multiplying by negative part of the discrete derivative of the solution
- For discrete problem get Lipschitz continuity in space and Hölder continuity in time from a priori estimate
- For continuous problem this follows from the optimal stopping problem
- Idea of N. V. Krylov of “Shaking the coefficients” - introduction of a optimal control in the space and time variables allows one to estimate the error introduced by mollifying w and $w_{\tau,h}$
- Taylor's theorem applied to the mollified versions of w and $w_{\tau,h}$

Artificial Boundary Conditions

Aim: estimate error arising from computation on finite Q e.g. $Q = \mathcal{M}_T \cap B_R$ for some $R > 0$.
Take R, R_1, R_2 such that $R > R_1 > R_2 > 0$:

- Let g_{R_1} be equal to g inside B_{R_1}, equal to 0 outside B_{R_1+1} and Lipschitz continuous
- Solve on $Q = \mathcal{M}_T \cap B_R$, get $w_{\tau,h}^{R,R_1}$
- We get error estimate inside B_{R_2}

There are $\mu > 0$ and $\gamma \in (0,1)$ such that on $[0,T] \times B_{R_2}$

$$|w_{\tau,h} - w_{\tau,h}^{R,R_1}| \leq C \left(e^{-\mu R_1^2 + R_2^2/2} + e^{\gamma(R_1 - R)} \right)$$

Ingredients for proof

- General estimate on distribution of exit times of a diffusion process from a ball. Let

  $$dx_t = \beta_t dt + \sigma_t dW_t, \quad x_0 = \xi$$

  Then, under a general monotonicity-like assumption, there exists $\mu > 0$ such that

  $$\mathbb{P}(T_R^* \leq T) \leq 3e^{-\mu R^2}(1 + \mathbb{E}e^{\xi^2/2}),$$

  where $T_R^* := \inf\{t \geq 0 : |x_t| \geq R\}$
- Corollary of comparison theorem for fully discrete problem

Summary

- On $[0, T] \times B_{R_2}$

$$|w - w_{\tau,h}^{R,R_1}| \leq C \left(e^{-\mu R_1^2 + R_2^2/2} + \tau^{1/4} + h^{1/2} + e^{\gamma(R_1 - R)} \right)$$

- Error estimate for finite difference approximation is optimal
- First error estimate for artificial boundary conditions
- Variable coefficients and stochastic volatility is allowed but under restrictive growth / regularity assumptions.

Bibliography and Thank you!

Gyöngy, I. and Šiška, D. (2009).
On the rate of convergence of finite-difference approximations for normalized Bellman equations with Lipschitz coefficients.
Appl. Math. Optim., 60(3), 297–339.

D. Šiška. (2012).
Error estimates for finite difference approximations of American put option price.
Comput. Methods Appl. Math., 11(1), 3–15.