Optimal investment with smooth trading strategies

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Outline

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- Market model and constraints
- Upper bound on the log-error and log-quadratic error
- Linear model and linear-quadratic criteria
- An open problem

General remarks

- **Main ideas**:
  (a) Constrain the trading strategies to be differentiable, and thus 'smooth'.
  (b) Track a given benchmark portfolio with the constrained portfolio.
  (c) Use discrete-time approximation of the portfolio model and single-step optimization.
- **Outcomes:**
  (a) The results are valid for a very general market model
  (b) Optimal trading strategies are in an explicit closed-form.
  (c) The constrained portfolio will have a significantly lower eventual proportional transaction cost.

1. Market model and constraints

Consider the following market model driven by an m-dimensional standard Brownian motion $W(t)$:

$$\begin{cases} dS_0(t) = S_0(t) r dt, \quad S_0(0) \quad \text{is given,} \\ dS_i(t) = S_i(t)[\mu_i dt + \sigma_i dW(t)], \\ S_i(0) \quad \text{is given,} \quad i = 1, ..., n. \end{cases}$$

Here r, μ_i, σ_i, are given coefficients. Let $v_i(t)$, $i = 0, ..., n$, denote the number of shares of asset i held by the investor at time t. We make the following two assumptions:

Assumption 1. $v_i(t) > 0$, $i = 0, 1, ..., n$.

Assumption 2. $d \ln v_i(t) = u_i(t) dt$, $i = 0, ..., n$.

Theorem 1. Let the Assumption 1 and Assumption 2 hold, and let $x_i(t) \equiv \ln[v_i(t)S_i(t)]$, $i = 0, ..., n$. For a self-financing portfolio the following holds:

$$\begin{cases} dx_0(t) = -\sum_{i=1}^{n} e^{x_i(t)-x_0(t)} u_i(t)dt + rdt, \\ dx_i(t) = [u_i(t) + \mu_i - 0.5\sigma_i\sigma_i']dt + \sigma_i dW(t), \quad i = 1, ..., n. \end{cases}$$

□

This is a nonlinear equation in the state variables, but linear in the control variables.

Using Euler approximation with a sampling time T_s, we obtain the discrete-time model:

$$\begin{cases} x_0(k+1) = x_0(k) - \sum_{i=1}^{n} e^{x_i(k)-x_0(k)} u_i(k) T_s + rT_s, \\ x_i(k+1) = x_i(k) + [u_i(k) + \mu_i - 0.5\sigma_i\sigma_i'] T_s + \sigma_i e(k+1)\sqrt{T_s}. \end{cases}$$

where $e(k+1) = [e_1(k+1), ..., e_m(k+1)]'$ is a vector of independent standard normal random variables. In the more convenient vector-matrix notation, this equation can be written as

$$x(k+1) = x(k) + A(k, x(k))u(k)T_s + D(k)T_s + \Sigma(k)e(k+1)\sqrt{T_s}.$$

where $x(k) = [x_0(k), ..., x_n(k)]'$, $u(k) = [u_1(k), ..., u_n(k)]'$.
The aim to be achieved with this constrained portfolio is to either track closely or outperform some already designed reference portfolio that has a positive trading strategy.
This is done with the aim of obtaining a lower eventual proportional transaction cost due to the smooth trading strategy of the constrained portfolio.

Upper bound on the log-error and log-quadratic errors
One choice for the criterion for the quality of tracking will be an upper bound on the *discrete-time logarithmic error* $e_l(k+1)$ between the two portfolios

$$e_l(k+1) = \ln[y_r(k+1)] - \ln[y(k+1)],$$

where $y_r(k+1)$ is the value of the self-financing reference portfolio and $y(k+1)$ is the value of the constrained tracking portfolio, i.e.

$$y(k+1) = \sum_{i=0}^{n} e^{x_i(k+1)}.$$

An upper bound on $e_l(k+1)$ can be found using Jensen's inequality as follows. Let $\gamma_i(k+1)$, $i = 0, 1, ..., n$, be such that $0 \leq \gamma_i(k+1) \leq 1$, and $\gamma_0(k+1) + ... + \gamma_n(k+1) = 1$. Then, Jensen's inequality gives the following for each k

$$\ln[y(k+1)] \geq \sum_{i=0}^{n} \gamma_i(k+1) \ln[y_i(k+1)].$$

where $y_i(k+1) = v_i(k+1)S_i(k+1)$. An upper bound on the logarithmic error $e_u(k+1) \geq e_l(k+1)$ can thus be expressed as

$$\begin{aligned} e_u(k+1) &= \ln[y_r(k+1)] - \sum_{i=0}^{n} \gamma_i(k+1) \ln[y_i(k+1)] \\ &= \ln[y_r(k+1)] - \sum_{i=0}^{n} \gamma_i(k+1) x_i(k+1). \end{aligned}$$

The aim now is to minimize this upper error bound. One possibility is to minimize its mean and variance. We do so by first selecting $\gamma_i(k+1)$, $i = 0, 1, ..., n$, such that the conditional variance of $e_u(k+1)$ is zero, and then minimize its mean.

Theorem 2. Let the reference portfolio $y_r(k+1)$ be a self-financing portfolio with a positive trading strategy, and with the fraction of wealth invested in asset i at step k denoted by $\alpha_i^r(k)$, $i = 0, 1, ..., n$. For $k = 0, 1, ...$, the conditional variance $Var_k[e_u(k+1)]$ is equal to zero if

$$\gamma_i(k+1) = \alpha_i^r(k), \quad i = 0, 1, ..., n.$$

If the volatility matrix $\sigma = [\sigma_1', ..., \sigma_n']'$ is square and non-singular, then this condition is also necessary. $\square$.

The expected value of $e_u(k+1)$ will be minimized if we maximize the following

$$\mathbb{E}_k \sum_{i=0}^{n} \gamma_i(k+1) \ln[y_i(k+1)] = \mathbb{E}_k \sum_{i=0}^{n} \alpha_i^r(k) x_i(k+1), = \mathbb{E}_k[\alpha'(k) x(k+1)].$$

In order to give the investor the means for trade off between a lower eventual transaction cost and a higher profit, we also include a quadratic penalty on the logarithmic rates of change of trading strategies $u_i(k)$. The resulting criterion is:

$$J(u(k)) = \mathbb{E}_k \left[\frac{1}{2} u'(k) B u(k) T_s - \alpha'(k) x(k+1) \right],$$

where $B \in \mathbb{R}^{n \times n}$ be a given symmetric and positive definite matrix. The solution to the problem of minimizing $J(u(k))$ with respect to $u(k)$ is easily found to be

$$u^*(k) = B^{-1} A'(k, x(k)) \alpha(k).$$

Example. The reference portfolio in this example is selected to be the log-optimal portfolio with no consumption. Let us consider a market having a bank account $S_0(t)$ and a single stock $S_1(t)$ with the following dynamics

$$\begin{aligned} dS_0(t) &= rS_0(t)dt, \\ dS_1(t) &= S_1(t)[\mu_1 dt + \sigma_1 dW_1(t)]. \end{aligned}$$

We assume that the parameters are constant and have these numerical values: $r = 0.04$, $\mu_1 = 0.05$, and $\sigma = \sigma_{11} = 0.25$. The initial investors wealth and the initial asset prices are assumed as $y(0) = S_0(0) = S_1(0) = 1$. The fraction of wealth invested in the stock for the log-optimal portfolio $\alpha_1^r(k)$ is given as:

$$\alpha_1^r(k) = \alpha_1^r = \frac{\mu_1 - r}{\sigma_1^2} = 0.16,$$

The initial selection for both portfolios (the log-optimal and the tracking) will thus be $v_0^*(0) = 0.84$, $v_1^*(0) = 0.16$. The optimal logarithmic rate of change, with a sampling time of $T_s = 0.004$, becomes

$$u_1^*(k) = \frac{1}{b_1}\left[0.16 - 0.84\frac{v_1(k)S_1(k)}{v_0(k)S_0(k)}\right].$$

Let us also have two different values for the penalty coefficient, $b_1^{(1)} = 0.05$ and $b_1^{(2)} = 0.5$. In a market with no transaction cost, for one realization of the stock price, the trading of the stock for the log-optimal and the tracking portfolios are shown in figures to follow.

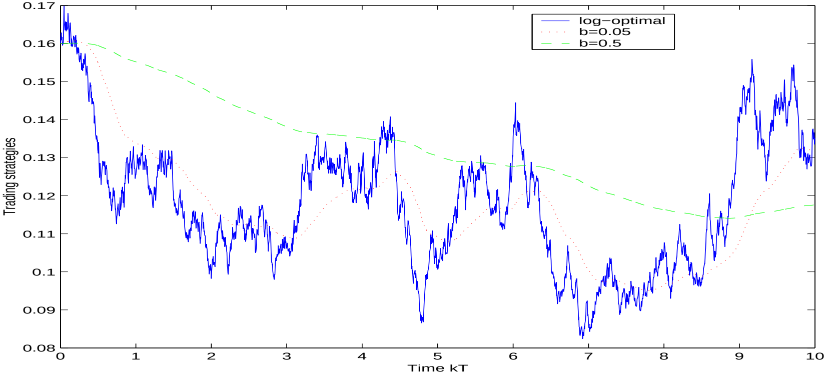

Figure : Trading strategies for the stock.

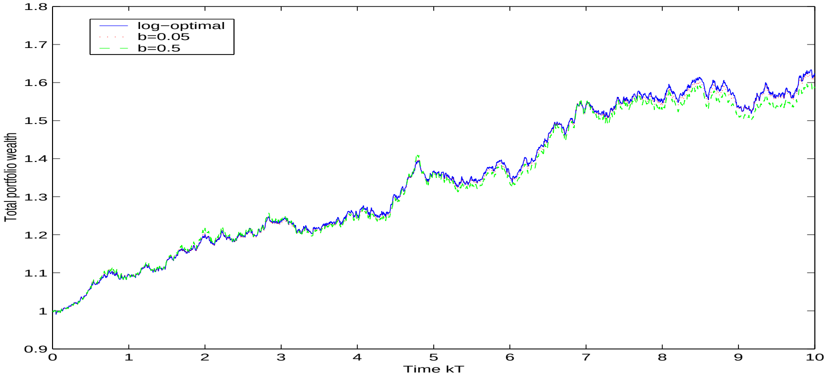

Figure : Total portfolio wealth during the trading period.

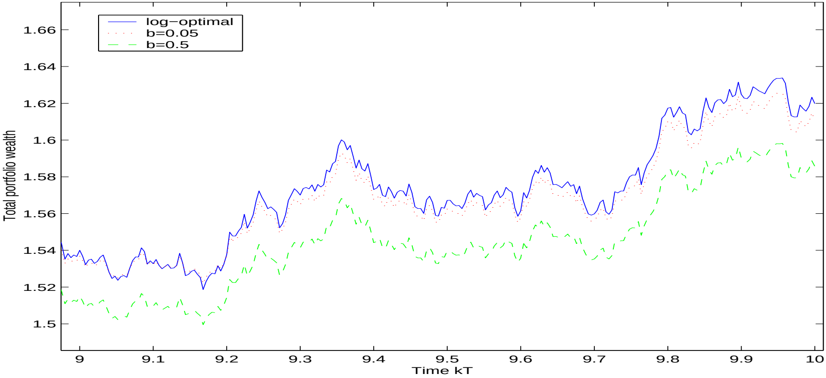

Figure : Total portfolio wealth at the end of the trading period.

We assume that there is a charge of 1% of the total transaction value of buying or selling the stock, and no transaction cost for the bank account.
The wealth y_f and the eventual proportional transaction cost C_f at the end of the trading period are:

$$\begin{aligned} \textbf{Log}-\textbf{optimal} &: \quad y_f = 1.61983, \quad C_f = 0.05743 \\ \mathbf{b_1^{(1)}} &: \quad y_f = 1.61128, \quad C_f = 0.00488 \\ \mathbf{b_1^{(2)}} &: \quad y_f = 1.58572, \quad C_f = 0.00258 \end{aligned}$$

This shows that for almost the same final wealth, the eventual transaction cost is more than 11 and 22 times smaller for the constrained portfolios. Moreover, the difference between the final wealth and the total eventual transaction cost $(y_f - C_f)$, is higher for the constrained portfolios.
We can also use the **log-quadratic errors** as criterion. Again explicit closed-form solution can be obtained. For illustration, the tracking of a Black-Scholes replicating portfolio for a European Call option is given in the following figure.

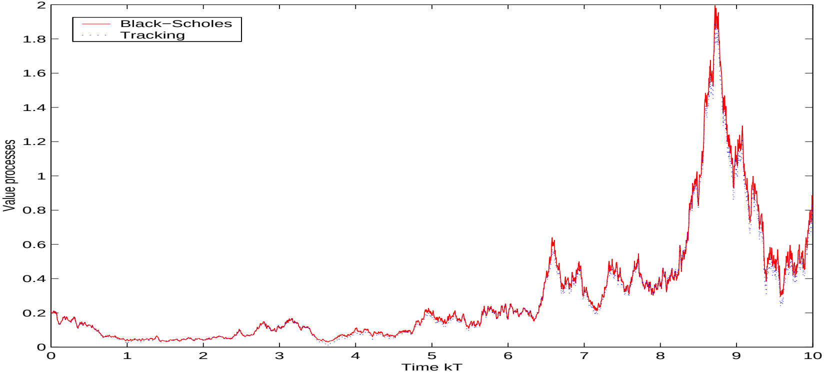

Figure : Value processes for the Black-Scholes and the tracking portfolios.

Linear model and linear-quadratic criteria

Assumption 3. $dv_i(t) = q_i(t)dt$, $i = 0, 1, 2, ..., n$.
Theorem 3. Let $x_i(t) = v_i(t)S_i(t)$, $i = 0, 1, 2, ..., n$. For a self-financing portfolio that satisfies Assumption 3, the following holds

$$\begin{cases} dx_0(t) = rx_0(t)dt - \sum_{i=1}^{n} S_i(t)q_i(t)dt, \\ \\ dx_i(t) = [\mu_i x_i(t) + q_i(t)S_i(t)]dt + x_i(t)\sigma_i dW(t), \\ \\ i = 1, 2, ..., n. \end{cases}$$

This is clearly a linear model in both state and control variables.

Discrete-time.
Tracking portfolio as in the previous slides, but with quadratic errors as criterion.
Continuous-time.
a) Linear utility and a quadratic penalty on $q_i(t)$. Explicit solution is found for both finite and infinite horizon.
b) When $q_i(t)S_i(t)$ is used as a control, and the non-negativity of wealth is not taken into consideration, the LQ regulator was used to find the solution for several linear-quadratic criteria.

An open problem

In the linear model case, and when using a quadratic criteria, if we include the non-negativity of the total wealth at terminal time, we end up with an **LQ control problem with a hard inequality terminal constraint**. This is an open problem.
In the deterministic control systems, provided the system ic completely controllable, and using the solution to the minimum-energy control problem and quadratic programming, complete solution to this kind of problems is available.

Naturally, one approach is to extend such results to the stochastic setting. Some progress has been achieved. In particular, the exact controllability and minimum energy control for a general class of linear stochastic control systems are settled.
However, these are still far from solving the mentioned optimal investment problem, since the self-financing portfolio is not even exactly controllable!