Analysis of the Gerber-Shiu function in a perturbed delayed risk model

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1. Introduction

Consider a surplus process of an insurance company at time t

$$U(t) = u + ct - S(t) + \sigma B(t), \qquad u \geq 0, \tag{1}$$

- $U(0) = u$ is the initial surplus,
- c is the constant premium rate,
- $\{B(t); t \geq 0\}$ is a standard Wiener process with mean 0 and volatility σ, (the extra diffusion term reflects the fluctuations in the insurance surplus, which may happen due to uncertainty of premium income or the economic environment)
- and $\{S(t); t \geq 0\}$ is the aggregate claim amount process independent of $\{B(t); t \geq 0\}$.

We assume that $\{S(t); t \geq 0\}$ generates two kind of claims: the main claims and the by-claims.

Let $\{N(t); t \geq 0\}$ to be a Poisson claim number process with intensity $\lambda(> 0)$ and main claim occurrence epochs $\{T_i\}_{i=0}^{\infty}$ (i.i.d exponentials rv), with $T_0 = 0$.

(delayed assumption) In every epoch T_i of the Poisson process

- a main claim X_i [d.f. $F(x)$] will occur and this will induce a by-claim Y_i [d.f. $Q(x)$],
- Y_i may occur simultaneously with X_i with probability θ, or may be delayed to T_{i+1} with probability $1 - \theta$,

notation: the claim amount of the occurrence of one claim and one by-claim has d.f. $F_2(x) = (F_1 \star Q)(x)$ and density $f_2(x)$, while the claim amount of the appearance of one claim and two by-claim has d.f. $F_3(x) = (F_1 \star Q \star Q)(x)$ and density $f_3(x)$. Also, with $\widehat{}$ it will denoting the LT of the corresponding function.

Based in the above set up, the aggregate claim amount process is given by

$$S(t) = \sum_{i=1}^{N(t)} X_i + R(t), \qquad t \geq 0, \tag{2}$$

where $R(t)$ is the sum of all by-claims Y_i that occurred before time t.

Practical application of the model in portfolios with IBNR claims (IBNR claims are claims that not yet know to the insurer, but it is believed that will exist at the reserving date)
Also, let $T = \inf\{t \geq 0 : U(t) < 0\}$ be the time of ruin, and for $\delta \geq 0$, we define the ultimate Gerber-Shiu expected discounted penalty function [Gerber and Shiu (1998), (2005)] as

$$\begin{aligned} \phi(u) &= \mathbb{E}\big(e^{-\delta T} w(U(T-), |U(T)|) 1_{(T<\infty)} \big| U(0) = u\big) \qquad (3) \\ &= \mathbb{E}\big(e^{-\delta T} w(U(T-), |U(T)|) 1_{(T<\infty), U(T)<0} \big| U(0) = u\big) \\ &\quad + \mathbb{E}\big(e^{-\delta T} 1_{(T<\infty), U(T)=0} \big| U(0) = u\big), \qquad u \geq 0, \end{aligned}$$

with $\phi(0) = 1$ due to oscillating sample paths of $U(t)$. Here δ is interpreted as the force of interest, $U(T-)$ is the surplus immediately before ruin, $|U(T)|$ is the deficit at ruin, $T-$ is the left limit of T, $w\colon [0, \infty) \times (0, \infty) \to [0, \infty)$ is representing the penalty at ruin and $1_{(\cdot)}$ represents the indicator function.

Further note that in T_1 we have the following two possible events:
(a) both main claims and by-claims occur simultaneously, and the surplus process renews itself at these points
(b) in T_1 a main claim occurs, while the by-claim delays his occurrence until T_2, and the surplus process does not renew itself.

Thus, for the event (b) [similar to Xie and Zou (2011)] we consider an auxiliary process in which at T_1 instead of having simultaneously occurred a main claim and a by-claim, another by-claim is added to the first epoch. The corresponding Gerber-Shiu function for the auxiliary surplus process is denoting by $\phi_1(u)$ for $u \geq 0$ with $\phi_1(0) = 1$.

Using well known properties of the first passage time of the Brownian motion [see for e.g. Revuz and Yor (1991)], one can show that both $\phi(u)$ and $\phi_1(u)$ are twice continuously differentiable in u over $(0, \infty)$.

Integro-differential equations and their solution

Applying Itô's Lemma for jump diffusion processes we have:

Theorem 1. *For $u \geq 0$, the Gerber-Shiu functions $\phi(u)$, $\phi_1(u)$ satisfy the following integro-differential equation system*

$$\begin{aligned}
\frac{\sigma^2}{2}\phi''(u) + c\phi'(u) - (\lambda+\delta)\phi(u) =& -\lambda\theta\Big(\int_0^u \phi(u-x)dF_2(x) + w_2(u)\Big) \\
& -\lambda(1-\theta)\Big(\int_0^u \phi_1(u-x)dF_1(x) + w_1(u)\Big), \\
\frac{\sigma^2}{2}\phi_1''(u) + c\phi_1'(u) - (\lambda+\delta)\phi_1(u) =& -\lambda\theta\Big(\int_0^u \phi(u-x)dF_3(x) + w_3(u)\Big) \\
& -\lambda(1-\theta)\Big(\int_0^u \phi_1(u-x)dF_2(x) + w_2(u)\Big),
\end{aligned} \tag{4}$$

where $w_k(x) = \int_x^\infty w(x, y-x) f_k(y)\, dy = \int_0^\infty w(x,y) f_k(x+y)\, dy$, $k = 1,2,3.$

Taking the LT on both sides of the two equations in (4) and solving the resulting system w.r.t. $\widehat{\phi}(s)$, $\widehat{\phi}_1(s)$ we get that

$$\widehat{\phi}(s) = \frac{\widehat{B}(s)}{\left(\frac{\sigma^2}{2}s^2 + cs - \lambda - \delta\right)^2 + \lambda\widehat{f}_2(s)\left(\frac{\sigma^2}{2}s^2 + cs - \lambda - \delta\right)}$$

$$\widehat{\phi}_1(s) = \frac{\widehat{B}_1(s)}{\left(\frac{\sigma^2}{2}s^2 + cs - \lambda - \delta\right)^2 + \lambda \widehat{f}_2(s)\left(\frac{\sigma^2}{2}s^2 + cs - \lambda - \delta\right)},$$

where

$$\begin{aligned}
\widehat{B}(s) &= \Big(\frac{\sigma^2}{2}s^2 + cs - \lambda - \delta + \lambda(1-\theta)\widehat{f}_2(s)\Big)\Big(\frac{\sigma^2}{2}(s+\phi'(0)) + c - \widehat{w}(s)\Big) \\
&\quad -\lambda(1-\theta)\widehat{f}_1(s)\Big(\frac{\sigma^2}{2}(s+\phi_1'(0)) + c - \widehat{w}^*(s)\Big), \\
\widehat{B}_1(s) &= \Big(\frac{\sigma^2}{2}s^2 + cs - \lambda - \delta + \lambda\theta\widehat{f}_2(s)\Big)\Big(\frac{\sigma^2}{2}(s+\phi_1'(0)) + c - \widehat{w}^*(s)\Big) \\
&\quad -\lambda\theta\widehat{f}_3(s)\Big(\frac{\sigma^2}{2}(s+\phi'(0)) + c - \widehat{w}(s)\Big),
\end{aligned}$$

with $\widehat{w}(s) = \lambda\big(\theta\widehat{w}_2(s) + (1-\theta)\widehat{w}_1(s)\big)$ and $\widehat{w}^*(s) = \lambda\big(\theta\widehat{w}_3(s) + (1-\theta)\widehat{w}_2(s)\big)$.

To complete the solution of $\widehat{\phi}(s)$ and $\widehat{\phi}_1(s)$ in Eq. (5) we need to determine the initial values $\phi'(0)$ and $\phi_1'(0)$.

By using standard arguments, it can be proved that $\phi'(0)$ and $\phi_1'(0)$ are given by

$$\phi'(0) = \sum_{i=1}^{2}(-1)^{i+1}\widehat{f}_1(r_{i-(-1)^i})\frac{[y(r_i)\big(b_i - \frac{2}{\sigma^2}\widehat{w}(r_i)\big) - \lambda(1-\theta)\widehat{f}_1(r_i)\big(b_i - \frac{2}{\sigma^2}\widehat{w}^*(r_i)\big)}{y(r_2)\widehat{f}_1(r_1) - y(r_1)\widehat{f}_1(r_2)}, \quad (5)$$

$$\phi_1'(0) = y(r_2)\frac{b_2 - \frac{2}{\sigma^2}\widehat{w}(r_2) + \phi'(0)}{\lambda(1-\theta)\widehat{f}_1(r_2)} - \big(b_2 - \frac{2}{\sigma^2}\widehat{w}^*(r_2)\big), \quad (6)$$

where $y(s) = \frac{\sigma^2}{2}s^2 + cs - \lambda - \delta + \lambda(1-\theta)\widehat{f}_2(s)$, $b_i = r_i + \frac{2c}{\sigma^2}$, $i = 1, 2$, and $r_i(\delta) \equiv r_i$, $i = 1, 2$ are the only two roots in the right-half complex plane of the characteristic equation

$$\big(\frac{\sigma^2}{2}s^2 + cs - \lambda - \delta\big)^2 + \lambda\widehat{f}_2(s)\big(\frac{\sigma^2}{2}s^2 + cs - \lambda - \delta\big) = 0. \quad (7)$$

Since we are interested in the initial/ultimate Gerber-Shiu function, $\phi(u)$, using the roots of the characteristic equation and the Lagrange interpolation formula it can be proved that $\phi(u)$ satisfies a defective renewal equation.

Theorem 2. *For $u \geq 0$, the Gerber-Shiu function $\phi(u)$ satisfies the following defective renewal equation*

$$\boxed{\phi(u) = \frac{1}{1+\xi}\int_0^u \phi(u-x)dG(x) + h(u),} \tag{8}$$

where $G(x) = (1+\xi)\int_0^x g(y)dy$ is a proper d.f. with $g(y) = \frac{\lambda}{\sigma^2/2}\big(m_2 \star T_{r_2} f_2\big)(y)$, $m_k(y) = e^{-b_k y}$, $k = 1, 2$, ξ is such that $\frac{1}{1+\xi} = \int_0^\infty g(y)dy = 1 - \frac{\delta}{\frac{\sigma^2}{2} r_2 b_2} < 1$,

$$\begin{aligned}
h(u) &= \frac{1}{\sigma^2/2}\big(m_2 \star T_{r_2} w\big)(u) + e^{-b_2 u} + \frac{\lambda(1-\theta)}{(\sigma^2/2)^2}\sum_{k=1}^{2}\frac{\big(m_k \star T_{r_k}\eta\big)(u)}{\prod_{j=1, j\neq k}^{2}(r_k - r_j)(r_k + b_j)},\\
T_{r_k}\eta(u) &= \frac{\sigma^2}{2} b_k\big(T_{r_k} f_1(u) - T_{r_k} f_2(u)\big) - \frac{\sigma^2}{2}\big(f_1(u) - f_2(u)\big) + \big(T_{r_k} A_2(u) - T_{r_k} A_1(u)\big)\\
&\quad + \frac{\sigma^2}{2}\big(\phi_1'(0) T_{r_k} f_1(u) - \phi'(0) T_{r_k} f_2(u)\big),
\end{aligned}$$

with

$$T_r f(x) = \int_x^\infty e^{-r(y-x)} f(y)dy, \qquad x \geq 0, \Re(r) > 0.$$

the Dickson-Hipp integral operator and $A_1(u) = \big(f_1 \star w^*\big)(u)$, $A_2(u) = \big(f_2 \star w\big)(u)$ ($\star$ denotes the convolution operator).

Remark 1. *When $\theta = 1$, in any time period the main claim and the by-claim occur simultaneously, and the delayed risk model given in Eqs.* (1)-(2) *is reduced to the classical perturbed by diffusion compound Poisson risk model with claim amounts $\{X_i + Y_i\}_{i=1}^{\infty}$. In this case, Eq.* (8) *is simplified to to the defective renewal equation for the classical risk model perturbed by diffusion, see Eq. (2.10) of Tsai and Willmot (2002) and Eq. (17) of Gerber and Landry (1998).*

The solution of the previous defective renewal equation is given in terms an associated compound geometric d.f. For that reason we define $K(u) = 1 - \overline{K}(u)$ by

$$\overline{K}(u) = \frac{\xi}{1+\xi} \sum_{n=1}^{\infty} \left(\frac{1}{1+\xi}\right)^n \overline{G}^{\star n}(u), \qquad u \geq 0,$$

where $\overline{G}^{\star n}(u)$ is the tail of the n-fold convolution of $\overline{G}(u) = 1 - G(u)$ with itself. Then, an explicit expression for the solution of the defective renewal equation (8) can be derived by applying Theorem 2.1 of Lin and Willmot (1999).

Proposition 1. *For $u \geq 0$, the Gerber-Shiu function $\phi(u)$ satisfying the defective renewal equation* (8)*, can be expressed as*

$$\boxed{\phi(u) = \frac{1+\xi}{\xi} \int_0^u h(u-x)dK(x) + h(u).} \tag{9}$$

The d.f. of the associated compound geometric distribution, when the main claims and the by-claims are rationally distributed, i.e.

$$\widehat{f}_1(s) = \frac{p_{1,k_1-1}(s)}{p_{1,k_1}(s)}, \quad p_{1,k_1-1}(0) = p_{1,k_1}(0), \quad \text{and} \quad \widehat{q}(s) = \frac{p_{2,k_2-1}(s)}{p_{2,k_2}(s)}, \quad p_{2,k_2-1}(0) = p_{2,k_2}(0), \tag{10}$$

(this is a very wide class of distributions, containing among others the exponential, the Erlang, the Coxian, the phase-type distribution, as well as and the mixtures of them) is be is given by

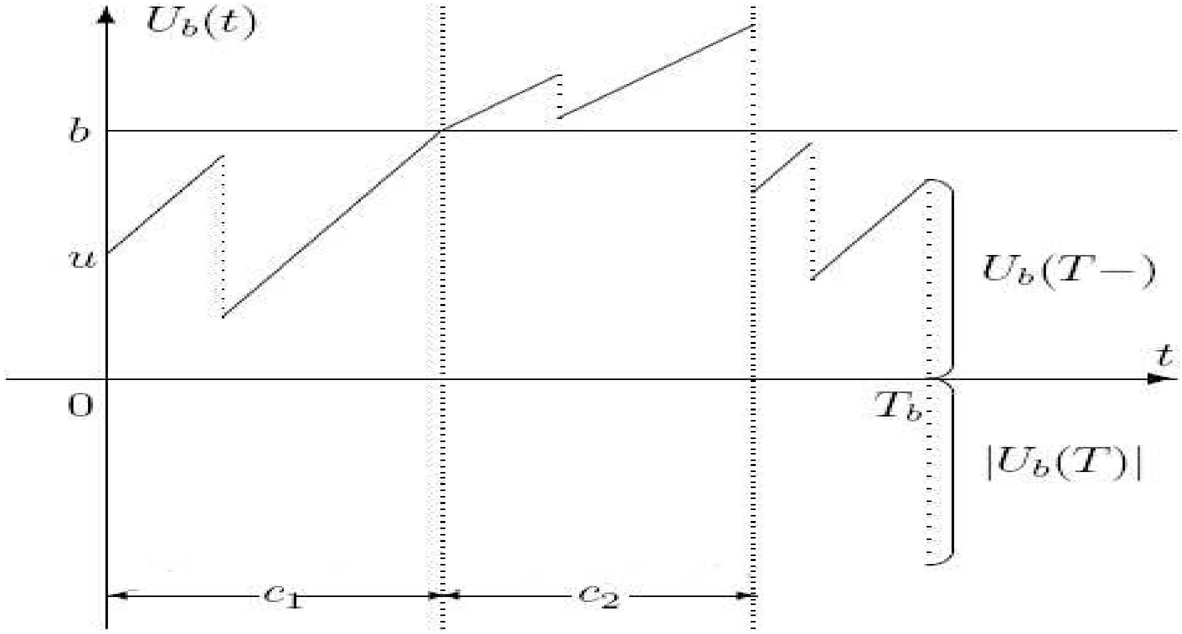
$$K(u) = 1 - \sum_{j=1}^{k+1} a_j e^{-R_j u}, \quad a_j = \frac{R_1 \cdots R_{k+1}(b_2 - R_j)}{R_j \prod_{k=1,k\neq j}^{k+1}(R_k - R_j)} p_k(-R_j) \frac{1}{p_k(0) b_{2,i}}, \tag{11}$$

where $-R_j$ with $\Re(R_j) > 0$, $j = 1, \ldots, k+1$, are the roots of Eq. $J_{k+2}(s) = 0$ with $J_{k+2}(s) = \left(\frac{\sigma^2}{2} s^2 + cs - \lambda - \delta\right) p_k(s) + \lambda p_{k-1}(s)$.

The multi-layer dividend strategy

Now, similar to Albrecher and Hartinger (2007), Lin and Sendova (2008), Yang and Zhang (2008, 2009a, 2009b), we consider the following modification:

• Let a multi-layer dividend strategy with n-layers, $0 = \beta_0 < \beta_1 < \cdots < \beta_n < \beta_{n+1} = \infty$, $0 = \beta_0$, $\beta_{n+1} = \infty$.



• We assume that whenever the surplus process is in the layer i (i.e., between two successive layers β_{i-1} and β_i, $i = 1, \ldots, n+1$) the insurer pays dividends to the shareholders at rate d_i and the corresponding net premium rate (within the i-th layer) is $c_i = c - d_i$, $i = 1, \ldots, n+1$, where $c = c_1 > \cdots > c_n > c_{n+1} \geq 0$.

• Let $\boldsymbol{\beta} = \{\beta_0, \beta_1, \ldots, \beta_n, \beta_{n+1}\}$ and $\{U_\beta(t)\}_{t\geq 0}$ to be the surplus process at time t, with initial surplus $U_\beta(0) = u$, where its dynamics are given by

$$dU_\beta(t) = c_i dt - dS(t) + \sigma_i dB(t), \qquad \beta_{i-1} \leq U_\beta(t) \leq \beta_i, \tag{12}$$

where $\{B(t)\}_{t=0}^{\infty}$ is a standard Brownian motion with mean 0 and dispersion parameter σ_i (within the i-th layer) and $S(t)$ is defined exactly as before [i.e. in the delayed set up].

Let T_β to be the ruin time for this process and for $\delta \geq 0$, we define the Gerber-Shiu function as

$$\phi(u, \boldsymbol{\beta}) = \mathbb{E}\big(e^{-\delta T_\beta} w(U_\beta(T_\beta -), |U_b(T_\beta)|) 1_{(T_\beta < \infty)} \big| U_\beta(0) = u\big) = \begin{cases} \phi_1(u), & 0 \leq u < b_1 \\ \quad \vdots & \\ \phi_n(u), & b_n \leq u < \infty \end{cases}, \tag{13}$$

with $\phi(0, \boldsymbol{\beta}) = 1$ due to oscillating sample paths of $U_{\boldsymbol{\beta}}(t)$. Here δ and $w(x, y)$ have the same definitions and interpretation as before, $U_{\boldsymbol{\beta}}(T_{\boldsymbol{\beta}}-)$ is the surplus immediately before ruin, $|U_{\boldsymbol{\beta}}(T_{\boldsymbol{\beta}})|$ is the deficit at ruin and $T_{\boldsymbol{\beta}}-$ is the left limit of $T_{\boldsymbol{\beta}}$ for the modified surplus process $\{U_{\boldsymbol{\beta}}(t); t \geq 0\}$.

For the same reasons explained before, we consider an auxiliary process in which at T_1 instead of having simultaneously a main claim and a by-claim, another by-claim is added to the first epoch. Then, the corresponding Gerber-Shiu function for this auxiliary process under the multi-layer dividend modification is defined as

$$\phi_1(u, \boldsymbol{\beta}) = \begin{cases} \phi_{1,1}(u), & 0 \leq u < b_1 \\ \vdots & \\ \phi_{1,n}(u), & b_n \leq u < \infty \end{cases}, \tag{14}$$

Theorem 3. *For $\beta_{i-1} < u < \beta_i$, $i = 1, \ldots, n+1$, the Gerber-Shiu functions $\phi_i(u)$, $\phi_{1,i}(u)$ satisfy the following integro-differential equations system*

$$\frac{{\sigma_i}^2}{2}\phi_i''(u) + c_i\phi_i'(u) - (\lambda+\delta)\phi_i(u) = -\lambda\theta\Big(\int_0^{u-\beta_{i-1}} \phi_i(u-x)dF_2(x) + \xi_{2,i}(u)\Big)$$
$$- \lambda(1-\theta)\Big(\int_0^{u-\beta_{i-1}} \phi_{1,i}(u-x)dF_1(x) + \zeta_{1,i}(u)\Big),$$

$$\frac{{\sigma_i}^2}{2}\phi_{1,i}''(u) + c_i\phi_{1,i}'(u) - (\lambda+\delta)\phi_{1,i}(u) = -\lambda\theta\Big(\int_0^{u-\beta_{i-1}} \phi_i(u-x)dF_3(x) + \xi_{3,i}(u)\Big)$$

$$- \lambda(1-\theta)\Big(\int_0^{u-\beta_{i-1}} \phi_{1,i}(u-x)dF_2(x) + \zeta_{2,i}(u)\Big),$$

with boundary conditions

$$\phi(0,\boldsymbol{\beta}) = \phi_1(0) = 1, \qquad \phi_1(0,\boldsymbol{\beta}) = \phi_{1,1}(0) =, 1$$

$$\phi_{i-1}(\beta_{i-1}-) = \phi_i(\beta_{i-1}+), \qquad \phi_{1,i-1}(\beta_{i-1}-) = \phi_{1,i}(\beta_{i-1}+), \quad i = 2, \ldots, n+1,$$

$$\frac{\sigma_{i-1}^2}{2}\phi_{i-1}''(\beta_{i-1}-) + c_{i-1}\phi_{i-1}'(\beta_{i-1}-) = \frac{\sigma_i^2}{2}\phi_i''(\beta_{i-1}+) + c_i\phi_i'(\beta_{i-1}+),\ i = 2, \ldots, n+1,$$

$$\frac{\sigma_{i-1}^2}{2}\phi_{1,i-1}''(\beta_{i-1}-) + c_{i-1}\phi_{i-1}'(\beta_{i-1}-) = \frac{\sigma_i^2}{2}\phi_{1,i}''(\beta_{i-1}+) + c_i\phi_{1,i}'(\beta_{i-1}+),\ i = 2, \ldots, n+1,$$

$$\lim_{u\to\infty}\phi(u,\boldsymbol{\beta}) = \lim_{u\to\infty}\phi_{n+1}(u) = 0, \quad \lim_{u\to\infty}\phi_1(u,\boldsymbol{\beta}) = \lim_{u\to\infty}\phi_{1,n+1}(u) = 0,$$

where $w_j(u)$, $j = 1, 2, 3$ *defined as before and*

$$\xi_{j,i}(u) = \sum_{k=1}^{i-1}\int_{u-\beta_k}^{u-\beta_{k-1}} \phi_k(u-x)dF_j(x) + w_j(u), \qquad j = 2, 3,$$

$$\zeta_{j,i}(u) = \sum_{k=1}^{i-1}\int_{u-\beta_k}^{u-\beta_{k-1}} \phi_{1,k}(u-x)dF_j(x) + w_j(u), \qquad j = 1, 2,$$

A system Volterra-type integral equations of second kind

To solve the second order non-homogeneous integro-differential equation system of Theorem 3 we consider a more general second order non-homogeneous system.
For $u \geq \beta$, let $\Phi(u)$ and $\Phi_1(u)$ satisfy the following non-homogeneous system of integro-differential equations

$$\begin{aligned}
\frac{\sigma^2}{2}\Phi''(u) + c\Phi'(u) - (\lambda+\delta)\Phi(u) &= -\lambda\Big(\int_0^{u-\beta} \Phi(u-x)dF_2(x) + \xi_2(u)\Big) \\
&\quad - \lambda(1-\theta)\Big(\int_0^{u-\beta} \Phi_1(u-x)dF_1(x) + \zeta_1(u)\Big), \\
\frac{\sigma^2}{2}\Phi_1''(u) + c\Phi_1'(u) - (\lambda+\delta)\Phi_1(u) &= -\lambda\Big(\int_0^{u-\beta} \Phi(u-x)dF_3(x) + \xi_3(u)\Big) \\
&\quad - \lambda(1-\theta)\Big(\int_0^{u-\beta} \Phi_1(u-x)dF_2(x) + \zeta_2(u)\Big),
\end{aligned}$$

where $\xi_i(u)$, for $i = 2, 3$, and $\zeta_i(u)$ for $i = 1, 2$, are some arbitrary integrable functions representing the non-homogeneous terms.

Now, changing the variable $u - \beta = x$ and setting $\Lambda(x) = \Phi(u)$, $\Lambda_1(x) = \Phi_1(u)$, $\xi_{j,\beta}(x) = \xi_j(x+\beta)$ for $j = 2, 3$ and $\zeta_{j,\beta}(x) = \zeta_j(x+\beta)$ for $j = 1, 2$, the above non-homogeneous system of integro-differential equations become, for $x \geq 0$,

$$
\begin{aligned}
\frac{\sigma^2}{2}\Lambda''(x) + c\Lambda'(x) - (\lambda+\delta)\Lambda(x) =& -\lambda\theta\Big(\int_0^x \Lambda(x-y)dF_2(y) + \xi_{2,\beta}(x)\Big)\\
& -\lambda(1-\theta)\Big(\int_0^x \Lambda_1(x-y)dF_1(y) + \zeta_{1,\beta}(x)\Big),\\
\frac{\sigma^2}{2}\Lambda_1''(x) + c\Lambda_1'(x) - (\lambda+\delta)\Lambda_1(x) =& -\lambda\theta\Big(\int_0^x \Lambda(x-y)dF_3(y) + \xi_{3,\beta}(x)\Big)\\
& -\lambda(1-\theta)\Big(\int_0^x \Lambda_1(x-y)dF_2(y) + \zeta_{2,\beta}(x)\Big).
\end{aligned} \tag{15}
$$

Remark 2. *The usual approach for solving the above second order non-homogeneous integro-differential equation system is to fi nd a particular solution plus two linearly independent solutions to the associate (to* (15)*) homogeneous intgero-differential equation system. BUT, one can prove that the later homogeneous integro-differential equation system has always a trivial solution and thus there are no exist two linearly independent solutions.*

- Hence, we transform the second order non-homogeneous system in (15) into a Volterra-type integral equation.

Using LTs and the results on theory of defective renewal equations in the absence of multilayer strategy we get the following solution.

Theorem 4. *For* $x \geq 0$*, the general solution to the non-homogeneous integro-differential equations system* (15) *may be expressed as*

$$\boxed{\begin{aligned} \Lambda(x) &= \Lambda(0) + x\Lambda'(0) + \int_0^x (x-t)l(t)dt \\ \Lambda_1(x) &= \Lambda_1(0) + x\Lambda_1'(0) + \int_0^x (x-t)l_1(t)dt, \end{aligned}} \tag{16}$$

with

$$l(x) = \frac{1+\xi}{\xi}\int_0^x v(x-t)dK(t) + v(x), \quad l_1(x) = \frac{1+\xi}{\xi}\int_0^x v_1(x-t)dK(t) + v_1(x), \tag{17}$$

$K(u) = 1 - \overline{K}(u)$ *the d.f. of the associated compound geometric distribution,* ξ *some constants that can determined,* $v(x)$ $v_1(x)$ *some real valued functions in terms of* $w_A(x) = \lambda\big(\theta\xi_{2,\beta}(x) + (1-\theta)\zeta_{1,\beta}(x)\big)$*,* $w_B(x) = \lambda\big(\theta\xi_{3,\beta}(x) + (1-\theta)\zeta_{2,\beta}(x)\big)$*, and also of the initial values* $\Lambda(0)$*,* $\Lambda'(x)$*,* $\Lambda_1(0)$*,* $\Lambda_1'(0)$*.*

Theorem 5. *(i) For $\beta_{i-1} \leq u < \beta_i$, $i = 1, \ldots, n+1$, the Gerber-Shiu functions $\phi_i(u)$ and $\phi_{1,i}(u)$ can be calculated recursively as:*

$$\boxed{\phi_i(u) = \Phi_i(u) = \Lambda_i(u - \beta_{i-1}), \quad \textit{and} \quad \phi_{1,i}(u) = \Phi_{1,i}(u) = \Lambda_{1,i}(u - \beta_{i-1}),} \tag{18}$$

where $\Lambda_i(u)$ and $\Lambda_{1,i}(u)$ are defined by the two Eqs in (16) with the help of (17), with the added index i whenever appropriate. In particular, β is replaced by β_{i-1} and $\xi_{j,\beta}(u)$, $j = 2, 3$, and $\zeta_{j,\beta}(u)$, $j = 1, 2$, in Theorem 4 are replaced by some recursive function in the of the $\sum_{k=1}^{i-1} \int_{u+\beta_{i-1}-\beta_k}^{u+\beta_{i-1}-\beta_{k-1}} \phi_k(u + \beta_{i-1} - x) dF_j(x) + w_j(u + \beta_{i-1})$, $j = 2, 3$ and $\sum_{k=1}^{i-1} \int_{u+\beta_{i-1}-b_k}^{u+b_{i-1}-\beta_{k-1}} \phi_{1,k}(u + \beta_{i-1} - x) dF_j(x) + w_j(u + \beta_{i-1})$, $j = 1, 2$, respectively.

(ii) The initial values $\Lambda_i(0)$, $\Lambda_{1,i}(0)$, $\Lambda_i'(0)$ and $\Lambda_{1,i}'(0)$, for $i = 1, \ldots, n+1$, are uniquely determined by the initial and boundary conditions of Theorem 3 (by solving a $4n + 4$ linear equation system).

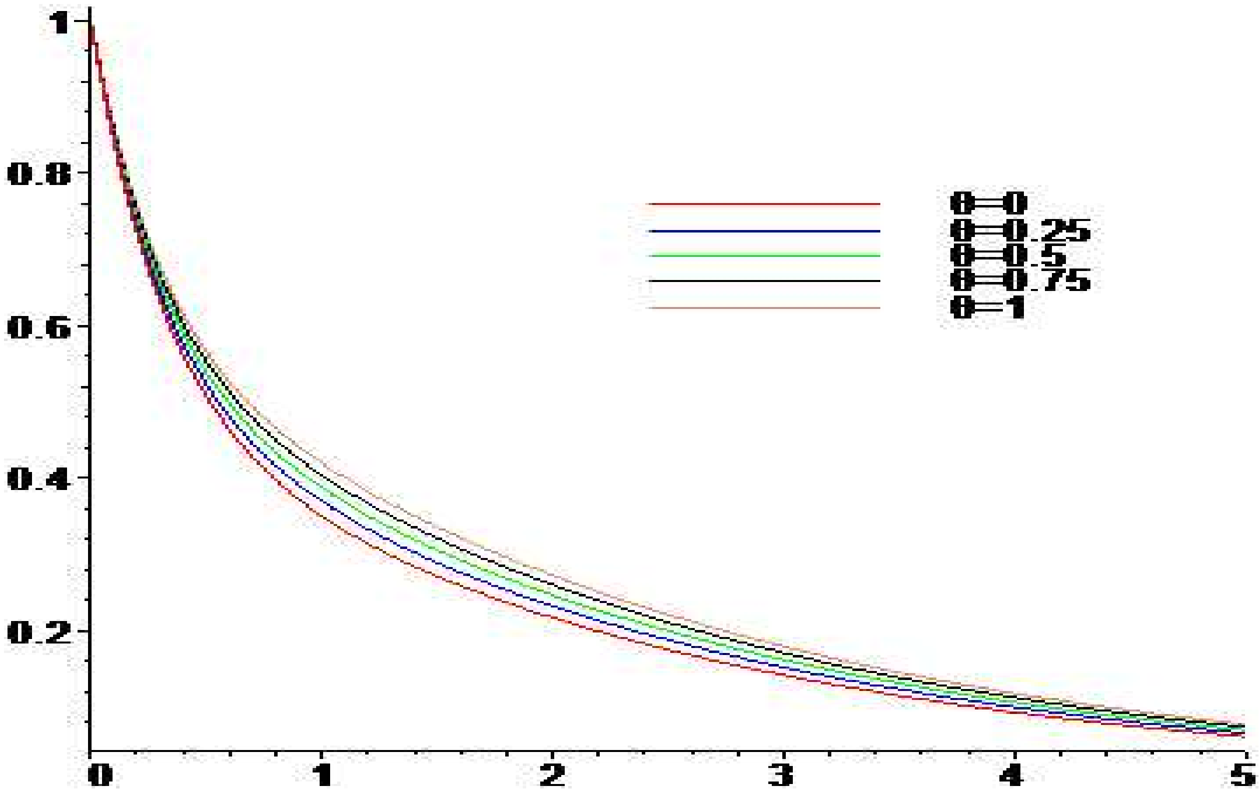
An example

Remark 3. *Note that from Theorems 4 and 5, the solution of $\phi_i(u)$ and $\phi_{1,i}(u)$ depend on the d.f. of the same associated compound geometric distribution as in dividend free environment (with the added index i whenever appropriate).*

To illustrate the applicability of our results, let

- $\delta = 0$ and $w(x, y) = 1$,
- $\phi(u, \boldsymbol{\beta})$ and $\phi_1(u, \boldsymbol{\beta})$ are reduced to the ruin probabilities, say $\psi(u, \boldsymbol{\beta})$ and $\psi_1(u, \boldsymbol{\beta})$, respectively,
- also we assume the existence of three layers $[0, \beta_1)$, $[\beta_1, \beta_2)$ and $[\beta_2, \infty)$, i.e. $\boldsymbol{\beta} = \{0, \beta_1, \beta_2, \infty\}$,
- and the main claim and sub-claim amounts are exponential distributed with parameters α_1 and α_2, respectively

Considering the following set of parameters $c_1 = 3.3$, $c_2 = 3.1$, $c_3 = 2.5$, $\sigma_1 = 1.5$, $\sigma_2 = 1.3$, $\sigma_3 = 1$, $a_1 = 1$, $a_2 = 2$, $\lambda = 1$, $\theta = 0.5$, $\beta_1 = 2$, and $\beta_2 = 3$, the ruin probabilities can be seen graphically by the following plot



From the above figure, we see that the ruin probability $\psi(u, \boldsymbol{\beta})$ decreases as the probability of the by-claim increases.

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THANK YOU !