Dr. Anna Pratoussevitch

Exercises 1

Submit your solutions to questions marked with **[HW]** in the lecture on Monday 30/09/2013. Questions or parts of questions without **[HW]** are for practice and will not be marked. This week there are 3 **[HW]** questions.

Problem 1

Find the Cartesian form $x + iy$ of the complex numbers

$$\text{(i) } e^{4\pi i/3}, \quad \text{(ii) } \frac{1+2i}{2+i}, \quad \text{(iii) } [\mathbf{HW}]\ (1+i)^{-5}.$$

Problem 2

Find the polar form $re^{i\theta}$ of the complex number

$$\text{(i) } -1+i, \quad \text{(ii) } 1+i\sqrt{3}, \quad \text{(iii) } \frac{-1+i}{1+i\sqrt{3}}, \quad \text{(iv) } \left(\cos\frac{1}{2} - i\sin\frac{1}{2}\right)^2, \quad \text{(v) } [\mathbf{HW}]\ ie^{\pi i/6}.$$

Problem 3

Find the modulus of the complex number $\frac{8+i}{4-7i}$.

Problem 4

Solve the quadratic equation $z^2 - 4iz + 5 = 0$.

Problem 5

For which complex numbers z is it true that $z^2 = |z|^2$?

Problem 6

Make a sketch and give a geometrical description of each of the following sets. Explain clearly which points on the boundary are included in the set and which are not included. For each set state whether it is open, closed, both or neither. Give brief reasons for your answers (geometrical answers are acceptable).

(i) $\{z \in \mathbb{C} : |z - i| = |z + i|\}$;

(ii) $\{z \in \mathbb{C} : |z - i| \neq |z + i|\}$;

(iii) $\{z \in \mathbb{C} : |z| < 2 \text{ and } \operatorname{Re}(z) > 0\}$;

(iv) $\{z \in \mathbb{C} : 1 \leqslant |z| < 2\}$;

(v) The domain of the function f defined by $f(z) = \frac{z}{z^2 - 4iz + 5}$;

(vi) **[HW]** $\{z \in \mathbb{C} : |z| > 2 \text{ and } |z - 2| < 3\}$.

Challenge Problems

Problem 7

This problem demonstrates one of many applications of complex numbers to geometry: Show that, for any complex numbers a and b,

$$|a-b|^2 + |a+b|^2 = 2(|a|^2 + |b|^2).$$

Interpret this result geometrically in terms of the lengths of the sides and diagonals of a parallelogram.

Problem 8

Make a sketch and give a geometrical description of the following set:

$$\{z \in \mathbb{C} : |z|^2 \geqslant z + \bar{z}\}.$$

Explain clearly which points on the boundary are included in the set and which are not included. State whether the set is open, closed, both or neither. Give brief reasons for your answers (geometrical answers are acceptable).

Solutions to Practice Problems

Solution to Problem 1: The Cartesian forms $x + iy$ are:

(i) $e^{4\pi i/3} = \cos(4\pi/3) + i\sin(4\pi/3) = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$;

(ii)

$$\frac{1+2i}{2+i} = \frac{(1+2i)(2-i)}{(2+i)(2-i)} = \frac{4+3i}{|2+i|^2} = \frac{4+3i}{5} = \frac{4}{5} + \frac{3}{5}i.$$

Solution to Problem 2:

(i) For $z = -1 + i$ we have $z = x + iy$ with $x = -1$, $y = 1$. The modulus of z is

$$r = \sqrt{x^2 + y^2} = \sqrt{(-1)^2 + 1^2} = \sqrt{2}.$$

The point z is to the left of the imaginary axis ($x < 0$), see Figure 1, hence the argument of z is

$$\theta = \pi + \tan^{-1}\left(\frac{y}{x}\right) = \pi + \tan^{-1}(-1) = \pi + \left(-\frac{\pi}{4}\right) = \frac{3\pi}{4}.$$

The polar form is $-1 + i = re^{i\theta} = \sqrt{2}e^{3\pi i/4}$.

(ii) For $z = 1 + i\sqrt{3}$ we have $z = x + iy$ with $x = 1$, $y = \sqrt{3}$. The modulus of z is

$$r = \sqrt{x^2 + y^2} = \sqrt{1^2 + (\sqrt{3})^2} = \sqrt{4} = 2.$$

The point z is to the right of the imaginary axis ($x > 0$), see Figure 1, hence the argument of z is

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}.$$

The polar form is $1 + i\sqrt{3} = re^{i\theta} = 2e^{\pi i/3}$.

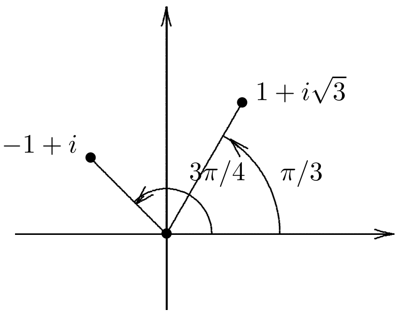

Figure 1: Problem 2

(iii) To find the polar form of $z = \frac{-1+i}{1+i\sqrt{3}}$, we first compute the polar forms of the numerator and denominator, $-1+i = \sqrt{2}e^{3\pi i/4}$ and $1+i\sqrt{3} = 2e^{\pi i/3}$, to obtain

$$z = \frac{-1+i}{1+i\sqrt{3}} = \frac{\sqrt{2}e^{3\pi i/4}}{2e^{\pi i/3}} = \frac{\sqrt{2}}{2}e^{3\pi i/4-\pi i/3} = \frac{1}{\sqrt{2}}e^{5\pi i/12}.$$

Another method would be to compute the cartesian form of this complex number first and then to find the polar form. In this example the second method would lead to more complicated computations than the first method.

(iv) To find the polar form of $z = (\cos(1/2) - i\sin(1/2))^2$, we first compute the polar form of $w = \cos(1/2) - i\sin(1/2)$. Remembering that for a real number t we have $e^{it} = \cos(t) + i\sin(t)$ and hence $e^{-it} = \cos(t) + i\sin(-t) = \cos(t) - i\sin(t)$, we recognise that $w = \cos(1/2) - i\sin(1/2) = e^{-i/2}$. If you do not see this immediately, you can find the polar form of w as follows: We have $w = x + iy$ with $x = \cos(1/2)$, $y = -\sin(1/2)$. The modulus of w is $r = \sqrt{x^2+y^2} = \sqrt{\cos^2(1/2)+\sin^2(1/2)} = 1$. The point w is to the right of the imaginary axis ($x > 0$), hence the argument of w is

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{-\sin(1/2)}{\cos(1/2)}\right) = \tan^{-1}(-\tan(1/2)) = -1/2.$$

The polar form is $w = re^{i\theta} = e^{-i/2}$. Using the polar form of w we compute

$$\left(\cos\frac{1}{2} - i\sin\frac{1}{2}\right)^2 = (e^{-\frac{i}{2}})^2 = e^{2\cdot\left(-\frac{i}{2}\right)} = e^{-i}.$$

Solution to Problem 3: To compute the modulus of the complex number $z = \frac{8+i}{4-7i}$ we use the fact that $|a/b| = |a|/|b|$ for any complex numbers a and b. We obtain

$$|z| = \left|\frac{8+i}{4-7i}\right| = \frac{|8+i|}{|4-7i|} = \frac{\sqrt{8^2+1^2}}{\sqrt{4^2+7^2}} = \frac{\sqrt{65}}{\sqrt{65}} = 1.$$

Another method would be to compute the cartesian form of $\frac{8+i}{4-7i}$ first,

$$z = \frac{8+i}{4-7i} = \frac{(8+i)(4+7i)}{(4-7i)(4+7i)} = \frac{25+60i}{4^2+7^2} = \frac{25}{65} + \frac{60}{65}i,$$

and then to find the modulus, $|z| = \sqrt{(25/65)^2 + (60/65)^2} = 1$. In this example the second method leads to more complicated computations than the first method.

Solution to Problem 4: The equation is of the form $z^2 + pz + q = 0$ with $p = -4i$ and $q = 5$. By a general formula for quadratic equations the solutions are

$$-\frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q} = -(-2i) \pm \sqrt{(-2i)^2 - 5} = 2i \pm \sqrt{-4-5} = 2i \pm \sqrt{-9} = 2i \pm 3i.$$

We get two solutions, $z = 5i$ or $z = -i$.

Solution to Problem 5:

Method 1: We have $z^2 = |z|^2 = z\bar{z}$, that is, $0 = z^2 - z\bar{z} = z(z - \bar{z})$. The equation $z(z-\bar{z}) = 0$ implies that either $z = 0$ or $z - \bar{z} = 0$. The equation $z = \bar{z}$ is equivalent to $\mathrm{Im}(z) = 0$ which is satisfied exactly for those z which are on the real axis. Notice that $z = 0$ is also contained in the real axis. Hence the equation $z^2 = |z|^2$ holds if and only if z is real.

Method 2: Let $z = re^{i\theta}$ be the polar form of z. Then $z^2 = (re^{i\theta})^2 = r^2e^{2i\theta}$ and $|z|^2 = r^2$, hence $z^2 = |z|^2$ is equivalent to $r^2e^{2i\theta} = r^2$, i.e. $e^{2i\theta} = 1$. The latter equation is satisfied if and only if 2θ is a multiple of 2π, i.e. θ is a multiple of π. This is the case if and only if $z = re^{i\theta}$ is on the real axis. Hence the equation $z^2 = |z|^2$ holds if and only if z is real.

Solution to Problem 6:

(i) $S_1 = \{z \in \mathbb{C} : |z - i| = |z + i|\}$ is the set of points equidistant from i and $-i$, which coincides with the real axis. The set S_1 is not open: for any $z \in S_1$, there is no open disc centred at z contained in S_1. The set S_1 is closed since the complement of S_1 in $\mathbb{C}$ is the set $S_2 = \{z \in \mathbb{C} : |z - i| \neq |z + i|\}$, which is open (for the proof, see next subsection).

(ii) $S_2 = \{z \in \mathbb{C} : |z - i| \neq |z + i|\}$ is the complement in $\mathbb{C}$ of the real axis. The set S_2 is open: for any $z \in S_2$, the disc $D(z, |\mathrm{Im}(z)|)$ is in S_2. The set S_2 is not closed since the complement of S_2 in $\mathbb{C}$ is the set S_1, which is not open.

(iii) $S_3 = \{z \in \mathbb{C} : |z| < 2 \text{ and } \mathrm{Re}(z) > 0\}$ is the semi-circle obtained as the intersection of the disc $D(0, 2)$ and upper half-plane $\mathrm{Re}(z) > 0$. The set S_3 is open as an intersection of two open sets. The set S_3 is not closed, for example for $z = 2i \notin S_3$ any open disc centred at z intersects S_3.

(iv) $S_4 = \{z \in \mathbb{C} : 1 \leqslant |z| < 2\}$ is the ring with inner radius 1 and outer radius 2. Note that the inner circle $S(0, 1)$ is included in the set S_4, while the outer circle $S(0, 2)$ is not included. The set S_4 is not open, for example for $z = 1 \in S_4$ there is no open disc centred at z contained in S_4. The set S_4 is not closed, for example for $z = 2 \notin S_4$ any open disc centred at z intersects S_4.

(v) Solving the equation $z^2 - 4iz + 5 = 0$ we get $z = 5i$ or $z = -i$ (see Problem 4). So the function $f = \frac{z}{z^2-4iz+5}$ has domain $S_5 = \mathbb{C}\backslash\{5i, -i\}$. The set S_5 is open. Indeed, for any point $z \in S_5$, the set S_5 contains the open disc $D(z, \delta)$, where $\delta = \min\{|z - 5i|, |z + i|\}$. (Similarly, the complement of any *finite* subset of $\mathbb{C}$ is open.) The set S_5 is not closed, for example for $z = 5i \notin S_5$ any open disc centred at z intersects S_5.

Dr. Anna Pratoussevitch

Exercises 3

Submit your solutions to questions marked with [**HW**] in the lecture on Monday 14/10/2013. Questions or parts of questions without [**HW**] are for practice and will not be marked. This week there are 3 [**HW**] questions.

Problem 1

(a) Let f be defined on an open set S and let $f(x+iy) = u(x,y) + iv(x,y)$, where x, y, u, v are real. Write down the Cauchy-Riemann equations which hold if f is holomorphic in S.

(b) Show that function $f(z) = |z|^2 - \bar{z}^2$ satisfies the Cauchy-Riemann equations only at $z = 0$.

(c) Find all holomorphic functions on $\mathbb{C}$ with imaginary part $v(x,y) = 2xy + x$.

(d) Verify that the function $u(x,y) = e^y \cos x$ satisfies the Laplace equation $u_{xx} + u_{yy} = 0$. Find all holomorphic functions on $\mathbb{C}$ with real part $u(x,y)$.

(e) For which values of z does the function $f(z) = (z - \bar{z})(z - 1)$ satisfy the Cauchy-Riemann equations? [**HW**]

(f) Find a constant k such that the function $v(x,y) = y^3 - 4xy + kx^2y$ can be the imaginary part of a holomorphic function f on $\mathbb{C}$. Find all such holomorphic functions f. [**HW**]

Problem 2

Using Cauchy-Riemann equations or otherwise, determine where the following functions are holomorphic and compute their derivatives:

(i) $\sin(z)$, (ii) $\cos(z)$, (iii) $\tan(z)$, (iv) $\cot(z)$, (v) [**HW**] e^z.

Problem 3

State where each of the following functions is holomorphic:

(i) $z^8 + 20z^5 - \pi z + 10^{45} z - 1$;

(ii) $\frac{e^z}{z(z-1)(z-2)}$;

(iii) $(z^5 - 1)^{-5}$.

Challenge Problems

Problem 4

Let $f(z) = e^{\bar{z}}$ for all $z \in \mathbb{C}$. Find the real and imaginary parts u and v of the function $f : \mathbb{C} \to \mathbb{C}$ and determine whether or not f is holomorphic on $\mathbb{C}$.

P.T.O.

Problem 5

In the lectures we proved that if a function is differentiable, then it satisfies the Cauchy-Riemann equations. However, in general, a function which satisfies the Cauchy-Riemann equations does not need to be differentiable. In this problem we discuss an example of such a function: Let $f(z) = z^5/|z|^4$ for $z \neq 0$ and let $f(0) = 0$.

(i) Show that $f(h)/h$ does not have a limit as $h \to 0$. Deduce that $f'(0)$ does not exist.

(ii) Let $u = \mathrm{Re}(f)$ and $v = \mathrm{Im}(f)$. Show that $u(x,0) = x$, $v(0,y) = y$, $u(0,y) = v(x,0) = 0$ for all real x and y (including 0).

(iii) Deduce that the partial derivatives of u and v exist at $(0,0)$. Check that the Cauchy-Riemann equations hold at $(0,0)$. *[Hence Cauchy-Riemann equations do not imply that f' exists.]*

Problem 6

Verify that, for every positive integer r,

$$(1+2z)(1-2z+4z^2-\cdots+(-1)^r(2z)^r) = 1+(-1)^r(2z)^{r+1}.$$

Hence show that, if $|z| < \frac{1}{2}$, then $\sum\limits_{n=0}^{\infty}(-1)^n(2z)^n$ is convergent and

$$\sum_{n=0}^{\infty}(-1)^n(2z)^n = \frac{1}{1+2z}.$$

Assuming that term-by-term differentiation is valid, differentiate the above series. For $|z| < \frac{1}{2}$, find

$$\sum_{n=1}^{\infty}(-1)^n n(2z)^{n-1}.$$

Solutions to Practice Problems

Solution to Problem 1:

(a) If $f(x+iy) = u(x,y)+iv(x,y)$ is holomorphic in S, then the following Cauchy-Riemann equations hold:

$$u_x = v_y, \quad u_y = -v_x.$$

(b) For the function $f(z) = |z|^2 - \bar{z}^2$ we have

$$f(x+iy) = |x+iy|^2 - (x-iy)^2 = (x^2+y^2) - (x^2-y^2-2ixy) = 2y^2 + i\cdot 2xy,$$

that is, $u(x,y) = 2y^2$ and $v(x,y) = 2xy$. The partial derivatives are $u_x = 0$, $u_y = 4y$, $v_x = 2y$ and $v_y = 2x$. The Cauchy-Riemann equations provide $0 = u_x = v_y = 2x$ and $4y = u_y = -v_x = -2y$, hence $2x = 0$ and $4y = -2y$. The equation $2x = 0$ implies $x = 0$. The equation $4y = -2y$ implies $6y = 0$ and hence $y = 0$. Therefore the Cauchy-Riemann equations are satisfied if and only if $x = y = 0$, i.e. if and only if $z = x+iy$ is equal zero.

(c) Assume that the imaginary part of a holomorphic function on $\mathbb{C}$ is $v(x,y) = 2xy + x$. According to the Cauchy-Riemann equations $v_y = 2x = u_x$ and $u_y = -v_x = -2y - 1$.

Method 1: The equality $u_x = 2x$ implies that $u(x,y) = x^2 + g(y)$ for some function $g(y)$. Then $u_y(x,y) = g'(y)$. The equality $g'(y) = u_y(x,y) = -2y - 1$ implies that $g(y) = -y^2 - y + C$ for some $C \in \mathbb{R}$. Thus $u(x,y) = x^2 + g(y) = x^2 - y^2 - y + C$ for some $C \in \mathbb{R}$.

Method 2: The equality $u_y = -2y - 1$ implies that $u(x,y) = -y^2 - y + h(x)$ for some function $h(x)$. Then $u_x(x,y) = h'(x)$. The equality $h'(x) = u_x(x,y) = 2x$ implies that $h(x) = x^2 + C$ for some $C \in \mathbb{R}$. Thus $u(x,y) = -y^2 - y + h(x) = -y^2 - y + x^2 + C$ for some $C \in \mathbb{R}$.

Both methods give the same answer for $u(x,y)$. [It is sufficient to work out $u(x,y)$ using one of the methods, you do not need to use both methods.]

For $z = x + iy$ we have

$$\begin{aligned} f(z) &= u(x,y) + iv(x,y) = (x^2 - y^2 - y + C) + i(2xy + x) \\ &= (x^2 - y^2 + 2xyi) + (-y + ix) + C = z^2 + iz + C, \quad C \in \mathbb{R}. \end{aligned}$$

(d) Indeed $u_x = -e^y \sin x$, $u_{xx} = -e^y \cos x$, $u_y = e^y \cos x$, $u_{yy} = e^y \cos x$, hence

$$u_{xx} + u_{yy} = -e^y \cos x + e^y \cos x = 0.$$

To find a function f with $\mathrm{Re}(f) = u$ we consider the Cauchy-Riemann equations: $u_x = -e^y \sin x = v_y$ and $u_y = e^y \cos x = -v_x$.

Method 1: The equality $v_y = -e^y \sin x$ implies that $v(x,y) = -e^y \sin x + g(x)$ for some function $g(x)$. Then $v_x = -e^y \cos x + g'(x)$. The equality $-e^y \cos x + g'(x) = v_x = -e^y \cos x$ implies $g'(x) = 0$ and therefore $g(x) = C$ for some $C \in \mathbb{R}$. Thus $v(x,y) = -e^y \sin x + g(x) = -e^y \sin x + C$ for some $C \in \mathbb{R}$.

Method 2: The equality $v_x = -e^y \cos x$ implies that $v(x,y) = -e^y \sin x + h(y)$ for some function $h(y)$. Then $v_y = -e^y \sin x + h'(y)$. The equality $-e^y \sin x + h'(y) = v_y = -e^y \sin x$ implies $h'(y) = 0$ and therefore $h(y) = C$ for some $C \in \mathbb{R}$. Thus $v(x,y) = -e^y \sin x + h(y) = -e^y \sin x + C$ for some $C \in \mathbb{R}$.

Both methods give the same answer for $v(x,y)$. [It is sufficient to work out $v(x,y)$ using one of the methods, you do not need to use both methods.]

For $z = x + iy$ we have

$$\begin{aligned} f(z) &= u(x,y) + iv(x,y) = e^y \cos x + i(-e^y \sin x + C) \\ &= e^y(\cos x - i \sin x) + iC = e^y \cdot e^{-ix} + iC = e^{-iz} + iC, \quad C \in \mathbb{R}. \end{aligned}$$

Solution to Problem 2:

(i) For $f(z) = \sin(z)$ we have

$$f(x+iy) = \sin(x+iy) = \sin x \cosh y + i \cos x \sinh y,$$

hence the real and imaginary parts of $f(x+iy)$ are $u(x,y) = \sin x \cosh y$ and $v(x,y) = \cos x \sinh y$. We have $u_x = \cos x \cosh y = v_y$ and $u_y = \sin x \sinh y = -v_x$, so the partial derivatives are continuous and the Cauchy-Riemann equations hold for any z, hence the function is holomorphic on the whole $\mathbb{C}$. Using $f' = u_x + iv_x$ we have

$$f'(z) = (u_x + iv_x)(x,y) = \cos x \cosh y - i \sin x \sinh y = \cos(x+iy) = \cos(z).$$

(ii) For $f(z) = \cos(z)$ we have

$$f(x+iy) = \cos(x+iy) = \cos x \cosh y - i \sin x \sinh y,$$

hence the real and imaginary parts of $f(x+iy)$ are $u(x,y) = \cos x \cosh y$ and $v(x,y) = -\sin x \sinh y$. We have $u_x = -\sin x \cosh y = v_y$ and $u_y = \cos x \sinh y = -v_x$, so the partial derivatives are continuous and the Cauchy-Riemann equations hold for any z, hence the function is holomorphic on the whole $\mathbb{C}$. Using $f' = u_x + iv_x$ we have

$$f'(z) = (u_x + iv_x)(x,y) = -\sin x \cosh y - i \cos x \sinh y = -\sin(x+iy) = -\sin(z).$$

(iii) The function $\tan(z) = \sin(z)/\cos(z)$ is the quotient of two functions, $\sin(z)$ and $\cos(z)$, both of which are holomorphic on $\mathbb{C}$, hence the function is holomorphic where the denominator is not equal to zero, i.e. we have to exclude points where $\cos(z) = 0$, i.e. the points where $e^{iz} + e^{-iz} = 0$. But $e^{iz} + e^{-iz} = e^{-iz} \cdot (e^{2iz} + 1)$, hence $e^{iz} + e^{-iz} = 0$ if and only if $e^{2iz} = -1$. This means that $|e^{2iz}| = |-1| = 1$ and $\arg(e^{2iz}) = \arg(-1) = \pi + 2\pi k$, where $k \in \mathbb{Z}$. If $z = x + iy$, then

$$e^{2iz} = e^{2i(x+iy)} = e^{-2y+2xi} = e^{-2y}(\cos(2x) + i \sin(2x)),$$

hence $|e^{2iz}| = e^{-2y}$ and $\arg(e^{2iz}) = 2x$. The equality $1 = |e^{2iz}| = e^{-2y}$ implies $y = 0$. The equality $\pi + 2\pi k = \arg(e^{2iz}) = 2x$ implies $x = \frac{\pi}{2} + \pi k$, where $k \in \mathbb{Z}$. Hence for $z = x + iy$ we have $\cos(z) = 0$ if and only if $z = \frac{\pi}{2} + \pi k + i0$. Thus we see that the function $\tan(z)$ is holomorphic on $\{z \in \mathbb{C} : z \neq \frac{\pi}{2} + \pi k,\ k \in \mathbb{Z}\}$. We compute the derivative of $\tan(z)$ using the quotient rule

$$\tan'(z) = \left(\frac{\sin(z)}{\cos(z)}\right)' = \frac{\sin'(z)\cos(z) - \sin(z)\cos'(z)}{\cos^2(z)} = \frac{\cos^2(z) + \sin^2(z)}{\cos^2(z)} = \frac{1}{\cos^2(z)}.$$

(iv) The function $\cot(z) = \cos(z)/\sin(z)$ is the quotient of two functions, $\cos(z)$ and $\sin(z)$, both of which are holomorphic on $\mathbb{C}$, hence the function is holomorphic where the denominator is not equal to zero, i.e. we have to exclude points where $\sin(z) = 0$, i.e. the points where $e^{iz} - e^{-iz} = 0$. But $e^{iz} - e^{-iz} = e^{-iz} \cdot (e^{2iz} - 1)$, hence $e^{iz} - e^{-iz} = 0$ if and only if $e^{2iz} = 1$. This means that $|e^{2iz}| = |1| = 1$ and $\arg(e^{2iz}) = \arg(1) = 2\pi k$, where $k \in \mathbb{Z}$. If $z = x + iy$, then

$$e^{2iz} = e^{2i(x+iy)} = e^{-2y+2xi} = e^{-2y}(\cos(2x) + i \sin(2x)),$$

hence $|e^{2iz}| = e^{-2y}$ and $\arg(e^{2iz}) = 2x$. The equality $1 = |e^{2iz}| = e^{-2y}$ implies $y = 0$. The equality $2\pi k = \arg(e^{2iz}) = 2x$ implies $x = \pi k$, where $k \in \mathbb{Z}$. Hence for $z = x + iy$ we have $\sin(z) = 0$ if and only if $z = \pi k$. Thus we see that the function $\tan(z)$ is holomorphic on $\{z \in \mathbb{C} : z \neq \pi k,\ k \in \mathbb{Z}\}$. We compute the derivative of $\cot(z)$ using the quotient rule

$$\begin{aligned}\cot'(z) &= \left(\frac{\cos(z)}{\sin(z)}\right)' = \frac{\cos'(z)\sin(z) - \cos(z)\sin'(z)}{\sin^2(z)} \\ &= \frac{-\sin^2(z) - \cos^2(z)}{\sin^2(z)} = -\frac{1}{\sin^2(z)}.\end{aligned}$$

Solution to Problem 3:

(i) Constant functions and $f(z) = z$ are holomorphic on $\mathbb{C}$ and sums and products of functions holomorphic on $\mathbb{C}$ are holomorphic on $\mathbb{C}$, hence all polynomials are holomorphic on $\mathbb{C}$, including $z^8 + 20z^5 - \pi z + 10^{45}z - 1$.

(ii) The function $\frac{e^z}{z(z-1)(z-2)}$ is the quotient of two functions, e^z and $z(z-1)(z-2)$, both of which are holomorphic on $\mathbb{C}$, hence the function is holomorphic where the denominator is not equal to zero, i.e. on $\mathbb{C} \setminus \{0, 1, 2\}$.

(iii) The function $(z^5 - 1)^{-5}$ can be written as the quotient of two functions, 1 and $(z^5 - 1)^5$, both of which are holomorphic on $\mathbb{C}$, hence the function is holomorphic where the denominator is not equal to zero, is holomorphic on $\mathbb{C} \setminus \{1, e^{2\pi i/5}, e^{4\pi i/5}, e^{6\pi i/5}, e^{8\pi i/5}\}$.