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Second Semester 2011

# Class Test

Wednesday 16/03/2011

The test consists of 4 problems, two on the front of this paper and two on the back. Answer as many questions as you can. The time allowed is 40 minutes. This test counts 15% towards your final mark on MATH349. Good Luck!

**Problem 1** [5 points]

Let $\gamma : \mathbb{R} \to \mathbb{R}^2$ be given by $\gamma(t) = (t + t^3, t^2 + 1)$.

(i) Compute the velocity of $\gamma$.

(ii) Compute the speed of $\gamma$.

(iii) Show that $\gamma$ is a regular plane curve.

(iv) Compute the unit tangent $T$ and the unit normal $U$ of $\gamma$.

(v) Compute the curvature $\kappa$ of $\gamma$.

(vi) For $t = 1$, find the centre of curvature of $\gamma$.

**Problem 2** [10 points]

Let $\gamma : \mathbb{R} \to \mathbb{R}^3$ be given by $\gamma(t) = (2t + \sin t, \cos t, t)$.

(i) Compute the velocity of $\gamma$.

(ii) Compute the speed of $\gamma$.

(iii) Show that $\gamma$ is a regular space curve.

(iv) Compute the unit tangent $T$ of $\gamma$.

(v) Compute the vector $\gamma' \times \gamma''$.

(vi) Compute the binormal $B$ of $\gamma$.

(vii) Compute the principal normal $P$ of $\gamma$.

(viii) Compute the curvature $\kappa$ and the torsion $\tau$ of $\gamma$.

You need not attempt to simplify the expressions obtained.

**Problem 3** [10 points]

Let
$$X(u,v) = \left(u+v, u-v, \frac{u^2+v^2}{2}\right).$$

(i) Compute the partial derivatives $X_u$ and $X_v$.

(ii) Show that $X$ defines a regular surface patch. (This surface is a paraboloid of revolution).

(iii) Calculate the coefficients $E$, $F$, $G$ of the first fundamental form for this surface.

(iv) Write down an integral which gives the length of the curve $\gamma_1(t) = X(t,1)$ on this surface from $t=1$ to $t=2$. You do not need to evaluate this integral.

(v) Calculate the cosine of the angle between the coordinate curves
$$\gamma_1(t) = X(t,1) \quad \text{and} \quad \gamma_2(t) = X(1,t)$$
on the surface at the point $X(1,1) = (2,0,1)$, where the curves $\gamma_1$ and $\gamma_2$ meet.

**Problem 4** [10 points]

Let $U = \{(u,v) \in \mathbb{R}^2 \mid v > 0\}$. Let the surface patch $X : U \to \mathbb{R}^3$ be defined by
$$X(u,v) = (v\cos u, v\sin u, v).$$

This surface is a circular cone. Let the curve $\gamma$ on the cone be given by $\gamma(t) = X(t,c)$, where $c > 0$ is a constant number.

(i) Find a unit normal $N(u,v)$ to the surface.

(ii) Find a unit normal $N(t)$ along the curve $\gamma$.

(iii) Find the unit tangent $T$ of $\gamma$.

(iv) Find $U = N \times T$ along the curve $\gamma$.

(v) Find the geodesic curvature $\kappa_g$, the normal curvature $\kappa_n$ and the geodesic torsion $\kappa_t$ for the curve $\gamma$ using the formulae
$$\kappa_g = \frac{T' \bullet U}{|\gamma'|}, \quad \kappa_n = \frac{T' \bullet N}{|\gamma'|}, \quad \kappa_t = \frac{U' \bullet N}{|\gamma'|}.$$

**Solution of Problem 1:** Let $\gamma : \mathbb{R} \to \mathbb{R}^2$ be given by $\gamma(t) = (t + t^3, t^2 + 1)$.

(i) The velocity is $\gamma'(t) = (1 + 3t^2, 2t)$.

(ii) The speed is $|\gamma'(t)| = \sqrt{1 + 10t^2 + 9t^4}$.

(iii) The first component of the velocity vector $\gamma'(t) = (1 + 3t^2, 2t)$ is $1 + 3t^2 \geq 1$, hence the velocity of $\gamma$ is never equal to $(0, 0, 0)$, i.e. the curve $\gamma$ is regular. Alternatively, the speed $|\gamma'(t)| = \sqrt{1 + 10t^2 + 9t^4} \geq 1$ is never equal to zero, hence the curve $\gamma$ is regular.

(iv) The unit tangent is

$$T(t) = \frac{\gamma'(t)}{|\gamma'(t)|} = \frac{1}{\sqrt{1 + 10t^2 + 9t^4}} \cdot (1 + 3t^2, 2t).$$

The unit normal $U$ is obtained by rotating $T$ anti-clockwise through $90°$:

$$U(t) = \frac{1}{\sqrt{1 + 10t^2 + 9t^4}} \cdot (-2t, 1 + 3t^2).$$

(v) We compute $\gamma''(t) = (6t, 2)$. The curvature $\kappa$ is

$$\kappa(t) = \frac{\det(\gamma'(t), \gamma''(t))}{|\gamma'(t)|^3} = \frac{\begin{vmatrix} 1 + 3t^2 & 6t \\ 2t & 2 \end{vmatrix}}{(\sqrt{1 + 10t^2 + 9t^4})^3} = \frac{2 - 6t^2}{(1 + 10t^2 + 9t^4)^{3/2}}.$$

(vi) The centre of curvature of $\gamma$ at $t$ is given by

$$\gamma(t) + \frac{1}{\kappa(t)} \cdot U(t).$$

For $t = 1$, we have $\gamma(1) = (2, 2)$, $U(1) = \frac{1}{\sqrt{20}} \cdot (-2, 4)$ and $\kappa(1) = -\frac{1}{5\sqrt{20}}$, hence the centre of curvature of $\gamma$ at $t = 1$ is

$$\gamma(1) + \frac{1}{\kappa(1)} \cdot U(1) = (2, 2) - 5\sqrt{20} \cdot \frac{1}{\sqrt{20}} \cdot (-2, 4) = (2, 2) - 5 \cdot (-2, 4) = (12, -18).$$

**Solution of Problem 2:** Let $\gamma : \mathbb{R} \to \mathbb{R}^3$ be given by $\gamma(t) = (2t + \sin t, \cos t, t)$.

(i) The velocity is $\gamma'(t) = (2 + \cos t, -\sin t, 1)$.

(ii) The speed is

$$|\gamma'(t)| = \sqrt{(2 + \cos t)^2 + \sin^2 t + 1} = \sqrt{4 + 4\cos t + \cos t^2 + \sin^2 t + 1} = \sqrt{6 + 4\cos t}.$$

(iii) The third component of the velocity vector $\gamma'(t) = (2 + \cos t, -\sin t, 1)$ is 1, hence the velocity of $\gamma$ never vanishes, i.e. the curve $\gamma$ is regular. Alternatively, the speed $|\gamma'(t)| = \sqrt{6 + 4\cos t} \geq \sqrt{2}$ is never equal to zero, hence the curve $\gamma$ is regular.

(iv) The unit tangent is

$$T(t) = \frac{\gamma'(t)}{|\gamma'(t)|} = \frac{1}{\sqrt{6 + 4\cos t}} \cdot (2 + \cos t, -\sin t, 1).$$

(v) We compute $\gamma''(t) = (-\sin t, -\cos t, 0)$. The vector product is

$$\begin{aligned}(\gamma' \times \gamma'')(t) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2+\cos t & -\sin t & 1 \\ -\sin t & -\cos t & 0 \end{vmatrix} \\ &= \mathbf{i}\cdot \cos t - \mathbf{j}\cdot \sin t + \mathbf{k}\cdot(-2\cos t - \cos^2 t - \sin^2 t) \\ &= (\cos t, -\sin t, -2\cos t - 1).\end{aligned}$$

(vi) We compute

$$|(\gamma' \times \gamma'')(t)| = \sqrt{\cos^2 t + \sin^2 t + 4\cos^2 t + 4\cos t + 1.} = \sqrt{2 + 4\cos t + 4\cos^2 t}.$$

The binormal is

$$B(t) = \frac{(\gamma' \times \gamma'')(t)}{|(\gamma' \times \gamma'')(t)|} = \frac{1}{\sqrt{2 + 4\cos t + 4\cos^2 t}} \cdot (\cos t, -\sin t, -2\cos t - 1).$$

(vii) The principal normal is

$$\begin{aligned}P(t) &= B(t) \times T(t) \\ &= \frac{1}{\sqrt{2 + 4\cos t + 4\cos^2 t}} \cdot \frac{1}{\sqrt{6 + 4\cos t}} \cdot \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos t & -\sin t & -2\cos t - 1 \\ 2+\cos t & -\sin t & 1 \end{vmatrix},\end{aligned}$$

where

$$\begin{aligned}&\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos t & -\sin t & -2\cos t - 1 \\ 2+\cos t & -\sin t & 1 \end{vmatrix} \\ &= \mathbf{i}\cdot(-\sin t - 2\sin t\cos t - \sin t) \\ &- \mathbf{j}\cdot(\cos t + 4\cos t + 2 + 2\cos^2 t + \cos t) \\ &+ \mathbf{k}\cdot(-\sin t\cos t + 2\sin t + \sin t\cos t) \\ &= (-2\sin t(1+\cos t), -2(1 + 3\cos t + \cos^2 t), 2\sin t),\end{aligned}$$

hence

$$\begin{aligned}&P(t) \\ &= \frac{1}{\sqrt{2 + 4\cos t + 4\cos^2 t}} \cdot \frac{1}{\sqrt{6 + 4\cos t}} \cdot (-2\sin t(1+\cos t), -2(1 + 3\cos t + \cos^2 t), 2\sin t) \\ &= \frac{1}{\sqrt{1 + 2\cos t + 2\cos^2 t} \cdot \sqrt{3 + 2\cos t}} \cdot (-\sin t(1+\cos t), -1 - 3\cos t - \cos^2 t, \sin t).\end{aligned}$$

(viii) The curvature is

$$\kappa(t) = \frac{|(\gamma' \times \gamma'')(t)|}{|\gamma'(t)|^3} = \frac{\sqrt{2 + 4\cos t + 4\cos^2 t}}{(\sqrt{6 + 4\cos t})^3} = \frac{\sqrt{1 + 2\cos t + 2\cos^2 t}}{2(\sqrt{3 + 2\cos t})^3}.$$

We then compute $\gamma'''(t) = (-\cos t, \sin t, 0)$,

$$\begin{aligned}[\gamma', \gamma'', \gamma'''](t) = ((\gamma' \times \gamma'') \bullet \gamma''')(t) &= (\cos t, -\sin t, -2\cos t - 1) \bullet (-\cos t, \sin t, 0) \\ &= -\cos^2 t - \sin^2 t = -1,\end{aligned}$$

and the torsion

$$\tau(t) = \frac{[\gamma', \gamma'', \gamma'''](t)}{|(\gamma' \times \gamma'')(t)|^2} = -\frac{1}{2 + 4\cos t + 4\cos^2 t}.$$

**Solution of Problem 3:** Let $X(u,v) = \left(u+v, u-v, \frac{u^2+v^2}{2}\right)$.

(i) We compute

$$X_u = (1,1,u), \quad X_v = (1,-1,v).$$

(ii) There are different ways to show that $X$ is a regular surface patch.

**Alternative 1:** We will prove that the vectors $X_u$ and $X_v$ are linearly independent. Assume that there are $\lambda, \mu \in \mathbb{R}$ such that $\lambda \cdot X_u + \mu \cdot X_v = (0,0,0)$. We compute

$$\lambda \cdot X_u + \mu \cdot X_v = \lambda \cdot (1,1,u) + \mu \cdot (1,-1,v) = (\lambda+\mu, \lambda-\mu, \lambda \cdot u + \mu \cdot v).$$

Thus the equality

$$\lambda \cdot X_u + \mu \cdot X_v = (0,0,0)$$

implies that $\lambda - \mu = \lambda + \mu = 0$, therefore $\lambda = \mu = 0$. This means that whenever $\lambda \cdot X_u + \mu \cdot X_v = 0$ for some $\lambda, \mu \in \mathbb{R}$ we obtain $\lambda = \mu = 0$. Therefore the vectors $X_u$ and $X_v$ are linearly independent.

**Alternative 2:** We will prove that $X_u \times X_v \neq 0$. We compute

$$X_u \times X_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & u \\ 1 & -1 & v \end{vmatrix} = \mathbf{i} \cdot (u+v) - \mathbf{j} \cdot (v-u) + \mathbf{k} \cdot (-2) = (u+v, u-v, -2).$$

The last component of $X_u \times X_v$ is equal to $-2$, i.e. is never equal to zero, hence $X_u \times X_v$ is never equal to $(0,0,0)$.

(iii) The coefficients of the first fundamental form are

$$E = X_u \bullet X_u = 2 + u^2, \quad F = X_u \bullet X_v = uv, \quad G = X_v \bullet X_v = 2 + v^2.$$

(iv) The curve $\gamma_1 = X(t,1)$ is of the form $\gamma_1(t) = X(u(t), v(t))$, where

$$(u(t), v(t)) = (t, 1), \quad \text{hence } (u'(t), v'(t)) = (1, 0).$$

The coefficients of the first fundamental form along the curve $\gamma_1$ are

$$E_t = E(t,1) = 2 + t^2, \quad F_t = F(t,1) = t, \quad G_t = G(t,1) = 3.$$

The speed of $\gamma_1$ can be computed as

$$|\gamma_1'(t)|^2 = (u'(t)\ v'(t)) \cdot \begin{pmatrix} E_t & F_t \\ F_t & G_t \end{pmatrix} \cdot \begin{pmatrix} u'(t) \\ v'(t) \end{pmatrix} = (1\ 0) \cdot \begin{pmatrix} E_t & F_t \\ F_t & G_t \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = E_t = 2 + t^2.$$

The curve $\gamma_1(t) = X(t,1)$ from $t_0 = 1$ to $t_1 = 2$ has length

$$\int_{t_0}^{t_1} |\gamma_1'(t)|\, dt = \int_1^2 \sqrt{2+t^2}\, dt.$$

(v) The curves $\gamma_1(t) = X(t,1)$ and $\gamma_2(t) = X(1,t)$ meet at the point

$$\gamma_1(1) = \gamma_2(1) = X(1,1) = (2,0,1).$$

The coefficients of the first fundamental form at this point are

$$E_0 = E(1,1) = 3, \quad F_0 = F(1,1) = 1, \quad G_0 = G(1,1) = 3.$$

The curve $\gamma_1$ is of the form $\gamma_1(t) = X(u_1(t), v_1(t))$, where

$$(u_1(t), v_1(t)) = (t, 1), \quad \text{hence } (u_1'(t), v_1'(t)) = (1, 0).$$

The curve $\gamma_2$ is of the form $\gamma_2(t) = X(u_2(t), v_2(t))$, where

$$(u_2(t), v_2(t)) = (1, t), \quad \text{hence } (u_2'(t), v_2'(t)) = (0, 1).$$

The speeds of $\gamma_1$ and $\gamma_2$ can be computed as

$$|\gamma'(1)|^2 = (u_1'(1)\ v_1'(1)) \cdot \begin{pmatrix} E_0 & F_0 \\ F_0 & G_0 \end{pmatrix} \cdot \begin{pmatrix} u_1'(1) \\ v_1'(1) \end{pmatrix} = (1\ 0) \cdot \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 3,$$
$$|\gamma_2'(1)|^2 = (u_2'(1)\ v_2'(1)) \cdot \begin{pmatrix} E_0 & F_0 \\ F_0 & G_0 \end{pmatrix} \cdot \begin{pmatrix} u_2'(1) \\ v_2'(1) \end{pmatrix} = (0\ 1) \cdot \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 3.$$

The cosine of the angle $\theta$ between the curves $\gamma_1$ and $\gamma_2$ can be computed as

$$\cos\theta = \frac{(u_1'(1)\ v_1'(1)) \cdot \begin{pmatrix} E_0 & F_0 \\ F_0 & G_0 \end{pmatrix} \cdot \begin{pmatrix} u_2'(1) \\ v_2'(1) \end{pmatrix}}{|\gamma_1'(1)| \cdot |\gamma_2'(1)|} = \frac{(1\ 0) \cdot \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}}{\sqrt{3} \cdot \sqrt{3}} = \frac{1}{3}.$$

**Solution of Problem 4:** Let $U = \{(u, v) \in \mathbb{R}^2 \mid v > 0\}$. Let the surface patch $X : U \to \mathbb{R}^3$ be defined by $X(u, v) = (v\cos u, v\sin u, v)$. Let the curve $\gamma$ be given by $\gamma(t) = X(t, c)$, where $c > 0$ is a constant number.

(i) We compute

$$\begin{aligned}
&X_u(u, v) = (-v\sin u, v\cos u, 0), \\
&X_v(u, v) = (\cos u, \sin u, 1), \\
&(X_u \times X_v)(u, v) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -v\sin u & v\cos u & 0 \\ \cos u & \sin u & 1 \end{vmatrix} \\
&= \mathbf{i} \cdot v\cos u - \mathbf{j} \cdot (-v\sin u) + \mathbf{k} \cdot (-v\sin^2 u - v\cos^2 u) \\
&= (v\cos u, v\sin u, -v), \\
&|(X_u \times X_v)(u, v)|^2 = v^2\cos^2 u + v^2\sin^2 u + v^2 = 2v^2, \\
&|(X_u \times X_v)(u, v)| = \sqrt{2v^2} = |v|\sqrt{2} = v\sqrt{2},
\end{aligned}$$

thus

$$N(u, v) = \frac{(X_u \times X_v)(u, v)}{|(X_u \times X_v)(u, v)|} = \frac{1}{v\sqrt{2}} \cdot (v\cos u, v\sin u, -v) = \frac{1}{\sqrt{2}} \cdot (\cos u, \sin u, -1).$$

(ii) Along the curve $\gamma(t) = X(t, c)$, the unit normal is

$$N(t) = N(t, c) = \frac{1}{\sqrt{2}} \cdot (\cos t, \sin t, -1).$$

(iii) The velocity of the curve $\gamma(t) = X(t, c) = (c\cos t, c\sin t, c)$ is

$$\gamma'(t) = (-c\sin t, c\cos t, 0),$$

the speed is $|\gamma'(t)| = \sqrt{c^2} = |c| = c$, the unit tangent is

$$T(t) = \frac{\gamma'(t)}{|\gamma'(t)|} = \frac{1}{c} \cdot (-c\sin t, c\cos t, 0) = (-\sin t, \cos t, 0).$$

(iv) We compute

$$
\begin{aligned}
U(t) = N(t) \times T(t) &= \frac{1}{\sqrt{2}} \cdot \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos t & \sin t & -1 \\ -\sin t & \cos t & 0 \end{vmatrix} \\
&= \frac{1}{\sqrt{2}} \cdot (\mathbf{i} \cdot \cos t - \mathbf{j} \cdot (-\sin t) + \mathbf{k} \cdot (\cos^2 t + \sin^2 t)) \\
&= \frac{1}{\sqrt{2}} \cdot (\cos t, \sin t, 1).
\end{aligned}
$$

(v) We have

$$
\begin{aligned}
T'(t) &= (-\cos t, -\sin t, 0), \\
U'(t) &= \frac{1}{\sqrt{2}} \cdot (-\sin t, \cos t, 0), \\
|\gamma'(t)| &= c,
\end{aligned}
$$

thus

$$
\begin{aligned}
\kappa_g &= \frac{T'(t) \bullet U(t)}{|\gamma'(t)|} = \frac{\frac{1}{\sqrt{2}} \cdot (-\cos^2 t - \sin^2 t)}{c} = -\frac{1}{c\sqrt{2}}, \\
\kappa_n &= \frac{T'(t) \bullet N(t)}{|\gamma'(t)|} = \frac{\frac{1}{\sqrt{2}} \cdot (-\cos^2 t - \sin^2 t)}{c} = -\frac{1}{c\sqrt{2}}, \\
\kappa_t &= \frac{U'(t) \bullet N(t)}{|\gamma'(t)|} = \frac{\frac{1}{2} \cdot (-\sin t \cos t + \sin t \cos t)}{c} = 0.
\end{aligned}
$$