ARC-LIKE CONTINUA, JULIA SETS OF ENTIRE FUNCTIONS, AND EREMENKO'S CONJECTURE

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Abstract. Let f be a transcendental entire function, and assume that f is hyperbolic with connected Fatou set; we say that such a function is "of disjoint type". It is known that a disjoint-type function provides a model for the dynamics of all maps in the same parameter space near infinity; the goal of this article is to study the topological properties of the Julia set of f.

Indeed, we give an almost complete description of the possible topology of the components of Julia sets for entire functions of disjoint type. More precisely, let C be a component of such a Julia set, and consider the *Julia continuum* $\hat{C} := C \cup \{\infty\}$. We show that ∞ is a terminal point of $\hat{C}$, and that $\hat{C}$ has span zero in the sense of Lelek; under a mild additional geometric assumption the continuum $\hat{C}$ is arc-like. (Whether every span zero continuum is also arc-like was a famous question in continuum theory, posed by Lelek in 1961, and only recently resolved in the negative by work of Hoehn.) Conversely, every arc-like continuum X possessing at least one terminal point can occur as the Julia continuum of a disjoint-type entire function. In particular, the $\sin(1/x)$-curve, the Knaster buckethandle and the pseudo-arc can all occur as components of Julia sets of entire functions.

We also give similar descriptions of the possible topology of Julia continua that contain periodic points or points with bounded orbits, and we answer a question of Barański and Karpińska regarding the accessibility of components of the Julia set from the Fatou set. We also show that the Julia set of a disjoint-type entire function may have components on which the iterates tend to infinity pointwise, but not uniformly. This property is related to a famous conjecture of Eremenko concerning escaping sets of entire functions.

1. Introduction

We consider the iteration of transcendental entire functions; i.e. of non-polynomial holomorphic self-maps of the complex plane. This topic was founded by Fatou in a seminal article of 1921 [Fat26], and has received particular interest over the past decade or so, partly due to emerging connections with the fields of rational and polynomial dynamics. For example, work of Inou and Shishikura as well as of Buff and Chéritat implies that certain well-known features of transcendental dynamics occur naturally near non-linearizable fixed points of quadratic polynomials [Shi09]. These results and their proofs are motivated by properties first discovered in the context of transcendental dynamics. It is to be hoped that a better understanding of the transcendental case will lead to further insights also in the polynomial and rational setting.

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In this article, we consider a particular class of transcendental entire functions, namely those that are of *disjoint type*; i.e. hyperbolic with connected Fatou set. To provide the required definitions, recall that the *Fatou set* $F(f)$ of a transcendental entire function f consists of those points z for which the family of iterates

$$f^n := \underbrace{f \circ \cdots \circ f}_{n \text{ times}}.$$

is equicontinuous with respect to the spherical metric in a neighbourhood of z. (I.e., these are the points where small perturbations of the starting point z result only in small changes of $f^n(z)$, independently of n.) Its complement $J(f) := \mathbb{C} \setminus F(f)$ is called the *Julia set*; it is the set on which f exhibits "chaotic" behavior. We also recall that the set $S(f)$ of (finite) *singular values* is the closure of all critical and asymptotic values of f in $\mathbb{C}$. Equivalently, it is the smallest closed set S such that $f\colon \mathbb{C} \setminus f^{-1}(S) \to \mathbb{C} \setminus S$ is a covering map.

1.1. Definition (Hyperbolicity)**.**
An entire function f is called *hyperbolic* if the set $S(f)$ is bounded and every point in $S(f)$ tends to an attracting periodic cycle of f under iteration. If f is hyperbolic and furthermore $F(f)$ is connected, then we say that f is of *disjoint type*.

Hyperbolic dynamical systems (also referred to as *Axiom A*, using Smale's terminology) are those that exhibit the simplest type of dynamics; understanding the hyperbolic case is usually the first step in building a more general theory. In [Rem09, Theorem 5.2], it is shown that the dynamics of any hyperbolic entire function on its Julia set can be obtained, via a suitable semi-conjugacy, as a quotient of the dynamics of a disjoint-type entire function; this was later generalized to certain non-hyperbolic functions by Mihaljević-Brandt [MB12]. Furthermore, suppose that f belongs to the *Eremenko-Lyubich class*

$$\mathcal{B} := \{f\colon \mathbb{C} \to \mathbb{C} \text{ transcendental entire}\colon S(f) \text{ is bounded}\}.$$

(By definition, this class contains all hyperbolic entire functions, as well as the particularly interesting *Speiser class* $\mathcal{S}$ of functions for which $S(f)$ is finite.) Then the map

$$f_\lambda\colon \mathbb{C} \to \mathbb{C}; \quad z \mapsto \lambda f(z)$$

is of disjoint type for sufficiently small λ, and it is shown in [Rem09, Theorem 1.1] that the maps f and $f\lambda$ have the same dynamics *near infinity*.

Hence a good understanding of the possible dynamics of disjoint-type entire functions should be the first step to a general theory of entire functions in the classes $\mathcal{S}$ and $\mathcal{B}$. As a simple example, consider the maps

$$S_\lambda(z) := \lambda \sin(z), \qquad \lambda \in (0,1).$$

Already Fatou observed that $J(S_\lambda)$ contains infinitely many curves on which the iterates tend to infinity (namely, iterated preimages of an infinite piece of the imaginary axis), and asked whether this is true for more general classes of functions. It turns out that, in fact, the entire set $J(S_\lambda)$ can be written as an uncountable union of arcs, each connecting a finite endpoint with ∞.[1] Each point on such a curve, with the possibility of the finite endpoint, tends to infinity under iteration. This led Eremenko [Ere89] to strengthen Fatou's question by asking whether, for an arbitrary entire function f, every point of the *escaping set*

$$I(f) := \{z \in \mathbb{C}\colon f^n(z) \to \infty\}$$

could be connected to infinity by an arc in $I(f)$. We remark that, when $f \in \mathcal{B}$, the set $I(f)$ is always contained in the Julia set [EL92, Theorem 1].

It turns out that the situation is not as simple as suggested by this question. Indeed, while the answer to Eremenko's question is positive when $f \in \mathcal{B}$ has finite order of growth [RRRS11, Theorem 8.4], there exists a disjoint-type entire function $f \in \mathcal{B}$ for which $J(f)$ contains no arcs. (When f is of disjoint type, the result for finite-order functions was obtained independently by Barański [Bar07].) This suggests that the possible topological types of components of $J(f)$ can be rather varied, even for f of disjoint-type, and it is natural to ask what types of objects can arise. We shall give an almost complete solution to this problem. However, before describing the general results, let us consider two particularly interesting applications of our methods.

A famous continuum (i.e., non-empty compact, connected metric space) that contains no arcs is given by the *pseudo-arc* (see Definition 1.4), a certain hereditarily indecomposable continuum with the intriguing property of being homeomorphic to every one of its non-degenerate subcontinua. In view of the results of [RRRS11] mentioned above, it is tempting to ask whether the pseudo-arc can arise in the Julia set of a transcendental entire function. We show that this is indeed the case; as far as we are aware, this is the first time that a dynamically defined subset of the Julia set of an entire or meromorphic function has been shown to be homeomorphic to the pseudo-arc. Observe that the following theorem sharpens [RRRS11, Theorem 8.4].

1.2. Theorem (Pseudo-arcs in Julia sets)**.** *There exists a disjoint-type entire function $f\colon \mathbb{C} \to \mathbb{C}$ such that, for every connected component C of $J(f)$, the set $C \cup \{\infty\}$ is a pseudo-arc.*

A further motivation for studying the topological dynamics of disjoint-type functions comes from a second question asked by Eremenko in [Ere89]: *Is every connected component of $I(f)$ unbounded?* This problem is now known as *Eremenko's Conjecture*, and has remained open despite considerable attention. For disjoint-type maps, and indeed for any entire function with bounded postsingular set, it is known that the answer is positive [Rem07]. However, the disjoint-type case nonetheless has a role to play in the study of this problem. Indeed, as discussed in [Rem09, Section 7], we may strengthen the question slightly by asking which entire functions have the following property:

(UE) For all $z \in I(f)$, there is a connected and unbounded set $A \subset \mathbb{C}$ with $z \in A$ such that $f^n|_A \to \infty$ *uniformly.*

[1] To our knowledge, this fact was first proved by Aarts and Oversteegen [AO93, Theorem 5.7], at least for $\lambda < 0.85$. Devaney and Tangerman [DT86] had previously discussed at least the existence of "Cantor bouquets" of arcs in the Julia set, and the proof that the whole Julia set has this property is analogous to the proof for the disjoint-type exponential maps $z \mapsto \lambda e^z$ with $0 < \lambda < 1/e$, first established in [DK84, p. 50]; see also [DG87].

If there exists a counterexample f to Eremenko's Conjecture in the class $\mathcal{B}$, then clearly f cannot satisfy property (UE). It follows from [Rem09] that (UE) fails for *every* map of the form $f_\lambda := \lambda f$. As noted above, f_λ is of disjoint type for λ sufficiently small, so we see that any counterexample $f \in \mathcal{B}$ would need to be closely related to a disjoint-type function for which (UE) fails. It was stated in [Rem07, Rem09] that such an example indeed exists; in this article we provide the first proof of this assertion. In fact, as we discuss in more detail below, there is a surprisingly close relationship between the topology of Julia continua and the existence of points $z \in I(f)$ for which (UE) fails. Hence our results allow us to give a good description of the cirucmstances in which such points exist at all, which is likely to be important in any attempt to construct a counterexample to Eremenko's Conjecture. In particular, we can prove the following, which strengthens the examples alluded to in [Rem07, Rem09].

1.3. Theorem (Non-uniform escape to infinity)**.** *There is a disjoint-type entire function f and an escaping point $z \in I(f)$ with the following property. If $A \subset I(f)$ is connected and $\{z\} \subsetneq A$, then*

$$\liminf_{n\to\infty} \inf_{z\in A} |f^n(z)| < \infty.$$

Topology of Julia continua. If f is of disjoint type, then it is easy to see that the Julia set $J(f)$ is a union of uncountably many connected components, each of which is closed and unbounded. If C is such a component, we shall refer to the continuum $\hat{C} := C \cup \{\infty\}$ as a *Julia continuum* of f. In the case of $z \mapsto \lambda \sin(z)$ with $\lambda \in (0,1)$, every Julia continuum is an arc, while for the example in Theorem 1.2 every Julia continuum is a pseudo-arc. In order to discuss the possible topology of Julia continua in greater detail, we shall require a small number of well-known concepts from continuum theory.

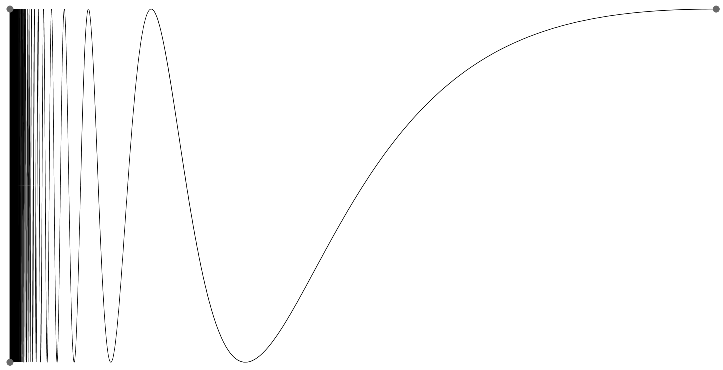
1.4. Definition (Terminal points; span zero; arc-like continua)**.**
Let X be a continuum (i.e., a compact, connected metric space).

(a) A point $x_0 \in X$ is called a *terminal point* of X if, for any two subcontinua $A, B \subset X$ with $x_0 \in A \cap B$, either $A \subset B$ or $B \subset A$.
(b) X is said to have *span zero* if any subcontinuum $\mathcal{A} \subset X \times X$ whose first and second coordinates both project to the same subcontinuum $A \subset X$ must intersect the diagonal. (I.e., if $\pi_1(\mathcal{A}) = \pi_2(\mathcal{A})$, then there is $x \in X$ such that $(x,x) \in \mathcal{A}$.)
(c) X is said to be *arc-like* if, for every $\varepsilon > 0$, there exists a continuous function $g\colon X \to [0,1]$ such that $\operatorname{diam}(g^{-1}(t)) < \varepsilon$ for all $t \in [0,1]$.
(d) X is called a *pseudo-arc* if X is arc-like and also *hereditarily indecomposable* (i.e., every point of X is terminal).

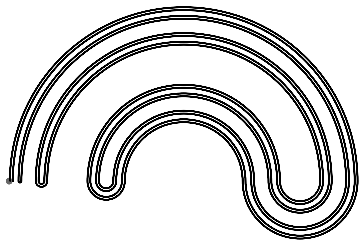
For the benefit of those readers who have not encountered these concepts before, let us make a few comments regarding their meaning. A few examples of arc-like continua and their terminal points are shown in Figure 1.

(1) One should think of terminal points as a natural analogue of the endpoints of an arc. However, as the example of the pseudo-arc shows, a continuum may contain far more than two terminal points.
(2) Roughly speaking, X has span zero if two points cannot exchange their position by travelling within X without meeting somewhere.

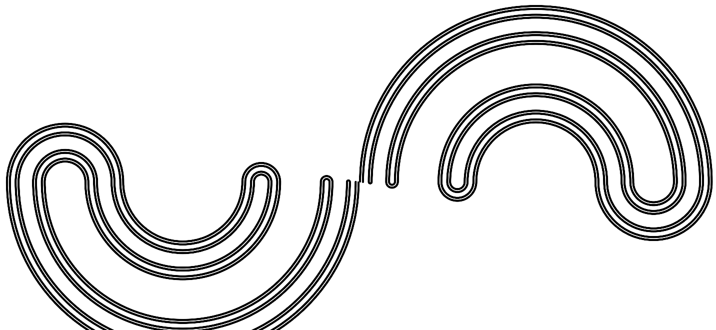
(a) The arc



(b) $The \sin(1/x)$-continuum



(c) Knaster bucket-handle



(d) A double bucket-handle

FIGURE 1. Some examples of arc-like continua; terminal points are marked by grey circles. (The numbers of terminal points in these continua are two, three, one and zero, respectively.)

(3) Intuitively, a continuum is arc-like if it looks like an arc at arbitrarily small scales. As we discuss in Section 7, there are a number of equivalent definitions, the most important of which is that X is arc-like if and only if it can be written as an inverse limit of arcs with surjective bonding maps.

(4) Any two pseudo-arcs, as defined above, are homeomorphic [Bin51a, Theorem 1]; for this reason, we also speak about *the pseudo-arc*. See Exercise 1.23 in [Nad92] for a construction that shows the existence of such an object, and the introduction to Section 12 in the same book for a short history.

(5) It is well-known [Lel64] that every arc-like continuum has span zero. A long-standing question, posed by Lelek [Lel71, Problem 1] in 1971 and featured on many subsequent problem lists in topology, asked whether every continuum with span zero must be arc-like. (It is known that this is true when the continuum is *hereditarily decomposable*; see Definition 2.10.) The question remained open for 40 years, until Hoehn [Hoe11] recently constructed a counterexample.

In order to make the most precise statements about the possible topology of Julia continua, we shall need to make a very mild function-theoretic restriction on the entire functions under consideration.

1.5. Definition (Bounded slope [RRRS11])**.**
An entire function is said to have *bounded slope* if there exists a curve $\gamma\colon [0,\infty) \to \mathbb{C}$ such that $|f(\gamma(t))| \to \infty$ as $t \to \infty$ and such that

$$\limsup_{t\to\infty} \frac{|\arg(\gamma(t))|}{\log|\gamma(t)|} < \infty.$$

Remark 1. Any function $f \in \mathcal{B}$ that is real on the real axis has bounded slope. So does the counterexample to Eremenko's question constructed in [RRRS11], as has, as far as we are aware, any specific example or family of functions $f \in \mathcal{B}$ whose dynamics has been considered in the past.

Remark 2. In all results stated in this section, "bounded slope" can be replaced by the considerably weaker condition of having "arc-like tracts", as per Definition 5.1.

With these preparations, we can state our main theorem.

1.6. Theorem (Topology of Julia continua)**.** *Let $\hat{C}$ be a Julia continuum of a disjoint-type entire function f. Then $\hat{C}$ has span zero and ∞ is a terminal point of $\hat{C}$. If, additionally, f has bounded slope, then $\hat{C}$ is arc-like.*

Conversely, there exists a disjoint-type entire function f of bounded slope with the following property. If X is any arc-like continuum having a terminal point $x_0 \in X$, then there exists a Julia continuum $\hat{C}$ of f and a homeomorphism $\psi\colon X \to \hat{C}$ such that $\psi(x_0) = \infty$.

Remark. The fact that ∞ is always a terminal point of $\hat{C}$ appears essentially already in [Rem07, Corollary 3.4] (though it is not quite stated there).

In particular, Theorem 1.6 gives a complete description of the possible topology of Julia continua for disjoint-type entire functions with bounded slope. The class of arc-like continua is extremely rich (e.g., there are uncountably many pairwise disjoint arc-like continua), and hence we see that, indeed, Julia sets of disjoint-type entire functions are topologically very varied. In the case where f does not have bounded slope (or, indeed, "arc-like tracts", which is a much more general condition), we do not obtain a complete classification. We note that any additional Julia continua would be of span zero but not arc-like, and hence of considerable topological interest in view of Lelek's question. Indeed, it is plausible that one could construct a disjoint-type entire function having a Julia continuum of this type, thus yielding a new proof of Hoehn's theorem mentioned above. However, we will not pursue this investigation here.

Nonescaping points and accessible points. Let us now turn to the behavior of points in a Julia continuum $\hat{C} = C \cup \{\infty\}$ under iteration. In the case of disjoint-type sine (or exponential) maps, and indeed for any disjoint-type entire function of finite order, each component C of the Julia set contains at most one point that does not tend to infinity under iteration, namely the finite endpoint of C. (Recall that $C \cup \{\infty\}$ is an arc in this case.) Furthermore, this finite endpoint is always accessible from the Julia set of f; no other point can be accessible from $F(f)$. (Compare [DG87].) This suggests the following questions:

(a) Can a Julia continuum contain more than one nonescaping point?

(b) Is every nonescaping point accessible from $F(f)$?
(c) Does every Julia continuum contain a point that is accessible from $F(f)$? This question is raised in [BK07, p. 393], where the authors prove that the answer is positive when a certain growth condition is imposed on the external address (see Definition 2.5) of the component C.

To answer these questions, let us introduce one more topological notion.

1.7. Definition (Irreducibility)**.**
Let X be a continuum, and let $x_0, x_1 \in X$. We say that X is *irreducible* between x_0 and x_1 if no proper subcontinuum of X contains both x_0 and x_1.

We shall apply this notion only in the case where x_0 and x_1 are terminal points of X. In this case, irreducibility of X between x_0 and x_1 means that, in some sense, the points x_0 and x_1 lie "on opposite ends" of X. For example, the $\sin(1/x)$-continuum of Figure 1(b) is irreducible between the terminal point on the right of the image and either of the two terminal points on the left, but not between the two latter points (since the limiting interval is a proper subcontinuum containing both of these).

1.8. Theorem (Nonescaping and accessible points)**.** *Let $\hat{C}$ be a Julia continuum of a disjoint-type entire function f. Any nonescaping point z_0 in $\hat{C}$ is a terminal point of $\hat{C}$, and $\hat{C}$ is irreducible between z_0 and ∞. The same is true for any point $z_0 \in \hat{C}$ that is accessible from $F(f)$.*

Furthermore, the set of nonescaping points in $\hat{C}$ has Hausdorff dimension zero. On the other hand, there exist a disjoint-type function having a Julia continuum for which the set of nonescaping points is a Cantor set and a disjoint-type function having a Julia continuum that contains a dense set of nonescaping points.

Note that, in particular, the two functions whose existence is asserted in this theorem will have nonescaping points that are not accessible from $F(f)$, since a Julia component can contain at most one accessible point. Furthermore, we can apply Theorem 1.6 to the bucket-handle continuum of Figure 1(c), which has only a single terminal point. Hence the corresponding Julia continuum contains neither nonescaping nor accessible points. In particular, this answers the question of Barański and Karpińska.

We remark that it is also possible to construct an inaccessible Julia continuum that *does* contain a finite terminal point z_0. Indeed, the examples mentioned in the second half of the preceding theorem must have this property (by the final statement of Theorem 3.10). Such an example can also be achieved by ensuring that the continuum is embedded in the plane in such a way that z_0 is not accessible from the complement of $\hat{C}$ (see Figure 2); we shall not discuss the details of such a construction here.

Bounded-address and periodic Julia continua. We now turn our attention to the different type of dynamics that f can exhibit on a Julia continuum. We shall see that each Julia continuum $\hat{C}$ in Theorem 1.6 can be constructed either such that $f^n|_C \to \infty$ uniformly, or such that $\min_{z\in C} |f^n(z)| < R$ for some $R > 0$ and infinitely many n. However, our construction will always require that also $\min_{z\in C} |f^{n_k}(z)| \to \infty$ for some subsequence f^{n_k} of iterates. In particular, the Julia continuum cannot be periodic.

We shall now consider when we can improve on this behaviour, in the following sense.

1.9. Definition (Periodic and bounded-address Julia continua).
Let $\hat{C} = C \cup \{\infty\}$ be a Julia continuum of a disjoint-type function f. We say that $\hat{C}$ is *periodic* if $f^n(C) = C$ for some $n \geq 1$.

We also say that $\hat{C}$ has *bounded address* if there is R such that, for every $n \in \mathbb{N}$, there is $z \in C$ such that $|f^n(z)| \leq R$.

With some reflection, it becomes evident that not every one of the continua in Theorem 1.6 can arise as a Julia continuum with bounded address. Indeed, it is easy to show that every Julia continuum $\hat{C}$ at bounded address contains a unique point with a bounded orbit (and hence that every periodic Julia continuum contains a periodic point). In particular, by Theorem 1.8, $\hat{C}$ contains some terminal point z_0 such that $\hat{C}$ is irreducible between z_0 and ∞. So if X is an arc-like continuum that does not contain two terminal points between which X is irreducible (such as the Knaster buckethandle), then X cannot be realized by a bounded Julia continuum. It turns out that this is the only restriction.

1.10. Theorem (Classification of bounded Julia continua). *There exists a bounded-slope, disjoint-type entire function f with the following property.*

Let X be an arc-like continuum, and let $x_0, x_1 \in X$ be two terminal points between which X is irreducible. Then there is a Julia continuum $\hat{C}$ of f with bounded address and a homeomorphism $\psi\colon X \to \hat{C}$ such that $\psi(x_0) = \infty$ and such that $\psi(x_1)$ has bounded orbit under f.

We also observe that not every continuum as in Theorem 1.10 can occur as a periodic Julia continuum. Indeed, if $\hat{C}$ is a periodic Julia continuum, then $f^p\colon \hat{C} \to \hat{C}$ is a homeomorphism, where p is the period of $\hat{C}$, and all but one point of $\hat{C}$ tends to ∞ under iteration by f^p. However, if X is, say, the $\sin(1/x)$-continuum from Figure 1, then every self-homeomorphism of X must map the limiting interval on the left to itself. Hence there cannot be any periodic Julia continuum $\hat{C}$ that homeomorphic to X. The following theorem describes exactly which continua can occur in this setting.

1.11. Theorem (Periodic Julia continua). *Let X be a continuum and let $x_0, x_1 \in X$. Then the following are equivalent:*

- *(a) There exists a bounded-slope, disjoint-type entire function f, a periodic Julia continuum $\hat{C}$ of f, say of period p, and a homeomorphism $\psi\colon X \to \hat{C}$ such that $\psi(x_0) = \infty$ and $f^p(\psi(x_1)) = \psi(x_1)$.*
- *(b) There is a continuous function $h\colon [0,1] \to [0,1]$ such that $h(0) = 0$, $h(1) = 1$ and $h(x) < x$ for all $x \in (0,1)$, and such that X is homeomorphic to the inverse limit space generated by h, with x_0 and x_1 corresponding to the points $1 \leftarrow 1 \leftarrow \dots$ and $0 \leftarrow 0 \leftarrow \dots$, respectively.*

Remark 1. Recall that the *inverse limit* generated by h is the space of all backward orbits under h, equipped with the product topology (Definition 2.12).

Remark 2. This result is slightly less satisfactory than Theorems 1.6 and 1.10. Indeed, both of those results can be stated in the following form: *Any (resp. any bounded-address) Julia continuum has a certain intrinsic topological property $\mathcal{P}$, and any arc-like continuum with property $\mathcal{P}$ can be realized as a Julia continuum (resp. bounded-address Julia continuum) of a disjoint-type, bounded-slope entire function.* It would be interesting to investigate whether Theorem 1.11 can also be phrased in such terms. However, we remark that there is e.g. no known natural topological classification of those arc-like continua that can be written as an inverse limit with a single bonding map.

Remark 3. By a classical result of Henderson [Hen64], the pseudo-arc can be written as an inverse limit as in (b). Hence we see from Theorem 1.11 that it can arise as an invariant Julia continuum of a disjoint-type entire function. It follows from the nature of our construction in the proof of Theorem 1.11 that, in this case, *all* Julia continua are pseudo-arcs (see Section 12), establishing Theorem 1.2 as stated at the beginning of this introduction.

(Non-)uniform escape to infinity. We now return to the question of rates of escape to infinity, and the "uniform Eremenko property" (UE). Recall that it is possible to construct a Julia continuum $\hat{C}$ that contains no finite terminal points, and hence has the property that $C \subset I(f)$. Also recall that we can choose $\hat{C}$ in such a way that the iterates of f do not tend to infinity uniformly on C. This easily implies that there is some point in C for which the property (UE) fails.

To study this type of question in greater detail, we make the following natural definition.

1.12. Definition (Uniformly escaping components)**.**
Let f be a transcendental entire function, and let $z \in I(f)$. The *uniformly escaping component* $\mu(z)$ is defined to be the union of all connected sets $A \supset z$ such that $f^n|_A \to \infty$ uniformly.

We also define $\mu(\infty)$ to be the union of all unbounded connected sets A such that $f^n|_A \to \infty$ uniformly.

There is an interesting connection between uniformly escaping components and the topology of Julia continua. Recall that the *composant* of a point x_0 in a continuum X is the union of all proper subcontinua of X containing x_0.

1.13. Theorem (Composants and uniformly escaping components)**.** *Let $\hat{C} = C \cup \{\infty\}$ be a Julia continuum of a disjoint-type entire function, and suppose that $f^n|_C$ does not tend to infinity uniformly. Then the composant of ∞ in $\hat{C}$ is given by $\{\infty\} \cup (\mu(\infty) \cap \hat{C})$.*

If $\hat{C}$ is periodic, then $\hat{C}$ is indecomposable if and only if $\hat{C} \cap I(f) \setminus \mu(\infty) \neq \emptyset$.

Any indecomposable continuum has uncountably many composants, all of which are pairwise disjoint. Hence we see that complicated topology of Julia continua automatically leads to the existence of points that cannot be connected to infinity by a set that escapes uniformly. However, our proof of Theorem 1.6 also allows us to construct Julia continua that have very simple topology, but nonetheless contain points in $I(f) \setminus \mu(\infty)$.

1.14. Theorem (A one-point uniformly escaping component)**.** *There exists a disjoint-type entire function f and a Julia continuum $\hat{C} = C \cup \{\infty\}$ of $J(f)$ such that:*

(a) $\hat{C}$ *is an arc, with one finite endpoint* z_0 *and one endpoint at* ∞*;*
(b) $C \subset I(f)$*, but* $\liminf_{n\to\infty} \min_{z\in C} |f^n(z)| < \infty$*. In particular, there is no nondegenerate connected set* $A \ni z_0$ *on which the iterates escape to infinity uniformly.*

Observe that this implies Theorem 1.3.

Number of tracts and singular values. So far, we have not said much about the nature of the functions f that occur in our examples, except that they are of disjoint type. Using recent results of Bishop [Bis12, Bis13], we can say considerably more:

1.15. Theorem (Class $\mathcal{S}$ and number of tracts)**.** *All examples of disjoint-type entire functions* f *mentioned in this section can be constructed in such a way that* f *has exactly two critical values and no finite asymptotic values, and such that all critical points of* f *have degree at most* 4*.*

Furthermore, with the exception of Theorem 1.10, the function f *can be constructed such that*

$$\mathcal{T}_R := f^{-1}(\{z \in \mathbb{C} \colon |z| > R\})$$

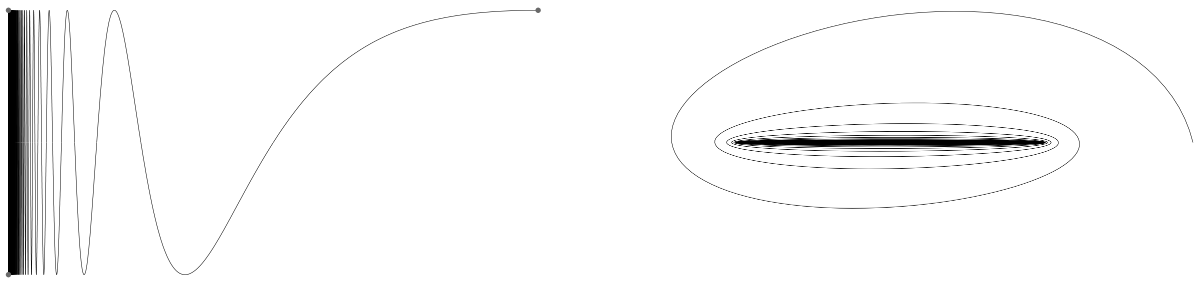
is connected for all R*. In Theorem 1.10, the function* f *can be constructed so that* $\mathcal{T}_R$ *has exactly two connected components for sufficiently large* R*.*

Remark. On the other hand, if $\mathcal{T}_R$ is connected for all R, then it turns out that every Julia component at a bounded address is homeomorphic to a periodic Julia component (Proposition 6.1). Hence it is indeed necessary to allow $\mathcal{T}_R$ to have two components in Theorem 1.10.

As pointed out in [BFR14], this leads to an interesting observation. By Theorem 1.15, the function f from Theorem 1.2 can be constructed such that $\#S(f) = 2$, such that f has no asymptotic values and such that all critical points have degree at most 4. Let v_1 and v_2 be the critical values of f, and let c_1 and c_2 be critical points of f over v_1 resp. v_2. Let $A \colon \mathbb{C} \to \mathbb{C}$ be the affine map with $A(v_1) = c_1$ and $A(v_2) = c_2$. Then the function $g := f \circ A$ has super-attracting fixed points at v_1 and v_2. By the results from [Rem09] discussed earlier, the Julia set $J(g)$ contains infinitely many invariant subsets, each of whose one-point compactification is homeomorphic to the pseudo-arc. On the other hand, $J(g)$ is locally connected by [BFR14, Corollary 1.9]. Hence we see that, in contrast to the polynomial case, local connectivity of Julia sets does not imply simple topological dynamics, even for hyperbolic functions.

Embeddings. Given an arc-like continuum X, there are usually different ways to embed X in the plane. That is, there might be continua $C_1, C_2 \subset \hat{\mathbb{C}}$ such that X is homeomorphic to C_1 and C_2, but such that no homeomorphism $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$ can map C_1 to C_2. (That is, C_1 and C_2 are not *ambiently* homeomorphic.) Our construction in Theorem 1.6 is rather flexible, and we can indeed construct different Julia continua that are homeomorphic but not *ambiently* homeomorphic. In particular, as briefly mentioned in the discussion of results concerning accessibility above, it would be possible to construct a Julia continuum $\hat{\mathbb{C}}$ that is homeomorphic to the $\sin(1/x)$-continuum, and such that the limiting arc is not accessible from the complement of $\hat{\mathbb{C}}$. (See Figure 2).

For a disjoint-type entire function which has bounded slope, every Julia continuum can be covered by a *chain* with arbitrarily small links such that every link is a connected



(a) (b)

FIGURE 2. Two embeddings of the $\sin(1/x)$-continuum that are not ambiently homeomorphic

subset of the Riemann sphere. (For the definition of a chain, compare the remark after Proposition 7.2.) It is well-known [Bin51b, Example 3] that there are embeddings of arc-like continua without this property.

It is natural to ask whether this is the only restriction on the continua that can arise by our construction, but we shall not investigate this question further here.

Structure of the article. In Section 2, we collect background on the dynamics of disjoint-type entire functions. In particular, we review the *logarithmic change of variable*, which will be used throughout the remainder of the paper. We also recall some basic facts from the theory of continua. Following these preliminaries, the article essentially splits into two parts, which can largely be read independently of each other:

- *General topology of Julia continua.* In the first part of the article, we study general properties of Julia continua of disjoint-type entire functions. More precisely, in Section 3 we show that each such continuum has span zero, and prove the results concerning terminal points stated earlier. In Section 4, we investigate the structure of uniformly escaping components. Section 5 studies conditions under which all Julia continua are arclike, and establishes one half of Theorem 1.11. Finally, Section 6 shows that, in certain circumstances, different Julia continua are homeomorphic to each other.
- *Constructing prescribed Julia continua.* The second part of the paper is concerned with the constructions that allow us to find entire functions having prescribed arc-like Julia continua, as outlined in the theorems stated in this section. We review topological background on arc-like continua in Section 7 and, in Section 8, give a detailed proof of a slightly weaker version of Theorem 1.6 (where the function f is allowed to depend on the arc-like continuum in question). Section 9 applies this general construction to obtain the examples from Theorems 1.3, 1.14 and 1.8. The construction of bounded-address continua is very similar to that in Section 8; we sketch it in Section 10. Sections 11 and 12 contain the proofs of Theorems 1.11 and 1.2. Finally, we briefly discuss the modifications of the construction necessary to prove Theorems 1.6 and 1.10 as stated in the introduction (Section 13), and how to obtain Theorem 1.15.

Basic notation. As usual, we denote the complex plane by $\mathbb{C}$ and the Riemann sphere by $\hat{\mathbb{C}}$. We also denote the unit disk by $\mathbb{D}$ and the right half-plane by $\mathbb{H}$.

We shall also continue to use the notations introduced throughout the introduction. In particular, the Fatou, Julia and escaping sets of an entire function are denoted by $F(f)$, $J(f)$ and $I(f)$, respectively. Euclidean distance is denoted by dist.

In order to keep the paper accessible to readers with a background in either continuum theory or transcendental dynamics, but not necessarily both, we aim to introduce all notions and results required from either area. For further background on transcendental iteration theory, we refer to [Ber93]. For a wealth of information on continuum theory, including the material treated here, we refer to [Nad92]. In particular, [Nad92, Chapter 12] contains a detailed treatment of arc-like continua.

We shall assume that the reader is familiar with plane hyperbolic geometry; see e.g. [BM07]. If $U \subset \mathbb{C}$ is simply-connected, then we denote the density of the hyperbolic metric by $\rho_U : U \to (0, \infty)$. In particular, we shall frequently use the *standard estimate* on the hyperbolic metric in a simply-connected domain:

$$\frac{1}{2\operatorname{dist}(z, \partial U)} \leq \rho_U(z) \leq \frac{2}{\operatorname{dist}(z, \partial U)} \tag{1.1}$$

We also denote hyperbolic diameter in U by diam_U, and hyperbolic distance by dist_U. Furthermore, the derivative of a holomorphic function f with respect to the hyperbolic metric is denoted by $\|Df(z)\|_U$. (Note that this is defined whenever $z, f(z) \in U$.)

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2. Preliminaries

Disjoint-type entire functions. Recall that a transcendental entire function $f : \mathbb{C} \to \mathbb{C}$ is of *disjoint type* if it is hyperbolic with connected Fatou set. The following (see [BK07, Lemma 3.1] or [MB12, Proposition 2.8]) provides an alternative definition of disjoint-type entire functions, which is the one that we shall work with.

2.1. Proposition (Characterization of disjoint-type functions)**.** *A transcendental entire function $f : \mathbb{C} \to \mathbb{C}$ is of disjoint type if and only if there exists a bounded Jordan domain D with $S(f) \subset D$ and $f(\overline{D}) \subset D$.*

Let f be of disjoint type, and consider the domain $W := \mathbb{C} \setminus \overline{D}$, with D as in Proposition 2.1. Since $S(f) \cap W = \emptyset$, if V is any connected component of $\mathcal{V} := f^{-1}(W)$, then $f : V \to W$ is a covering map. These components are called the *tracts* of f (over ∞). Since f is transcendental, it follows from the classification of covering maps of the punctured disc [For99, Theorem 5.10] that every tract V is simply-connected and that $f : V \to W$ is a universal covering map. In fact, ∂V is an unbounded Jordan domain, i.e. a Jordan domain whose boundary passes through ∞. (This follows by choosing a slightly larger domain $\tilde{D} \supset \overline{D}$ and applying the above observation: we see that ∂V is a preimage component of the simple closed curve ∂D under a universal covering map.)

Note that $\partial W \cap \partial \mathcal{V} = \emptyset$; this is why we use the term *disjoint type.* It follows easily that the Julia set $J(f)$ consists precisely of those points whose iterates remain in W

forever. Indeed, the latter set has no interior, compare [Rem09, Lemma 2.3], and $\overline{D}$ is clearly contained in the Fatou set (and indeed in the immediate basin of an attracting fixed point). For our purposes, we could also take this description as the *definition* of the Julia set of a disjoint-type entire function.

2.2. Proposition (Julia sets)**.** *If f is of disjoint type and D is as in Proposition 2.1,*

$$J(f) = \{z \in \mathbb{C}\colon f^n(z) \notin \overline{D} \text{ for all } n \geq 0\}.$$

The logarithmic change of variable. Following Eremenko and Lyubich [EL92], we study f using the *logarithmic change of variable.* To this end, let us assume for simplicity that $f(0) \in D$; this can always be achieved by conjugating f with a translation. Set $H := \exp^{-1}(W)$ and $\mathcal{T} := \exp^{-1}(\mathcal{V})$. Then there is a holomorphic function $F\colon \mathcal{T} \to H$ such that $f \circ \exp = \exp \circ F$. We may choose this map F to be $2\pi i$-periodic, in which case we refer to it as a *logarithmic transform* of f.

This representation is extremely convenient: for every component T of $\mathcal{T}$, the map $F\colon T \to H$ is now a conformal isomorphism (rather than a universal covering map as in the original coordinates). This makes it much easier to consider inverse branches. From now on, we shall always study the logarithmic transform of f. In fact, it turns out to be rather irrelevant that the map F has arisen from a globally defined entire function, which leads to the following definition, following [Rem09, RRRS11].

2.3. Definition (The classes $\mathcal{B}_{\log}$ and $\mathcal{B}_{\log}^{\mathrm{p}}$)**.**
The class $\mathcal{B}_{\log}$ consists of all holomorphic functions

$$F\colon \mathcal{T} \to H,$$

where F, $\mathcal{T}$ and H have the following properties:

(a) H is a $2\pi i$-periodic unbounded Jordan domain that contains a right half-plane.
(b) $\mathcal{T} \neq \emptyset$ is $2\pi i$-periodic and $\operatorname{Re} z$ is bounded from below in $\mathcal{T}$, but unbounded from above.
(c) Every component T of $\mathcal{T}$ is an unbounded Jordan domain that is disjoint from all its $2\pi i\mathbb{Z}$-translates. For each such T, the restriction $F\colon T \to H$ is a conformal isomorphism whose continuous extension to the closure of T in $\hat{\mathbb{C}}$ satisfies $F(\infty) = \infty$. (T is called a *tract of* F; we denote the inverse of $F|_T$ by F_T^{-1}.)
(d) The components of $\mathcal{T}$ have pairwise disjoint closures and accumulate only at ∞; i.e., if $z_n \in \mathcal{T}$ is a sequence of points all belonging to different components of $\mathcal{T}$, then $z_n \to \infty$.

If $F \in \mathcal{B}_{\log}$. then the *Julia set* and *escaping set* of F are defined by

$$J(F) := \{z \in H\colon F^n(z) \in \mathcal{T} \text{ for all } n \geq 0\} \qquad \text{and}$$
$$I(F) := \{z \in J(F)\colon \operatorname{Re} F^n(z) \to \infty \text{ as } n \to \infty\}.$$

If furthermore F is $2\pi i$-periodic, then we say that F belongs to the class $\mathcal{B}_{\log}^{\mathrm{p}}$. If $\overline{\mathcal{T}} \subset H$, then we say that F is of *disjoint type.*

Remark. If $F \in \mathcal{B}_{\log}$ has disjoint type, then, by conjugation with an isomorphism $H \to \mathbb{H}$ that commutes with translation by $2\pi i$, we obtain a disjoint-type function $G \in \mathcal{B}_{\log}$ that is conformally conjugate to F and whose range is the right half-plane $\mathbb{H}$. It is not difficult to see that all geometric properties discussed in this paper, such as bounded slope, are invariant under this transformation. Hence we could always assume that $H = \mathbb{H}$ in the following. However, we prefer to work directly with the above class, which retains a more direct connection to the original entire functions.

Any logarithmic transform F of a disjoint-type entire function, as described above, belongs to the class $\mathcal{B}_{\log}^{\mathrm{p}}$ and has disjoint type. The following result, due to Bishop, shows essentially that the converse also holds.

2.4. Theorem (Realization of disjoint-type functions)**.** *Let $G \in \mathcal{B}_{\log}^{\mathrm{p}}$ be of disjoint type and let g be defined by $g(\exp(z)) = \exp(G(z))$. Then there exists a disjoint-type function $f \in \mathcal{B}$ such that $f|_{J(f)}$ is topologically (and, in fact, quasiconformally) conjugate to $g|_{\exp(J(G))}$.*

Furthermore, there is a disjoint-type function $\tilde{f} \in \mathcal{S}$ such that every connected component of $J(G)$ is homeomorphic to a connected component of $J(\tilde{f})$ (but not necessarily vice-versa). The function f may be chosen to have exactly two critical values, no asymptotic values, and with all critical points of degree at most 4.

Proof. The first statement is Corollary 1.4 in [Bis12], which is a consequence of Theorem 1.2 in the same paper and [Rem09, Theorem 3.1]. The second statement follows in the same way, using Theorem 1.5 from [Bis12] rather than Theorem 1.2. ∎

Hence, in order to construct the examples of disjoint-type entire functions described in the introduction, it will be sufficient to construct suitable functions in the class $\mathcal{B}_{\log}^{\mathrm{p}}$. We remark that, with some extra care, the realization of our class $\mathcal{B}$ examples could also be carried out using the earlier approximation result in [Rem13, Theorem 1.9].

The combinatorics of Julia continua. Let $F \in \mathcal{B}_{\log}^{\mathrm{p}}$ be of disjoint type. The Markov partition provided by the tracts of F and their iterated preimages allows us to introduce a notion of symbolic dynamics as follows.

2.5. Definition (External addresses and Julia continua)**.**
Let $F \in \mathcal{B}_{\log}$ have disjoint type. An *external address* of F is a sequence $\underline{s} = T_0 T_1 T_2 \dots$ of tracts of F.

If $\underline{s}$ is such an external address, then we define

$$\begin{aligned} J_{\underline{s}}(F) &:= \{z \in J(F) \colon F^n(z) \in T_n \text{ for all } n\}, \\ \hat{J}_{\underline{s}}(F) &:= J_{\underline{s}}(F) \cup \{\infty\} \qquad \text{and} \\ I_{\underline{s}}(F) &:= I(F) \cap J_{\underline{s}}(F). \end{aligned}$$

When $J_{\underline{s}}(F)$ is nonempty, we say that $\underline{s}$ is *allowable* (for F). In this case, $\hat{J}_{\underline{s}}(F)$ is called a *Julia continuum* of F. An address $\underline{s}$ is called *bounded* if it contains only finitely many different tracts, and *periodic* if there is $p \geq 1$ such that $T_j = T_{j+p}$ for all $j \geq 0$.

Remark 1. By definition, we can write $\hat{J}_{\underline{s}}$ as a nested intersection of compact, connected sets (namely the pullback of $\overline{T_j} \cup \{\infty\}$ under the appropriate branch of F^{-j}) and hence $\hat{J}_{\underline{s}}$ is indeed a continuum.

Remark 2. It follows from [Rem07] that $J_{\underline{s}}(F)$ is always connected. We reprove this fact below, by showing that ∞ is a terminal point of $\hat{J}_{\underline{s}}(F)$. Indeed, a terminal point of a continuum X can never be a cut point of X, so $J_{\underline{s}}(F) = \hat{J}_{\underline{s}}(F) \setminus \{\infty\}$ is connected.

In particular, if f is an entire function of disjoint type and $F \in \mathcal{B}_{\log}^{\mathrm{p}}$ is a logarithmic transform of f, then every (arbitrary/bounded-address/periodic) Julia continuum of f, as defined in the introduction, is homeomorphic to a Julia continuum of F at an (allowable/bounded/periodic) address, and vice versa.

Hyperbolic expansion. In order to study disjoint-type functions, we shall use the fact that they are *expanding* on the Julia set, with respect to the hyperbolic metric on H. Recall that $\|DF(z)\|_H$ denotes the hyperbolic derivative of F, measured in the hyperbolic metric of H, and that diam_H denotes hyperbolic diameter in H.

2.6. Proposition (Expanding properties of F)**.** *Let $F\colon \mathcal{T} \to H$ be a disjoint-type function in $\mathcal{B}_{\log}$. Then there is a constant $\Lambda > 1$ such that*

$$\|DF(z)\|_H \geq \Lambda,$$

for all $z \in \mathcal{T}$; furthermore $\|DF(z)\|_H \to \infty$ as $\operatorname{Re} z \to \infty$.

Also, for every $R > 0$ there is $M > 0$ such that, for every $z \in H$:

$$\operatorname{diam}_H(\{w \in \mathcal{T}\colon |z-w| \leq R\}) \leq M.$$

Proof. The first fact is well-known and follows from standard estimates on the hyperbolic metric; see e.g. [BK07, Lemma 3.3] or [RRRS11, Lemma 2.1]. The second fact follows from the assumption that the closure of $\mathcal{T}$ is contained in H, that $\mathcal{T}$ is $2\pi i$-periodic, and the fact that the density $\rho_H(\zeta)$ of the hyperbolic metric of H tends to zero as $\operatorname{Re}\zeta \to \infty$. ∎

A simple consequence of hyperbolic expansion is the fact, mentioned in the introduction, that each Julia continuum at a bounded address contains a unique point with bounded orbit.

2.7. Proposition (Points with bounded orbits)**.** *Let $F \in \mathcal{B}_{\log}$, and let $\underline{s}$ be a bounded external address. Then there is a unique point $z_0 \in J_{\underline{s}}(F)$ with $\sup_{j\geq 0} \operatorname{Re} F^j(z_0) < \infty$. If $\underline{s}$ is periodic of period p, so is z_0.*

Proof. Uniqueness is clear from the expanding property of F, and the final claim in the statement follows from uniqueness. Hence it only remains to prove the existence of z_0.

Choose an arbitrary base point ζ_0 and set

$$D := \max_{i\geq 0} \operatorname{dist}_H(\zeta_0, F_{T_i}^{-1}(\zeta_0)),$$

where $\underline{s} = T_0 T_1 \dots$. Note that the maximum exists because $\underline{s}$ contains only finitely many different tracts.

Define $\delta := D \cdot \Lambda/(\Lambda - 1)$, where $\Lambda > 1$ is the constant from Proposition 2.6. Let Δ_0 be the closed hyperbolic disc of radius δ around ζ_0. If $z \in \Delta_0$, then we have

$$\operatorname{dist}_H(F_{T_i}^{-1}(z), \zeta_0) \leq \operatorname{dist}_H(F_{T_i}^{-1}(z), F_{T_i}^{-1}(\zeta_0)) + D \leq \frac{1}{\Lambda}\operatorname{dist}_H(z, \zeta_0) + D \leq \frac{\delta}{\Lambda} + D = \delta.$$

Hence $F_{T_i}^{-1}(\Delta_0) \subset \Delta_0$ for all $i \geq 0$. It follows that the compact sets

$$\Delta_j := F_{T_0}^{-1}(F_{T_1}^{-1}(\dots(F_{T_j}^{-1}(\Delta_0))\dots))$$

satisfy $\Delta_{j+1} \subset \Delta_j$, and hence their intersection contains some point z_0 with $F^j(z_0) \in \Delta_0$ for all $j \geq 0$. ■

The expanding property also implies that points within the same Julia continuum must eventually separate under iteration (see e.g. [RRRS11, Lemma 3.2]).

2.8. Lemma (Separation of real parts)**.** *Let $\underline{s}$ be an allowable external address, and let $z, w \in J_{\underline{s}}(F)$ with $z \neq w$. Then $|\operatorname{Re} F^n(z) - \operatorname{Re} F^n(w)| \to \infty$ as $n \to \infty$.*

Results from continuum theory. We shall frequently require the following fact.

2.9. Theorem (Boundary bumping theorem [Nad92, Theorem 5.6])**.** *Let X be a continuum, and let $E \subsetneq X$ be nonempty. If K is a connected component of $X \setminus E$, then $\overline{K} \cap \partial E \neq \emptyset$.*

We also recall some background on (in-)decomposable continua and composants. These are mainly used in Section 4.

2.10. Definition ((In-)decomposable continua)**.**
A continuum X is called *decomposable* if it can be written as the union of two proper subcontinua of X. Otherwise, X is *indecomposable*.

Furthermore X is called *hereditarily* (in-)decomposable if every non-degenerate subcontinuum of X is (in-)decomposable.

Recall that the *composant* of a point $x \in X$ is the union of all proper subcontinua containing X. We say that X is *irreducible* at a point $x \in X$ if there is some $y \in X$ such that X is irreducible between x and y (in the sense of Definition 1.7).

2.11. Proposition (Properties of composants)**.** *Let X be a continuum.*

- *(a) A point $x \in X$ is irreducible if and only if its composant is a proper subset of X.*
- *(b) A point $x \in X$ is terminal if and only if x is irreducible in K for every subcontinuum $K \ni x$.*
- *(c) A continuum is hereditarily indecomposable if and only if every point of x is a terminal point.*
- *(d) If C is a composant of X, then $X \setminus C$ is connected.*
- *(e) A decomposable continuum has either one or three different composants.*
- *(f) An indecomposable continuum has uncountably many composants, every two of which are disjoint, and each of which is dense in X.*

Proof. The first claim is immediate from the definition, and (b) is a simple exercise. By definition, X is hereditarily indecomposable if and only if any two subcontinua of X are either nested or disjoint. Clearly this is the case if and only if all points of X are terminal. The remaining statements can be found in Theorems 11.4, 11.13 and 11.17 and Exercise 5.20 of [Nad92]. ■

Finally, we recall the definition of *inverse limits*; see [Nad92, Chapter 2] for more information on this topic.

2.12. Definition (Inverse limits).
Let $(X_j)_{j\geq 0}$ be a sequence of continua, and let $f_j\colon X_j \to X_{j-1}$ be a continuous function for every $j \geq 1$.

Let X be the set of all "inverse orbits", $(x_0 \leftarrow\!\shortmid x_1 \leftarrow\!\shortmid x_2 \leftarrow\!\shortmid \dots)$, with $x_j \in X_j$ for all $j \geq 0$ and $f_j(x_j) = x_{j-1}$ for all $j \geq 1$. Then X, with the product topology, is called the *inverse limit* of the functions (f_j), and denoted $\varprojlim (f_j)_{j=1}^{\infty}$. The inverse limit is again a continuum. The maps f_j are called the *bonding maps* of the inverse limit X.

The introduction of some further topological background concerning arc-like continua will be delayed until Section 7, as it is only required in the second part of this article.

3. Topology of Julia continua

We now study the general topological properties of Julia continua for a function in the class $\mathcal{B}_{\log}$. In particular, we prove that every such Julia continuum has span zero. The idea of the proof is rather simple: Since each tract T cannot intersect its own $2\pi i$-translates, two points cannot exchange position by moving inside T without coming within distance 2π of each other. Now let $\underline{s}$ be an allowable external address. By applying the preceding observation to the tract T_j, for j large, and using the expanding property of F, we see that two points cannot cross each other within $J_{\underline{s}}(F)$ without passing within distance ε of each other, for every $\varepsilon > 0$. This establishes that $J_{\underline{s}}(F)$ has span zero. (This idea is similar in spirit to the proof of Lemma 2.2 and Corollary 3.4 in [Rem07], which we in fact recover below.)

However, some care is required, since the tract T can very well contain points whose imaginary parts differ by a large amount (see Figure 3). Hence we shall have to take some care in justifying the informal argument above, by studying the possible structure of *logarithmic tracts* somewhat more closely.

3.1. Definition (Logarithmic tracts).
A Jordan domain T that does not intersect its $2\pi i$ translates and that is unbounded to the right (i.e., $\operatorname{Re} z \to +\infty$ as $z \to \infty$ in T) is called a *logarithmic tract.* In particular, every tract of a function $F \in \mathcal{B}_{\log}$ is a logarithmic tract.

Within such a tract, we wish to understand when points can move around without having to come close to each other. To study this question, we introduce the following terminology.

3.2. Definition (Separation number).
For any $z \in \mathbb{C}$, we denote by I_z the line segment

$$I_z := \{z + i \cdot t \colon t \in [-2\pi, 2\pi]\}.$$

Let T be a logarithmic tract, and let $z \in T$. If $a, b \in T \setminus I_z$, then we define $\operatorname{sep}_T(a, b; z)$ to denote the smallest number of intersections of a curve connecting a and b in T with the segment I_z.

(The tract T will usually be fixed in the following, and we shall then suppress the subscript T in this notation.)

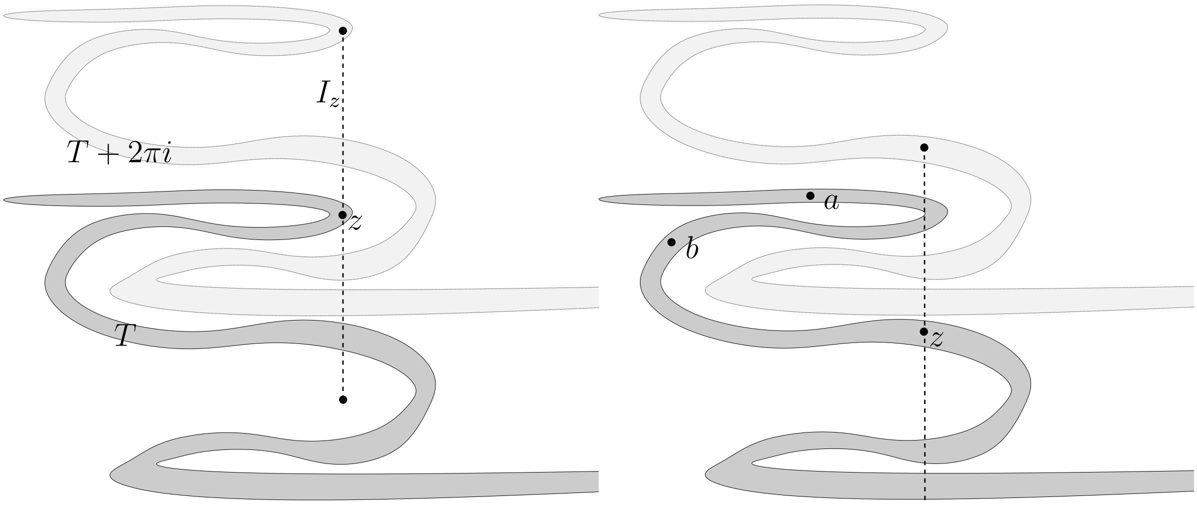


FIGURE 3. A tract containing points whose imaginary parts are further apart than 2π. Subfigure (a) illustrates the definition of the segment I_z, while the configuration in (b) shows that the number $\operatorname{sep}_T(a,b;z)$ can decrease under perturbation of z (it will change from 2 to 0 if we move the point z slightly to the right).

3.3. Proposition (Continuous parity of separation numbers)**.** *Let T be a logarithmic tract, let $a,b,z\in T$, and suppose that $a,b\notin I_z$. Then the parity of $\operatorname{sep}_T(a,b;z)$ varies continuously for small perturbations of a, b and z.*

That is, if X denotes the set of points $(a,b;z)\in T^3$ with $a,b\notin I_z$, then the function

$$\operatorname{sep}_T\colon X\to\mathbb{Z}_2;\quad (a,b;z)\mapsto\operatorname{sep}_T(a,b;z) \pmod 2$$

is continuous.

On the other hand, as a (or b) passes through the segment I_z transversally, the number $\operatorname{sep}_T(a,b;z)$ changes parity.

Remark. The function $\operatorname{sep}_T(a,b;z)$ itself need not be continuous in z, although it is always upper semi-continuous (i.e., under a small perturbation of z, the separation number might decrease.

Proof. Observe that $I_z\cap T$ is a union of vertical cross-cuts of the tract T. Clearly $\operatorname{sep}(a,b;z)$ is precisely the number of such cross-cuts that separate a from b in T. Recall that each cross-cut C separates T into precisely two components, one on each side of C. In particular, as the point a (or b) crosses I_z, keeping the other points fixed, the number $\operatorname{sep}(a,b;z)$ increases or decreases by 1. This proves the final claim.

Also observe that, if γ $\gamma\subset T$ is a curve connecting a and b, and γ intersects I_z only transversally, then the number of intersections between γ and I_z has the same parity as $\operatorname{sep}(a,b;z)$. Indeed, The curve γ must intersect every cross-cut that separates a from b in an odd number of points, and every cross-cut that does not separate a from b in an even number of points.

So let $a, b, z \in T$ with $a, b \notin I_z$. Clearly a sufficiently small perturbation of a or of b does not change the value (and hence the parity) of $\operatorname{sep}(a, b; z)$, so we only need to focus on what happens when we perturb z to a nearby point $\tilde{z}$.

Let γ be a curve, as above, conencting a and b and intersecting I_z only transversally. If $\tilde{z}$ is close enough to z, then γ also intersects $I_{\tilde{z}}$ only transversally, and in the same number of points. Hence we see that $\operatorname{sep}(a, b; z)$ and $\operatorname{sep}(a, b; \tilde{z})$ have the same parity, as claimed. ■

We are now ready to prove the statement alluded to at the beginning of the section, which then allows us to deduce that every Julia continuum has span zero.

3.4. Proposition (Bounded span of tracts)**.** *Let T be a logarithmic tract, and let $A \subset T \cup \{\infty\}$ be compact and connected. Suppose furthermore that $X \subset (T \cup \{\infty\})^2$ is a connected set whose first and second components both project to A.*

Then there is a point $(z, w) \in X \cap T^2$ such that $z \in I_w$. In particular, $|z - w| < 2\pi$.

Proof. We shall prove the contrapositive: suppose that $X \subset (T \cup \{\infty\})^2$ is any set whose first and second component both project to A, and such that $z \notin I_w$ whenever $(z, w) \in X \cap T^2$. We shall show that X is disconnected.

Let a be a left-most point of A; i.e. $\operatorname{Re} a = \min_{z \in A} \operatorname{Re} z$. Let U consist of the set of all points $(z, w) \in X$ such that $w \notin I_a$ and such that $\operatorname{sep}(a, z; w)$ is even. By Proposition 3.3, this set is open in X.

On the other hand, we claim that $V := X \setminus U$ is also open in X. Let $(z, w) \in V$. If $w \notin I_a$, then V contains a neighborhood of (z, w) in X by Proposition 3.3.

Now suppose that $w \in I_a$. Let $\tilde{z}, \tilde{w} \in T$ be chosen close to z and w (not necessarily in A). If $\operatorname{Re} \tilde{w} < \operatorname{Re} w = \operatorname{Re} a$, then clearly $\operatorname{sep}(a, \tilde{z}; \tilde{w}) = 0$. Hence it follows from Proposition 3.3 that $\operatorname{sep}(a, \tilde{z}; \tilde{w}) = 1$ when $\operatorname{Re} \tilde{w} > \operatorname{Re} a$. If $(\tilde{z}, \tilde{w}) \in X$, then we either have $\tilde{w} \in I_a$ or $\operatorname{Re} \tilde{w} > \operatorname{Re} a$ (provided the initial perturbation was small enough), and hence $(\tilde{z}, \tilde{w}) \in V$ in either case, as required.

Furthermore, both U and V are nonempty. Indeed, by assumption there are $z, w \in A$ is such that $(a, w), (z, a) \in X$. We have $(z, a) \in V$ by definition (since $a \in I_a$). Similarly, we have $w \notin I_a$ by assumption on X, and $\operatorname{sep}(a, a; w) = 0$ by definition. Hence $(a, w) \in U$.

We have shown that X is disconnected, as desired. ■

3.5. Theorem (Julia continua have span zero)**.** *Let $F \in \mathcal{B}_{\log}$ be of disjoint type, and let $\hat{C}$ be a Julia continuum of F. Then $\hat{C}$ has span zero.*

Proof. Suppose that $X \subset \hat{C}^2$ is a continuum whose projections to the first and second coordinates are the same set $A \subset T \cup \{\infty\}$. For $n \geq 0$, consider $A_n := F^n(A)$ and $X_n := \{(F^N(z), F^N(w)) : (z, w) \in X\}$.

Then for each n, A_n is contained in $T_n \cup \{\infty\}$ for some tract T_n of F. By Proposition 3.4, X_n contains a point (ζ_n, ω_n) such that $|\zeta_n - \omega_n| < 2\pi$. Let $z_n, w_n \in A$ such that $\zeta_n = F^n(z)$ and $\omega_n = F^n(w)$. Since the hyperbolic distance between ζ_n and ω_n is uniformly bounded by Proposition 2.6, and F uniformly expands the hyperbolic metric, it follows that the hyperbolic distance in T between ζ_n and ω_n tends to zero. Thus $|\zeta_n - \omega_n| \to 0$. Hence $(\zeta, \zeta) \in X$, where ζ is any limit point of (ζ_n). Hence we have shown that $\hat{C}$ has span zero. ■

We shall next prove the fact that infinity, as well as any nonescaping or accessible point, is terminal in each Julia continuum. To do so, it will be helpful to note down the following consequence of Proposition 3.3. (It is essentially an extension of the idea used in the proof of Proposition 3.4.)

3.6. Corollary (Moving along a connected set)**.** *Let T be a logarithmic tract, and let $A \subset T \cup \infty$ be compact and connected. Choose $a, b \in A$ be such that*

$$\operatorname{Re} a = \min_{z \in A} \operatorname{Re} z \quad \textit{and} \quad \operatorname{Re} b = \max_{z \in A} \operatorname{Re} z.$$

(Here we use the convention that $\operatorname{Re} \infty = +\infty$.)

Let $z \in A$ such that $a, b \notin I_z$. Then $\operatorname{sep}(a, b; z)$ is odd. In particular, I_z separates a from b in T.

Remark 1. Note that the number $\operatorname{sep}(a, \infty; z) = \lim_{\tilde{b} \to \infty}(a, \tilde{b}; z)$ is well-defined by Proposition 3.3.

Remark 2. The statement of this corollary means that, in order to move from the left-most point to the right-most point of A, we must pass along within distance at most 2π of all of A. This is the key statement we shall require in the following.

Proof. Let us assume, for simplicity, that $\infty \notin A$, so that b is finite (the case where $b = \infty$ follows easily by a limiting argument). By the previous proposition, the set of z such that $\operatorname{sep}(a, b; z)$ is odd is relatively open and closed in $A \setminus (I_a \cup I_b)$.

Let K be a component of $A \setminus (I_a \cup I_b)$. Then, by the boundary bumping theorem (Theorem 2.9), the closure of K intersects I_a or I_b; let us suppose without loss of generality that $z_0 \in I_a \cup \overline{K}$.

If $z \in K$ is sufficiently close to z_0, then $\operatorname{sep}(a, b; z) = 1$, just as in the proof of Proposition 3.4. By continuity, $\operatorname{sep}(a, b; z)$ is odd for all $z \in K$, as desired. ∎

3.7. Theorem (The role of ∞)**.** *Let $\hat{C}$ be a Julia continuum of a disjoint-type function $F \in \mathcal{B}_{\log}$. Then ∞ is a terminal point of $\hat{C}$.*

Remark. Theorems 3.5 and 3.7 together establish the first part of Theorem 1.6.

Proof. Let $\underline{s} = T_0 T_1 T_2 \dots$ be the address of $\hat{C}$. Suppose that $\hat{A}^1, \hat{A}^2 \subset \hat{C}$ are subcontinua both containing ∞. Let us set $\hat{A} := \hat{A}^1 \cup \hat{A}^2$. We also define $A := \hat{A} \setminus \{\infty\}$ and $A_n := F^n(A)$ for $n \geq 0$. The sets A_n^j, with $n \geq 0$ and $j \in \{1, 2\}$, are defined analogously.

For each n, let a_n be a left-most point of A_n as in Corollary 3.6. There is $j \in \{1, 2\}$ such that $a_{n_k} \in A_{n_k}^j$ for an infinite sequence (n_k). Without loss of generality, we may suppose that $j = 1$; we shall show that $A^2 \subset A^1$.

Indeed, let $z \in A^2$. We claim that $z_k := F^{n_k}(z)$ satisfies $\operatorname{dist}(z_k, A_{n_k}^1) \leq 2\pi$. If $z_k \in I_{a_{n_k}}$, then this is clearly true. Otherwise, I_{z_k} separates a_{n_k} from ∞ in the tract T_{n_k} by Corollary 3.6. Hence I_{z_k} intersects $A_{n_k}^1$, and thus indeed $\operatorname{dist}(z_k, A_{n_k}^1) \leq 2\pi$.

By the expanding property of F, it follows that $\operatorname{dist}(z, A^1) = 0$, and hence $z \in A^1$, as claimed. ∎

We now turn out attention to nonescaping points in Julia continua.

3.8. Theorem (Nonescaping points are terminal)**.** *Let $F \in \mathcal{B}_{\log}$ be of disjoint type, and let $\hat{C}$ be a Julia continuum of F. If $z_0 \in \hat{C} \setminus \{\infty\}$ is nonescaping, then z_0 is a terminal point of $\hat{C}$, and $\hat{C}$ is irreducible between z_0 and ∞.*

Proof. Let $\underline{s} = T_0T_1\dots$ be the address of $\hat{C}$. Since z_0 is a nonescaping point, there is a number $R > 0$ and a sequence (n_k) such that $\operatorname{Re} F^{n_k}(z_0) < R$ for all k.

Let $\hat{A}^1, \hat{A}^2 \subset \hat{C}$ be subcontinua both containing z_0. Similarly as in the preceding proof, let us set $\hat{A} := \hat{A}^1 \cup \hat{A}^2$, and let b_k be the right-most point of $A_k := F^{n_k}(\hat{A} \cap \mathbb{C})$. By relabelling, and by passing to a further subsequence if necessary, we may assume that $b_k \in A_k^1 := F^{n_k}(\hat{A}^1 \cap C$. We shall show that $A^2 \subset A^1$ (where $A^j = \hat{A}^j \cap \mathbb{C}$).

Recall that, up to translations in $2\pi i\mathbb{Z}$, only finitely many tracts intersect the vertical line at real part R. In particular, we can find a constant $Q > 0$, indepenent of k, with the following property. Any two points in T_{n_k} both of whose real parts are at most R can be connected by a curve $\gamma \subset T_{n_k}$ that consists entirely of points at real parts less than Q. (Simply choose Q sufficiently large to make sure that no bounded component of $T_{n_k} \setminus \{z \in \mathbb{C} : \operatorname{Re} z = R\}$ contains a point of real part greater than Q.)

Now let $z \in A^2$, and consider the point $z_k := F^{n_k}(z)$. Also let a_k be the left-most point of A_k. By Corollary 3.6, the segment I_{z_k} separates a_k from b_k. Furthermore, we have $|\operatorname{Re} z_k - \operatorname{Re} F^{n_k}(z_0)| \to \infty$ by Lemma 2.8, and hence $\operatorname{Re} z_k \geq Q$ when k is chosen sufficiently large. Hence I_{z_k} also separates $F^{n_k}(z_0)$ from b_k, and therefore intersects A_k^1. As before, it follows from the expansion of F that $z \in A^1$.

This proves that z_0 is a terminal point. Furthermore, if $\hat{A}^1 \subset \hat{C}$ is a continuum containing both z_0 and ∞, then we can choose $\hat{A}^2 = \hat{C}$ in the above argument, and conclude that $\hat{C} = \hat{A}^1$. Thus $\hat{C}$ is indeed irreducible between z_0 and ∞. ■

We remark that the set of nonescaping point in any given Julia continuum is geometrically rather small. (We refer to [Fal90] for the definition of Hausdorff dimension.)

3.9. Proposition (Hausdorff dimension of nonescaping points with a given address)**.** *Let $F \in \mathcal{B}_{\log}$ be of disjoint type, and let $\hat{C}$ be a Julia continuum of F. Then the Hausdorff dimension of the set of nonescaping points in $\hat{C}$ is zero.*

Proof. If z is a nonescaping point, then by definition there is $K > 0$ such that $\operatorname{Re} F^n(z) \leq K$ infinitely often. So the set of nonescaping points in $\hat{C} = J_{\underline{s}}(F)$ can be written as

$$J_{\underline{s}}(F) \setminus I(f) = \bigcup_{K>0} \bigcap_{n_0 \in \mathbb{N}} \bigcup_{n \geq n_0} F_{\underline{s}}^{-n}(\{z \in T_n : \operatorname{Re} z \leq R\}),$$

where $\underline{s} = T_0T_1\dots$ is the address of $\hat{C}$.

Since a countable union of sets of Hausdorff dimension zero has Hausdorff dimension zero, it is sufficient to fix $K > 0$ and show that the set

$$S(K) := \bigcap_{n_0 \in \mathbb{N}} \bigcup_{n \geq n_0} F_{\underline{s}}^{-n}(\{z \in T_n : \operatorname{Re} z \leq R\})$$

has Hausdorff dimension zero.

We again use the fact that, up to translation, there are only finitely many tracts that intersect the line $\{\operatorname{Re} z = K\}$. For each such tract, the set of points with real part $\leq K$ is a bounded subset of H, and hence has finite hyperbolic diameter (in H). In other words, for every K, there is a number C such that, for every tract T of F, the set of points in T with real part $\leq K$ has hyperbolic diameter bounded above by C.

Keeping in mind that the map $F\colon T_n \to H$ is a hyperbolic isometry, it follows that

$$\operatorname{diam}_{T_0}\bigl(F_{\underline{s}}^{-n}(\{z \in T_n : \operatorname{Re} z \leq R\})\bigr) \leq C \cdot \Lambda^{-(n-1)}$$

for $n \geq 1$, where $\Lambda > 1$ is the expansion constant from Proposition 2.6. Hence, by the standard estimate (1.1), the Euclidean diameter of this set is bounded by $2\pi \cdot C \cdot \Lambda^{-(n-1)}$.

Let $t > 0$. Then for every $n_0 \geq 1$ the t-dimensional Hausdorff measure of $S(K)$ is bounded by

$$\begin{aligned}\sum_{n\geq n_0} \operatorname{diam}(F_{\underline{s}}^{-n}(\{z \in T_n : \operatorname{Re} z \leq R\}))^t &\leq \sum_{n\geq n_0} \bigl(2\pi \cdot C \cdot \Lambda^{(-(n-1))}\bigr)^t \\ &= (2\pi C)^t \cdot \sum_{n\geq n_0 - 1} (\Lambda^t)^n.\end{aligned}$$

As this quantity tends to zero as $n_0 \to \infty$, we see that $\dim(S(K)) \leq t$. Since $t > 0$ was arbitrary, we have $\dim(S(K)) = 0$, as claimed. ∎

Our final topic in this section is the study of points in $J(F)$ that are accessible from $H \setminus J(F)$.

3.10. Theorem (Accessible points)**.** *Let $F \in \mathcal{B}_{\log}^{\mathrm{p}}$ be of disjoint type, and let $\hat{C} = C \cup \infty$ be the Julia continuum containing z_0. Suppose that $z_0 \in C$ is accessible from $\mathbb{C} \setminus J(F)$.*

Then z_0 is a terminal point of $\hat{C}$, and $\hat{C}$ is irreducible between z_0 and ∞. Furthermore, z_0 is the unique point of $\hat{C}$ that is accessible from $\mathbb{C} \setminus J(F)$, and $\hat{C} \setminus \{z_0\} \subset I(F)$.

Proof. Let γ be an arc that connects ∂H to z_0 without intersecting $J(F)$ in any other points. Then, for every $n \geq 0$, the image $F^n(\gamma)$ contains a piece that connects $F^n(z_0)$ to ∂H.

Let a_n be a left-most point of $C_n := F^n(C)$, and let γ_k be a piece of $F^n(\gamma)$ that connects $F^n(z_0)$ with a point of real part $\operatorname{Re} a_n$, containing no point of real part less than $\operatorname{Re} a_n$. Since γ_n does not intersect the $2\pi i\mathbb{Z}$-translates of C, it follows that the set

$$\tilde{C}_n := C_n \cup \gamma_n$$

is disjoint from its own $2\pi i\mathbb{Z}$-translates. It is not difficult to see that we can find a logarithmic tract $\tilde{T}_n$ with $\tilde{T}_n \supset \tilde{C}_n$. Note that $\tilde{T}_n$ is *not* a tract of F, but that we can nonetheless apply the methods of this section to its compact connected subsets.

With this observation, the proof that z_0 is terminal, and that $\hat{C}$ is irreducible between z_0 and ∞, is completely analogous to Theorem 3.8. Indeed, suppose that $\hat{A}^1, \hat{A}^2$ are subcontinua of $\hat{C}$ both containing z_0, and consider the sets

$$A_n := A_n^1 \cup A_n^2 \cup \gamma_k$$

(where A_n^j is again the image under F^n of $A^j := \hat{A}^j \setminus \{\infty\}$). Let b_n be the right-most point of A_n; we assume that the sets are labelled such that $b_n \in A_n^1$ for infinitely many n. Corollary 3.6 implies that $I_z \cap A_n^1 \cup \gamma_n \neq \emptyset$ for every $z \in A_n^2$. Hence $A^2 \subset A^1 \cup \gamma$. Since $\gamma \cap \hat{C} = \{z_0\}$, it follows that in fact $\hat{A}^2 \subset \hat{A}^1$, as required.

Now suppose that $\zeta_0 \in \hat{C}$ is a nonescaping point; say $\operatorname{Re} F^{n_k}(\zeta_0) \leq R$ for a suitable infinite sequence (n_k). Set $\zeta_k := F^{n_k}(\zeta_0)$ and $z_k := F^{n_k}(z_0)$; then $\operatorname{Re} z_k \to \infty$. Let ω_k

be the left-most point of γ_{n_k} (so $\operatorname{Re}\omega_k = \operatorname{Re} a_{n_k}$). By Corollary 3.6, either $\omega_k \in I_{\zeta_k}$, or the segment I_{ζ_k} separates the right-most point of γ from infinity. As in the proof of Theorem 3.8, this segment then also separates the former point from z_k, provided that $\operatorname{Re} z_k$ is sufficiently large. Hence we see that $I_{\zeta_k} \cap \gamma_{n_k} \neq \emptyset$ for all sufficiently large k. But this implies that $\zeta_0 \in \gamma$, which is a contradiction.

A similar argument shows that $\hat{C}$ cannot contain two different accessible points. We omit the details since this fact is well-known. Indeed, the set $\hat{C}$ is precisely the impression of a unique prime end of $\mathbb{C} \setminus J(F)$ (see e.g. [BK07], and hence contains at most one accessible point. (This also follows from the results of Section 6 below.) ■

Remark. A similar argument shows that, if $\hat{C}$ contains both an accessible point z_0 and a nonescaping point z_1, then $z_0 = z_1$. In particular, any Julia continuum containing more than one nonescaping point (such as those we construct later on in the paper) cannot contain any accessible points.

4. Uniform escape

We next discuss the connection between topological properties of Julia continua and uniformly escaping components.

4.1. Definition (Uniformly escaping component).
Let $F \in \mathcal{B}_{\log}$ be of disjoint type, and let $\underline{s}$ be an allowable external address. If $z \in I_{\underline{s}}(F)$, then the *uniformly escaping component* of z, denoted $\mu(z) := \mu_{\underline{s}}(z)$, is the union of all connected sets $A \subset J(F)$ with $z \in A$ for which $\operatorname{Re} F^n|_A$ converges to infinity uniformly.

We also define

$$\mu_{\underline{s}}(\infty) := \{z \in J_{\underline{s}}(F) : \text{there is an unbounded, closed, connected set } A \subset J_{\underline{s}}(F) \text{ such that } z \in A \text{ and } \operatorname{Re} F^n|_A \to \infty \text{ uniformly}\}.$$

Remark. The set $\mu_{\underline{s}}(\infty)$ appears implicitly in [Rem07, Corollary 3.4], which implies that it is always connected as a subset of the complex plane. In particular, if $z \in \mu_{\underline{s}}(\infty)$, then $\mu_{\underline{s}}(z) = \mu_{\underline{s}}(\infty)$.

For the remainder of the section, we shall fix a disjoint-type function $F \in \mathcal{B}_{\log}$ and an allowable external address. In [Rem07, Proposition 3.2], given any $z \in J_{\underline{s}}(F)$, an unbounded and connected subset of $I_{\underline{s}}(F)$ is constructed whose points escape "as fast as possible" in a certain sense. This shows that $\mu_{\underline{s}}(\infty)$ is non-empty, and suggests the following definition.

4.2. Definition ($\underline{s}$-fast escaping points).
Let $\underline{s} = T_0T_1\dots$ be an external address for F that is realized. We say that a point $z \in J_{\underline{s}}(F)$ belongs to the *$\underline{s}$-fast escaping set $A_{\underline{s}}(F)$* if there exists an open set D_0 intersecting $J_{\underline{s}}(F)$ with the following property: If we inductively define $D_{j+1} := F(T_j \cap D_j)$, then $F^j(z)$ belongs to the unbounded connected component of $T_j \setminus D_j$ for all j.

Remark. The definition is reminiscent of, and motivated by, the description of the *fast escaping set* $A(f)$ of an entire function that was given by Rippon and Stallard [RS05]. However, we note that there is no simple relation between the two sets. Indeed, it is not only possible that the $\underline{s}$-fast escaping set contains points that are not "fast" for the global function, but also that some points that are "fast" for the global function may not belong to $A_{\underline{s}}(F)$. We shall not discuss this relation further here.

4.3. Proposition (Existence of $\underline{s}$-fast escaping points)**.** *If $z \in A_{\underline{s}}(F)$, then there exists an unbounded closed connected set $A \subsetneq A_{\underline{s}}(F)$ that contains z, and on which the iterates escape to infinity uniformly. In particular, $A_{\underline{s}}(F) \subset \mu_{\underline{s}}(F)$. Furthermore, $A_{\underline{s}}(F)$ is dense in $J_{\underline{s}}(F)$.*

Proof. The first claim is clear from the definition. Indeed, let A_j be the unbounded connected component of $T_j \setminus D_j$ as in Definition 4.2. Then $F_{T_j}^{-1}(\overline{A_{j+1}}) \subset A_j$ by definition. It follows that

$$\hat{A} := \bigcap_{n\geq 0} F_{T_0}^{-1}(F_{T_1}^{-1}(\dots(F_{T_n}^{-1}(\overline{A_{n+1}}))\dots)) \cup \{\infty\}$$

is a compact and connected set containing both z and ∞. Furthermore, the set $A := \hat{A} \setminus \{\infty\}$ is contained in $A_{\underline{s}}(F)$ by definition, and it is connected since ∞ is a terminal point of $\hat{C}$. The fact that points in A escape uniformly follows from the fact that, for every R, the set D_n separates all points in T_n that have real part at most R from ∞. (Compare [Rem07, Lemma 3.1].)

To prove density of $A_{\underline{s}}(F)$, let $z \in J_{\underline{s}}(F)$, and let D_0 be a small disc around z. Then the set A, constructed in the preceding paragraph, must intersect ∂D_0. This proves that $A_{\underline{s}}(F)$ is dense in $J_{\underline{s}}(F)$. (See also [Rem07, Corollary 3.6].) This also shows that $A \subsetneq A_{\underline{s}}(F)$. ■

Interestingly, it turns out that we can define $A_{\underline{s}}(F)$ purely using the topology of $\hat{J}_{\underline{s}}(F)$:

4.4. Proposition (Composants and uniform escape)**.** *$\hat{A}_{\underline{s}}(F) := A_{\underline{s}}(F) \cup \{\infty\}$ is the composant of ∞ in $\hat{J}_{\underline{s}}(F)$.*

In other words, $\hat{J}_{\underline{s}}(F)$ is irreducible between z and ∞ if and only if $z \notin A_{\underline{s}}(F)$.

Proof. By Proposition 4.3, $\hat{A}_{\underline{s}}(F)$ is contained in the composant of ∞.

On the other hand, let $K \subsetneq \hat{J}_{\underline{s}}(F)$ be a proper subcontinuum containing ∞; we must show that $K \subset \hat{A}_{\underline{s}}(F)$. Since $\hat{J}_{\underline{s}}(F) \cap K \neq \emptyset$, by the final statement in Proposition 4.3, we see that $\hat{A}_{\underline{s}}(F) \setminus K \neq \emptyset$.Hence, by the first part of Proposition 4.3, there is a continuum $A \subset \hat{A}_{\underline{s}}(F)$ with $\infty \in A$ and $A \not\subset K$. Since ∞ is a terminal point of $\hat{J}_{\underline{s}}(F)$, we have $K \subset A \subset \hat{A}_{\underline{s}}(F)$, as desired. ■

4.5. Corollary (Characterisation of decomposability)**.** *The set $J_{\underline{s}}(F) \setminus A_{\underline{s}}(F)$ is non-empty and connected. Moreover, the following are equivalent:*

(a) $\hat{J}_{\underline{s}}(F)$ is a decomposable continuum;
(b) $J_{\underline{s}}(F) \setminus A_{\underline{s}}(F)$ is bounded.

Proof. Let us set $\hat{C} := \hat{J}_{\underline{s}}(F)$ and $A := A_{\underline{s}}(F) \cup \{infty\}$; that is, A is the composant of ∞ in $\hat{C}$.

The set $B := J_{\underline{s}}(F) \setminus A_{\underline{s}}(F) = \hat{C} \setminus A$ is nonempty because ∞ is a terminal point of $\hat{C}$, and hence irreducible by Proposition 2.11 (b). It is connected by Proposition 2.11 (d). If $\hat{C}$ is indecomposable, then B is unbounded by Proposition 2.11 (f).

On the other hand, suppose that $\hat{C}$ is decomposable, say $\hat{C} = X \cup Y$, where X and Y are proper subcontinua, say with $\infty \in X$. Then $X \subset A$ by definition, and hence $B \subset Y$. Since ∞ is a terminal point, we see that $\infty \notin Y$, and hence Y is bounded. ∎

In many instances, the following statement will allow us to infer that there exist points in $I_{\underline{s}}(F) \setminus \mu_{\underline{s}}(F)$; i.e., escaping points that can *not* be connected to infinity by a set that escapes uniformly.

4.6. Corollary (Existence of different uniformly escaping components)**.** *Suppose that the set $J_{\underline{s}}(F) \setminus A_{\underline{s}}(F)$ contains more than one point. Then $I_{\underline{s}}(F) \setminus A_{\underline{s}}(F) \neq \emptyset$.*

If additionally

$$\min_{z \in J_{\underline{s}}(F)} \operatorname{Re} F^n(z) \not\to \infty$$

as $n \to \infty$, then $I_{\underline{s}}(F) \setminus \mu_{\underline{s}}(F) \neq \emptyset$.

Proof. By Corollary 4.5, the set $X := J_{\underline{s}}(F) \setminus A_{\underline{s}}(F)$ is connected, and the set of nonescaping points has Hausdorff dimension zero by Proposition 3.9. Hence, if X contains more than one point, it must intersect $I(f)$. (Indeed, this intersection has Hausdorff dimension at least 1.)

To prove the second claim, suppose that the iterates do not escape uniformly on $J_{\underline{s}}(F)$. Since $A_{\underline{s}}(F)$ is the composant of ∞ in $\hat{J}_{\underline{s}}(F)$, we see that no point in $J_{\underline{s}}(F) \setminus A_{\underline{s}}(F)$ can belong to $\mu_{\underline{s}}(\infty)$. Hence $\mu_{\underline{s}}(F) = J_{\underline{s}}(F)$, and the claim follows from the first statement. ∎

Proof of Theorem 1.13. The first statement follows from Proposition 4.4, together with the final part of the preceding proof.

If $\underline{s}$ is a periodic address, then $J_{\underline{s}}(F)$ contains a unique periodic point, and every other point of $J_{\underline{s}}(F)$ escapes to ∞. Hence Corollaries 4.5 and 4.6 imply that either $\hat{J}_{\underline{s}}(F)$ is an indecomposable continuum, or $J_{\underline{s}}(F) \setminus A_{\underline{s}}(F)$ consists of a single periodic point. ∎

5. Arc-like tracts

We now turn to a class of functions for which we can say more about the topology of Julia continua.

5.1. Definition (Arc-like tracts)**.**
Let $F : \mathcal{T} \to H$ be a disjoint-type function in the class $\mathcal{B}_{\log}$. We say that a tract T is *arc-like* if there exists a continuous function $\varphi : T \to [0, \infty)$ with $\varphi(z) \to \infty$ as $z \to \infty$ and a constant $C > 0$ such that

$$\operatorname{diam}_H(\varphi^{-1}(t)) \leq C$$

for all t. If all tracts of F are arc-like with the same constant C, then we say that F *has arc-like tracts.*

The following two cases of arc-like tracts are particularly important.

5.2. Definition (Bounded slope and bounded decorations)**.**
Let $F : \mathcal{T} \to \mathbb{H}$ be a disjoint-type function in the class $\mathcal{B}_{\log}$. We say that F has *bounded*

slope if there exists a curve $\gamma : [0, \infty) \to \mathcal{T}$ and a constant C such that $\operatorname{Re}\gamma \to +\infty$ and

$$|\operatorname{Im}\gamma(t)| \leq C \cdot \operatorname{Re}\gamma(t)$$

for all t.

We say that F has *bounded decorations* if there is a constant C such that

$$\operatorname{diam}_{\mathbb{H}}(F_T^{-1}(\{z \in \mathbb{H} : |z| = \rho\})) \leq C$$

for all $\rho \geq 0$ and all tracts T of F.

Remark. Note that, if f has bounded slope, as defined in the Introduction, then any logarithmic transform of F has bounded slope in the sense defined here (and vice versa).

The following result makes it easy to verify the bounded decorations condition. (It will be used in the second part of the paper.)

5.3. Proposition (Characterization of bounded decorations). *Let T be a logarithmic tract with $\overline{T} \subset \mathbb{H}$, and let $F : T \to \mathbb{H}$ be a conformal isomorphism with $F(\infty) = \infty$.*

Set $\gamma^+ := F^{-1}(i \cdot [0, \infty))$ and $\gamma^- := F^{-1}(i \cdot (-\infty, 0]$. The following are equivalent:

(a) T has bounded decorations.
(b) Every point of γ^+ can be connected to some point of γ^- by a curve in T whose hyperbolic diameter (in $\mathbb{H}$) is uniformly bounded.

Proof. This follows from well-known results in geometric function theory (compare the appendix of [RRRS11].) ■

5.4. Observation (Examples of arc-like tracts). *If F has bounded-slope or bounded decorations, then F has arc-like tracts.*

Proof. The desired function φ is given by functions $\varphi(z) = \operatorname{Re} z$ and $\varphi(z) = |F(z)|$, respectively. ■

The key reason for the above definitions is given by the following observation, which (together with Theorems 3.5 and 3.7) completes the proof of the first half of Theorem 1.6.

5.5. Proposition (Arc-like tracts imply arc-like continua). *Suppose that F has arc-like tracts. Then every Julia continuum of F is arc-like.*

Proof. Let $T_0T_1T_2\dots$ be the external address of a Julia continuum $\hat{C}$. For each T_j, let φ_j be the corresponding function from the definition of arc-like tracts. We set

$$g_j : \hat{C} \to [0, \infty]; \quad g_j(z) := \begin{cases} \varphi_j(F^j(z)) & \text{if } z \in C \\ \infty & \text{if } z = \infty. \end{cases}.$$

By hyperbolic expansion (Proposition 2.6),

$$\dim_H(g_j^{-1}(t)) \leq \Lambda^{-j} \cdot \operatorname{diam}_H(\varphi_j^{-1}(t)) \leq \frac{C}{\Lambda^j}$$

for all $t \in [0, \infty)$. It follows that $\hat{C}$ is arc-like. ■

Our final result in this section proves one direction of Theorem 1.11, concerning the topology of periodic Julia continua.

5.6. Theorem (Invariant continua in arc-like tracts)**.** *Let $F \in \mathcal{B}_{\log}$ be of disjoint type with arc-like tracts, and suppose that $\hat{C}$ is a periodic Julia continuum of F.*

Then there is a continuous function $h : [0, \infty] \to [0, \infty]$ such that $h(0) = 0$, $h(\infty) = \infty$, $h(x) < x$ for $x \in (0, 1)$ and such that $\hat{C}$ is homeomorphic to $\varprojlim f$.

Under this homeomorphism, the point $\infty \leftarrow\!\shortmid \infty \leftarrow\!\shortmid \dots$ corresponds to $\infty \in \hat{C}$, and $0 \leftarrow\!\shortmid 0 \leftarrow\!\shortmid \dots$ corresponds to a periodic point in $\hat{C}$.

Proof. By passing to an iterate, we may assume that $\hat{C}$ is invariant. That is, we are in the situation where T is an arc-like logarithmic tract, $F : T \to H$ is a conformal isomorphism and $\hat{C}$ consists of all points that stay in T under iteration, together with ∞.

Let φ and C be as in the definition of arc-like tracts. We may assume that the map $\varphi : T \to [0, \infty)$ is surjective. We may also suppose that the unique fixed point p of F in T has $\varphi(p) = 0$, and that φ extends continuously to the boundary of T. (The latter can be achieved by restricting the function F to a slightly smaller domain.) Recall that the hyperbolic diameter of $\varphi^{-1}(t)$ is bounded by C, independently of t.

Let us define a sequence $\zeta_j \in T$ inductively as follows. Let $\zeta_0 = p$. For $j \geq 0$, let $\zeta_{j+1} \in \overline{T}$ be a point with $\operatorname{dist}_H(\zeta_j, \zeta_{j+1}) = 3C$ such that $\varphi(\zeta_{j+1}) > \varphi(\zeta_j)$ and such that $\varphi(\zeta_{j+1})$ is minimal with this property. To see that such a point exists, note that $\varphi^{-1}(\varphi(\zeta_j))$ is contained in the hyperbolic disc of radius $3C$ around ζ_j. Hence the boundary of the disc must contain some points of $\varphi^{-1}((\zeta_j, \infty))$ by continuity and surjectivity of φ, as well as connectedness of T.

We also observe that, again by continuity of φ, we must have $x_j := \varphi(\zeta_j) \to \infty$. Postcomposing φ with a homeomorphism $[0, \infty] \to [0, \infty]$, we may assume for simplicity that $x_j = j$ for all j. Observe that, by construction, any point in $\varphi^{-1}([j, j + 1])$ has hyperbolic distance at most $3C$ from ζ_j, and hence $\operatorname{diam}_H(\varphi^{-1}([j, j + 1])) \leq 6C$ for all $j \geq 0$.

For $n \geq 0$, we define a function $h_n : [0, \infty) \to [0, \infty)$ by setting

$$h_n(4j) := \varphi(F^{-n}(\zeta_{4j})),$$

for $j \geq 0$ and interpolating linearly between these points.

Claim 1. If n is sufficiently large, then

(a) $\operatorname{diam}(\varphi(F^{-n}(\varphi^{-1}([4j, 4(j + 1)])))) \leq 2$;
(b) $|h_n(x) - h_n(y)| \leq |x - y|/2$ for all $x, y \in \mathbb{R}$, and
(c) $h_n(x) < x$ for all $x > 0$.

Proof. The hyperbolic diameter (in H) of $A_j := \varphi^{-1}([4j, 4(j + 1)])$ is bounded by $24C$ (independently of j). Let n be sufficiently large to ensure that $\Lambda^{n-1} > 48$, where Λ is once more the expansion factor from Proposition 2.6. Then, in the hyperbolic metric of T, the diameter of $F^{-n}(A_j)$ is less than $C/2$. Let B be an open hyperbolic disc of T, of radius C and containing $F^{-n}(A_j)$. Then $\varphi(B) \subset [0, \infty)$ is connected. Furthermore, $\varphi(B)$ can contain at most one integer, since the hyperbolic distance between any point of $\varphi^{-1}(m)$ and any point of $\varphi^{-1}(m + 1)$ is at least C, by construction. Hence $\varphi(\gamma)$ has diameter at most 2, as claimed in (a).

In particular, we have $|h_n(x) - h_n(y)| \leq 2$ for $x = 4j$ and $y = 4(j+1)$. This implies that the slope of h_n on each interval of linearity is at most $1/2$, establishing (c). Claim (c) follows from (b), using $y = 0$. △

Let us set $h := h_n$, where n is as in the claim. Since $h(x) \to \infty$ as $x \to \infty$, we can extend h continuously to a self-map of $[0, \infty]$. This function has the following key property, which essentially says that h behaves like the map F^{-n} (using the translation between the two coordinates provided by φ).

Claim 2. For all $z \in \overline{T}$, $|h(\varphi(z)) - \varphi(F^{-n}(z))| \leq 4$.

Proof. Choose $j \geq 0$ such that $\varphi(z) \in [4j, 4(j+1)]$. Recall from Claim 1 that both $h([4j, 4(j+1)])$ and $\varphi(F^{-n}(\varphi^{-1}([4j, 4(j+1)])))$ have diameter at most 2. Hence both $h(\varphi(z))$ and $\varphi(F^{-n}(z))$ have distance at most 2 from the point $h(4j)$, and the claim follows. △

It remains to prove that $\hat{C}$ is homeomorphic to the inverse limit $\varprojlim h$. This is a standard dynamical conjugacy argument for expanding maps. To provide the details, define maps $\psi_j : \hat{C} \to [0, \infty]$ by $\psi_j(\infty) = \infty$ and

$$\psi_j(z) := \lim_{k \to \infty} h^{k-j}(\varphi(F^{kn}(z))). \tag{5.1}$$

Claim 3. For every j, the limit in (5.1) exists, and is uniform, with

$$|\psi_j(z) - \varphi(F^{jn}(z))| \leq 8$$

for all $z \in \hat{C}$.

Furthermore, $h(\psi_{j+1}(z)) = \psi_j(z)$ for all j. In particular, these coordinates define a continuous function $\psi : \hat{C} \to \varprojlim h$.

Proof. We claim that

$$|h^{k-j}(\varphi(F^{kn}(z))) - \varphi(F^{jn}(z))| \leq 8$$

for all $k \geq j$. Indeed, this is trivial for $k = j$. Moreover, if the inequality holds for j and k, then a simple application of Claim 2 and the contracting property of h shows that it is also true for k and $j - 1$. This inductively establishes the claim for all k and j.

In particular, it follows (again using the contracting property of h) that the sequence of maps defining ψ_j is a Cauchy sequence, and hence the limit indeed exists, and is uniform.

The fact that $h(\psi_{j+1}(z)) = \psi_j(z)$ is immediate from the definition. △

Note that we have $\psi(p) = 0 \leftarrow\!\shortmid 0 \leftarrow\!\shortmid \dots$ and $\psi(\infty) = \infty \leftarrow\!\shortmid \infty \leftarrow\!\shortmid \dots$. In particular, each coordinate function ψ_j is surjective, which implies that ψ itself is also surjective.

Since $\hat{C}$ is compact, it only remains to prove that ψ is injective. We observe that $|\psi_j(z) - \varphi(F^{jn}(z)|$ is uniformly bounded by Claim 3. On the other hand, if $z, w \in \hat{C}$ with $z \neq w$, then $|\varphi(F^{jn}(z)) - \varphi(F^{jn}(w))| \to \infty$ as $j \to \infty$, because the hyperbolic distance between $F^{nk}(z)$ and $F^{nk}(w)$ tends to infinity. Thus $\psi_j(z) \neq \psi_j(w)$ for sufficiently large j, and hence $\psi(z) \neq \psi(w)$, as desired. ■

6. Homeomorphic subsets of Julia continua

To conclude this part of the paper, we shall establish that any two bounded Julia continua of an entire function with a single tract are (ambiently) homeomorphic.

6.1. Proposition (Julia continua with similar addresses)**.** *Let $F \in \mathcal{B}_{\log}^{\mathrm{p}}$, and let $M \in \mathbb{N}$. Let $\underline{s}^1 = T_0^1 T_1^1 \dots$ and $\underline{s}^2 = T_0^2 T_1^2 \dots$ be two external addresses such that $T_j^2 = T_j^1 + 2\pi i m_j$ for some $m_j \in \mathbb{Z}$.*

Suppose that $K \subset \hat{J}_{\underline{s}^1}$ is compact, and that there is $\vartheta > 0$ such that

$$\operatorname{Re} f^j(z) \geq \vartheta \cdot |m_j|$$

for all $z \in K \cap \mathbb{C}$ and all $j \geq 0$.

Then there exists a compact subset $A \subset \hat{J}_{\underline{s}^2}$ and a homeomorphism $\psi : K \to A$, with the property that

$$d_H(F^n(z), F^n(\psi(z))) \leq C \tag{6.1}$$

for all $z \in K$ and $n \geq 0$. Here the constant C depends on F and ϑ, but not otherwise on K, $\underline{s}^1$ and $\underline{s}^2$.

Remark. The reader may wish to keep in mind the simplest case, where F has a single tract, $\underline{s}^1$ is a fixed address and $\underline{s}^2$ is a bounded address, so that the sequence $|m_j|$ is uniformly bounded. In this case we can take $K = \hat{J}_{\underline{s}^2}$, and it follows easily that $A = \hat{J}_{\underline{s}^2}$ (see Corollary 6.2 below).

We shall use the more general statement in Section 12 below, in order to prove Theorem 1.2.

Proof. This is essentially the same argument that appears in [Rem09] to construct conjugacies between subsets of Julia sets of different functions in the class $\mathcal{B}_{\log}^{\mathrm{p}}$. Indeed, we can think of the above statement as a *non-autonomous* version, where we allow the function that is applied to vary at different times.

More precisely, we define maps $\psi_n^j : F^n(K) \to T_n^2$ by $\psi_n^0(z) := z + 2\pi i m_n$ and

$$\psi_n^{j+1}(z) := F_{T_n^2}^{-1}(\psi_{n+1}^j(F(z))).$$

Let $\Lambda > 1$ be the expansion factor of the map F with respect to the hyperbolic metric. The assumption implies that there exists a constant $\rho > 0$ (depending only on F and ϑ) such that

$$\operatorname{dist}_H(F^j(z), F^j(z) + 2\pi i m_j) \leq \rho$$

for all $z \in K$ and all $j \geq 0$. Set $C_1 := \vartheta \cdot \frac{\Lambda}{\Lambda - 1}$.

Claim. $\operatorname{dist}_H(\psi_n^j(z), F^n(z)) \leq C_1$ for all $n, j \geq 0$ and all $z \in K$.

Proof. The proof is by induction on j. By choice of ρ, the claim is trivial for $j = 0$. Now suppose that the claim is true for j, and let $z \in K$. Then

$$\begin{aligned}\operatorname{dist}_H(\psi_n^{j+1}(z), F^n(z) + 2\pi i m_n) &= \operatorname{dist}_H(\psi_n^{j+1}(z), F_{T_n^2}^{-1}(F^{n+1}(z))) \\ &\leq \frac{\operatorname{dist}_H(\psi_{n+1}^j(z), F^{n+1}(z))}{\Lambda} \leq \frac{C}{\Lambda}.\end{aligned}$$

So

$$\mathrm{dist}_H(\psi_n^{j+1}(z), F^n(z)) \leq \vartheta + \frac{C}{\Lambda} = C.$$

△

It follows that the maps $(\psi_n^j)_{j\geq 0}$ form a Cauchy sequence for every n, and hence converge to a map ψ_n. These maps satisfy $F \circ \psi_n = \psi_{n+1} \circ F$. In particular, $\psi := \psi_0$ satisfies (6.1). By expansion, this also implies that ψ is injective, and (if K is unbounded) extends continuously to ∞ with $\psi(\infty) = \infty$. So ψ is a homeomorphism onto its image $A := \psi(K)$, and we are done. ■

6.2. Corollary (Homeomorphic Julia continua)**.** *With the notation of the preceding Proposition, suppose that the sequence (m_j) is uniformly bounded. Then $J_{\underline{s}^1}(F)$ and $J_{\underline{s}^2}(F)$ are homeomorphic.*

Proof. Apply the Proposition to $K := J_{\underline{s}^1}(F)$ to obtain a map $\psi_1 : J_{\underline{s}^1}(F) \to J_{\underline{s}^2}(F)$. Then switch the roles of $\underline{s}^1$ and $\underline{s}^2$, and similarly obtain a map $\psi_2 : J_{\underline{s}^2}(F) \to J_{\underline{s}^1}(F)$. By expansion, it follows that $\psi_1 \circ \psi_2 = \mathrm{id}$ and $\psi_2 \circ \psi_1 = \mathrm{id}$, and hence ψ_1 is a homeomorphism between $J_{\underline{s}^1}$ and $J_{\underline{s}^2}$. ■

6.3. Corollary (Bounded Julia continua are homeomorphic)**.** *Suppose that $F \in \mathcal{B}_{\log}^{\mathrm{p}}$ has only a single tract up to translation by integer multiples of $2\pi i$. Then any two bounded Julia continua of F are homeomorphic.* □

For completeness, we also note the following observation concerning the embedding of the Julia continua considered.

6.4. Proposition (Ambient homeomorphism)**.** *The sets K and A in Proposition 6.1 are* ambiently *homeomorphic; i.e., the map ψ extends to a homeomorphism $h : \mathbb{C} \to \mathbb{C}$.*

Moreover, as $\underline{s}^2 \to \underline{s}^1$ (for fixed ϑ), the maps $h = h_{\underline{s}^2}$ converge uniformly to the identity.

Remark. The proof will show that h can even be taken *quasiconformal.*

Proof. Let U be a sufficiently small complex neighbourhood of the segment $[0, 1]$. For $\mu \in U$, let us define the following modification of the maps ψ_n^j from the proof of Proposition 6.1:

$$\psi_n^0(z) := z + 2\pi i m_n \cdot \mu \qquad \text{and} \psi_n^{j+1}(z) \qquad := F_{T_n^1}^{-1}(\psi_{n+1}^j(F(z))) + 2\pi i m_n \cdot \mu.$$

Observe that, for $\mu = 0$, each map is the identity, while for $\mu = 1$ we recover the original definition.

As in Proposition 6.1, we see that the maps ψ_0^j converge uniformly to a homeomorphism ψ_0 from K to some set A_μ, for every μ. Here $A_0 = K$ and $A_1 = A$. Since $\psi_0(z)$ depends holomorphically on μ, these sets form a *holomorphic motion* of the set K. By the λ-lemma of Bers and Royden [BR86, Lemma 1], each of these maps extends to an orientation-preserving quasiconformal homeomorphism $\mathbb{C} \to \mathbb{C}$.

The second claim follows directly from the proof of Proposition 6.1 and the fact that F is expanding. ■

7. Background on arc-like continua

In the second part of the article, we are now going to discuss the construction of entire function with prescribed arc-like continua in the Julia set. In order to make these constructions, we shall need to collect some further background on arc-like continua. Let us begin by recalling their definition, and introduce some additional terminology.

7.1. Definition (ε-maps and arc-like continua)**.**
An *ε-map* from a metric space A to a topological space B is a continuous function $g : A \to B$ such that $g^{-1}(x)$ has diameter less than ε for every $x \in B$.

A continuum X is called *arc-like* if, for every $\varepsilon > 0$, there exists an ε-map g from X onto an arc.

Key to our construction of arc-like continua in Julia sets is the following characterization of arc-like continua in terms of inverse limits.

7.2. Proposition (Characterization of arc-like continua with terminal points)**.** *Let X be a continuum, and let $p \in X$. The following are equivalent.*

- *(a) X is arc-like and p is terminal;*
- *(b) for every $\varepsilon > 0$, there is an ε-map $g : X \to [0, 1]$ with $g(p) = 1$;*
- *(c) there is a sequence $f_j : [0, 1] \to [0, 1]$ of surjective and continuous functions with $f_j(1) = 1$ for all j such that there is a homeomorphism from X to $\varprojlim((f_j)_{j=1}^{\infty})$ which maps p to the point $(1 \leftarrow 1 \leftarrow 1 \leftarrow \dots)$.*

If any (and hence all) of these properties hold, and q is a second terminal point such that X is irreducible between p and q, then the maps f_j can be additionally chosen to fix 0, with the point q corresponding to the point $0 \mapsto 0 \mapsto \dots$. Similarly, any ε-map g can be chosen such that $g(q) = 0$.

Remark. An additional equivalent formulation is as follows: for every $\varepsilon > 0$, there is an ε-chain in X that covers X and such that p belongs to the final link of this chain. (That is, there is a finite sequence $U_1, \dots, U_n$ of nonempty open subsets ("links") of X whose union equals X, such that two links intersect if and only if they are adjacent, such that $p \in U_n$ and such that $\operatorname{diam}(U_j) < \varepsilon$ for all j.) See [Nad92, Definition 12.8] for a discussion of chainability.

Proof. This result is well-known. Without the reference to terminal points, the equivalence is proved in [Nad92, Theorem 12.19]. For completeness, let us briefly sketch the proof, referring to [Nad92] and [Bin51b] where necessary.

First observe that (c) clearly implies (b), as we can let g be the projection to the j-th coordinate, for j sufficiently large.

Conversely, it follows from the proof of Theorem 12.19 in [Nad92] that (b) implies (c). Indeed, that proof constructs a suitable inverse limit, and an inspection of the proof of Lemma 12.17 immediately show that the map φ constructed there, which is used in the construction of the inverse limit, satisfies $\varphi(1) = 1$. (Indeed, with the notation of that proof, we have $t_n = 1$ and $\varphi(1) = \varphi(t_n) = s_{i(n)} = i(n)/m$. Here the definition of $i(n)$ ensures that $i(n) = m$, provided that $g_1(p) = g_2(p) = 1$.)

That (b) implies (a) is elementary. Inded, suppose that $A, B \subset X$ are continua such that $p \in A \cap B$, but $A \not\subset B$ and $B \not\subset A$. Then, assuming ε is chosen sufficiently small, we also have $g(A) \not\subset g(B)$ and $g(B) \not\subset g(A)$ for any ε-map g. Since $g(A)$ and $g(B)$ are closed subintervals of $[0,1]$, it follows that $g(p) \neq 1$.

Finally, Bing [Bin51b, Theorem 13] showed that (a) is equivalent to the statement on chainable continua mentioned in the remark after the statement of the theorem. This, in turn, is easily seen to imply (b).

The final part of the proposition follows analogously. ■

Hence our challenge shall be to start with an arbitrary inverse limit Y as above, and construct a function $F \in \mathcal{B}_{\log}$ having a Julia continuum that is homeomorphic to Y. This will be achieved using the following result, which implies, in certain circumstances, that a continuum X is homeomorphic to the inverse limit Y. We shall construct our function precisely in such a way that it satisfies these requirements. The Proposition is essentially a special case of [Nad92, Proposition 12.18], which in turn originally appeared as [MS63, Lemma 5]. (The result holds for arbitrary inverse limits of nonempty compact metric spaces; for convenience, we are stating it only for inverse limits of arcs.)

7.3. Proposition (Continua homeomorphic to an inverse limit)**.** *Let Y be the inverse limit of continuous maps $f_k : [0,1] \to [0,1]$ $(i \geq 1)$, and let (X, d) be an inverse limit of nonempty compact metric spaces $(X_k, d_k)_{k\geq 0}$ with continuous bonding maps $\Psi_k : X_k \to X_{k-1}$ for $d \geq 1$.*

Suppose that, for each $j \geq 0$, there are $\delta_j > 0$ and $\varepsilon_j > 0$, tending to zero as $j \to \infty$, and a surjective continuous function $g_j : X_j \to [0,1]$, such that the following hold:

(a) *Let $k \geq 1$ and suppose that $A \subset [0,1]$ with* $\operatorname{diam}(A) \leq \delta_k$. *Then, for all $j < k$,*
$$\operatorname{diam}(f_{j+1}(f_{j+2}(\dots(f_k(A))\dots)) \leq \frac{\delta_j}{2^{k-j}}.$$
(b) *For $j \geq 0$, define $\tilde{g}_j : X \to [0,1]$ by $\tilde{g}_j := g_j \circ \pi_j$ (where π_j denotes projection to the j-th coordinate). If $x, x' \in X$ and k is such that $d(x,x') \geq 2\varepsilon_j$, then $|\tilde{g}_j(x) - \tilde{g}_j(x')| > 2\delta_j$.*

(c) $|g_{j-1}(\Psi_j(x)) - f_j(g_j(x))| \leq \delta_{j-1}/2$ *for all $x \in X_j$ and all $j \geq 1$.*

Then X and Y are homeomorphic. More precisely, there exists a homeomorphism $h : X \to Y$ such that

$$|\tilde{g}_j(x) - \pi_j(h(x))| \leq \delta_j \tag{7.1}$$

for all $j \geq 0$.

Proof. As mentioned above, this is essentially proved in [Nad92, Proposition 2.8]. However, the statement in in [Nad92, Proposition 2.8] is slightly different. In particular, there the result is formulated only in the case where X is a fixed continuum, rather than a direct limit (i.e., $X_k = X_{k-1}$ and $F_k(x) = x$ for all k and all $x \in X_k$). Furthermore, the final part of our statement does not appear in the statement of the Proposition in Nadler's book, but appears as property (c) in that proof.

For these reasons, let us sketch the proof, which is quite dynamical in nature; indeed, it is once more essentially a standard conjugacy argument. We inductively define maps $\vartheta_j^k : X_k \to [0,1]$, for $j \leq k$, by
$$\vartheta_k^k := g_k; \quad \vartheta_{j-1}^k := f_j \circ \vartheta_j^k.$$

That is, ϑ_j^k involves first applying g_k and then the maps f_k, f_{k-1}, ..., f_{j+1}.

Now consider $h_j^k : X \to [0,1]$, $h_j^k := \vartheta_j^k \circ \pi_k$. Let us fix some $j \geq 0$. It follows from (a) and (c) that

$$|h_j^{k+1}(x) - h_j^k(x)| \leq \frac{\delta_j}{2^{k-j}}$$

for all $x \in X$ and all $k > j$. In particular, the maps h_j^k form a Cauchy sequence as $k \to \infty$, and

$$|h_j^k(x) - \tilde{g}_j(x)| \leq \delta_j \tag{7.2}$$

for all $x \in X$ and all $k > j$. Let h_j be the limit of this sequence. By definition, we have $h_{j-1} = f_j \circ h_j$ for all j, so the maps h_j are the coordinates of a continuous function $h : X \to Y$.

Injectivity of the map h follows from (7.2) and condition (b). Hence h is a homeomorphism onto its image.

Finally, h is surjective, due to the surjectivity of the maps g_k. Indeed, let $t_0 \hookleftarrow t_1 \hookleftarrow t_2 \hookleftarrow \dots$ be a point of Y. Then, for every $k \geq 0$ there is $x_k \in X_k$ such that $g_k(\tilde{x}_k) = t_k$. Let $\hat{x}^k$ be the partial inverse orbit defined by $\hat{x}_j^k := \Psi_{j+1}(\Psi_{j+2}(\dots(\Psi_k(x_k))\dots))$, and let $\hat{x} \in X$ be an accumulation point of the sequence $\hat{x}^k$. By definition, $\vartheta_j^k(\hat{x}_k^k) = \vartheta_j^k(x_k) = t_j$ for all $j \leq k$. Arguing as in the proof of (7.2), we see from (a) that

$$\vartheta_j^k(\hat{x}_j^k) \to t_j$$

as $k \to \infty$ for fixed j. Hence $h_j(x) = t_j$ for all $j \geq 0$. ■

When applying Proposition 7.3, the following observations will be useful.

7.4. Observation (Modifications of Proposition 7.3). *Let X and Y be given as in Proposition 7.3.*

1. *If $(\delta_j)_{j\geq 0}$ is a sequence such that* (a) *holds, then the sequence $\tilde{\delta}_j := n \cdot \delta_j$ also satisfies this property, for every $n \in \mathbb{N}$.*
2. *Let $(\delta_j)_{j\geq 0}$ be a sequence of positive numbers and let $n \in \mathbb{N}$. If*

$$\tilde{\varepsilon}_j := \sup_{x\in X} \operatorname{diam}\{x' \in X : |\tilde{g}_j(x) - \tilde{g}_j(x')| \leq \delta_j/n\} \to 0$$

as $j \to \infty$, then the sequence $\varepsilon_j := \tilde{\varepsilon}_j/(2n)$ satisfies (b).

Proof. Let (δ_j) be as in(1), and set $\tilde{\delta}_j := n \cdot \delta_j$. To establish (a) for $\tilde{\delta}_j$, it is clearly enough to consider the case where A is an interval. If $\operatorname{diam}(A) \leq \tilde{\delta}_j$, where (δ_j) satisfies (a), then we can subdivide A into n intervals $A_1, \dots, A_n$ of length at most δ_j. The claim now follows by applying (a) (for the sequence (δ_j)) to each of these intervals. Part (2) is obvious. ■

Finally, the following observation will allow us to realize *all* arclike continua with terminal points at once as Julia continua of a *single* entire function.

7.5. Proposition (Countable generating set). *There exists a countable set $\mathcal{F}$ of surjective continuous functions $f : [0,1] \to [0,1]$ with $f(1) = 1$ such that all maps in Proposition 7.2 (c) can be chosen to belong to $\mathcal{F}$.*

Proof. For general arc-like continua, this is stated in [Nad92] (without the assumption that $f(1) = 1$); we remark that in fact two maps are sufficient to construct all arc-like continua.

The proof in our case is entirely analogous. Indeed, we can let $\mathcal{F}$ consist of all maps that are piecewise linear, with any point of linearity rational, and with rational values at rational points. Clearly this set is countable. Furthermore, let (f_j) be any sequence as in 7.2 (c). Clearly we can construct a sequence $(\tilde{f}_j)$ with $\tilde{f}_j \in \mathcal{F}$ such that $\tilde{f}_j$ is sufficiently close to f_j to ensure that Proposition 7.3 can be applied to the inverse limits $Y = \varprojlim(f_j)$ and $X = \varprojlim(\tilde{f}_j)$ (with $g_j = \mathrm{id}$ for all j). Hence X and Y are homeomorphic. If all maps f_j fix 0, then we can choose $\tilde{f}_j$ to have the same property. ∎

8. Every arc-like continuum is a Julia continuum

The goal of this section is to prove the main part of Theorem 1.6. More precisely, given an arc-like continuum X with a terminal point x_0, we construct a function $F \in \mathcal{B}_{\log}^{\mathrm{p}}$ and an external address such that the Julia continuum $\hat{C} = J_{\underline{s}}(F) \cup \{\infty\}$ is homeomorphic to X, with ∞ corresponding to the point x_0. Here, we shall allow the function F to depend on the continuum X. In Section 13, we discuss how to adopt this argument to show how the map F can be constructed independently of X, and hence complete the proof of Theorem 1.6.

Let us begin with a few remarks.

(a) Our function will have a single tract T, up to translation by $2\pi i\mathbb{Z}$. That is, the map F is uniquely determined by specifying the simply-connected domain T, with $\overline{T} \subset \mathbb{H}$, and a conformal isomorphism from T to $\mathbb{H}$.

For simplicity of notation, we shall identify this conformal isomorphism with the map $F \in \mathcal{B}_{\log}^{\mathrm{p}}$ generated by it.

(b) As in the previous section, we do not require that T is a Jordan domain; we only require that T is simply connected, does not intersect its $2\pi i\mathbb{Z}$-translates and that $F(z) \to \infty$ as $z \to \infty$ in T.

In order to obtain a function in $\mathcal{B}_{\log}^{\mathrm{p}}$, we replace T by $\{z \in T : \operatorname{Re} F(z) > \varepsilon\}$, where $\varepsilon > 0$ is sufficiently small to ensure that the resulting function is still of disjoint type.

(c) The tracts of the map F are all of the form $T+m$, where $m \in \mathbb{Z}$. We shall simplify notation by identifying an external address $\underline{s} = (T + s_0)(T + s_1)(T + s_2)\dots$ with the sequence $s_0 s_1 s_2 \dots$ of integers.

The aim of this section is to prove the following theorem.

8.1. Theorem. *Let $f_k : [0,1] \to [0,1]$, for $k \geq 1$, be a sequence of continuous and surjective maps with $f_k(1) = 1$, and let $Y = \varprojlim(f_k)$.*

Then there exists a logarithmic tract T with $\overline{T} \subset \mathbb{H}$, symmetric with respect to the real axis, a conformal isomorphism $F : T \to \mathbb{H}$, and an external address $\underline{s}$ such that $\hat{J}_{\underline{s}}(F)$ is homeomorphic to Y, with ∞ corresponding to the point $1 \leftarrow\!\shortmid 1 \leftarrow\!\shortmid \dots$.

More precisely, the address $\underline{s}$ is of the form

$$\underline{s} = 0^{N_1}\, s(1)\, 0^{N_2}\, s(2)\, 0^{N_3}\, s(3) \dots,$$

where $s(k) \in \mathbb{Z}$ and $N_k \geq 0$ for all $k \geq 1$. Set $n_k := k + \sum_{j \leq k} N_j$ for $k \geq 0$. Given any sequence (M_k) with $M_k \geq 5$ for all k, the construction can be carried out such that the following hold for all $z \in J_{\underline{s}}(F)$.

(a) $\operatorname{Re} F^n(z) \geq \operatorname{Re} F^{n_j}(z) \geq M_k - 1$ for $n_k \leq n < n_{k+1}$. (In particular, $z \in I(F)$ if and only if $\lim_{k\to\infty} \operatorname{Re} F^{n_k}(z) = \infty$.)

(b) Let $h_k : \hat{J}_{\underline{s}} \to [0,1]$ denote the k-th component of the homeomorphism $h : \hat{J}_{\underline{s}} \to Y$ whose existence is asserted in the first part of the theorem. Then

$$h_k(z) = 0 \quad \Longrightarrow \quad \operatorname{Re} F^{n_k}(z) \leq M_k + 1$$

and

$$\liminf_{k\to\infty} h_k(z) > 0 \quad \Longrightarrow \quad \operatorname{Re} F^{n_k}(z) - M_j \to \infty.$$

Remark. The key point here is that we can choose the sequence (M_j) to be constant, and in this case the Julia continuum will not escape to infinity uniformly. Moreover, the final statement in the theorem will allow us to construct examples where the set of nonescaping points is empty, and examples where this set is uncountable.

By Theorem 2.4, the existence of a function in $\mathcal{B}_{\log}^{\mathrm{p}}$ automatically yields a function $f \in \mathcal{B}$, and even a function $f \in \mathcal{S}$ (possibly with more than one tract) having a Julia continuum homeomorphic to Y.

We devote the remainder of the section to the proof of Theorem 8.1. Let us fix the maps f_j and the sequence (M_j) from now on.

Description of the tract T. The tract T is chosen as a subset of $\{z+iy : x > 1, |y| < \pi\}$. It consists of a central straight strip, to which a number of "side channels", domains U_k, are attached that mimic the behaviour of the maps f_k. Between the places where U_{k-1} and U_k are attached, the domain will be narrowed to a window of size χ_k. (See Figure 4.)

Let us be more precise. Define

$$S := \{x + iy : x > 1 \text{ and } |y| < \pi/2\}.$$

The tract T is determined by a sequence $(R_k)_{k\geq 1}$ with $R_{k+1} - 1 > R_k > 2$ for all k, a sequence $\chi_k \in (0, \pi]$, $k \geq 1$, as well as a sequence of domains $U_k \subset \mathbb{H}$, where

$$U_k \subset \{x + iy : R_{k-1} < x < R_k + 1 \text{ and } \pi > y > \pi/2\}$$

(where we use the convention that $R_0 = 2$) and

$$\{x + i\pi/2 : x \in [R_k, R_k + 1]\} \subset \partial U_k.$$

Then the tract T is defined by

$$T := \left(S \setminus \bigcup_{k\geq 1} \{R_k + iy : |y| \geq \chi_k/2\} \right) \cup \bigcup_{k\geq 1} \left(U_k \cup \tilde{U}_k \cup \{x + iy : R_k < x < R_k + 1 \text{ and } |y| = \pi/2\} \right).$$

Here $\tilde{U}_k = \{x + iy : x - iy \in U_k\}$ denotes the set of all complex conjugates of points in U_k.

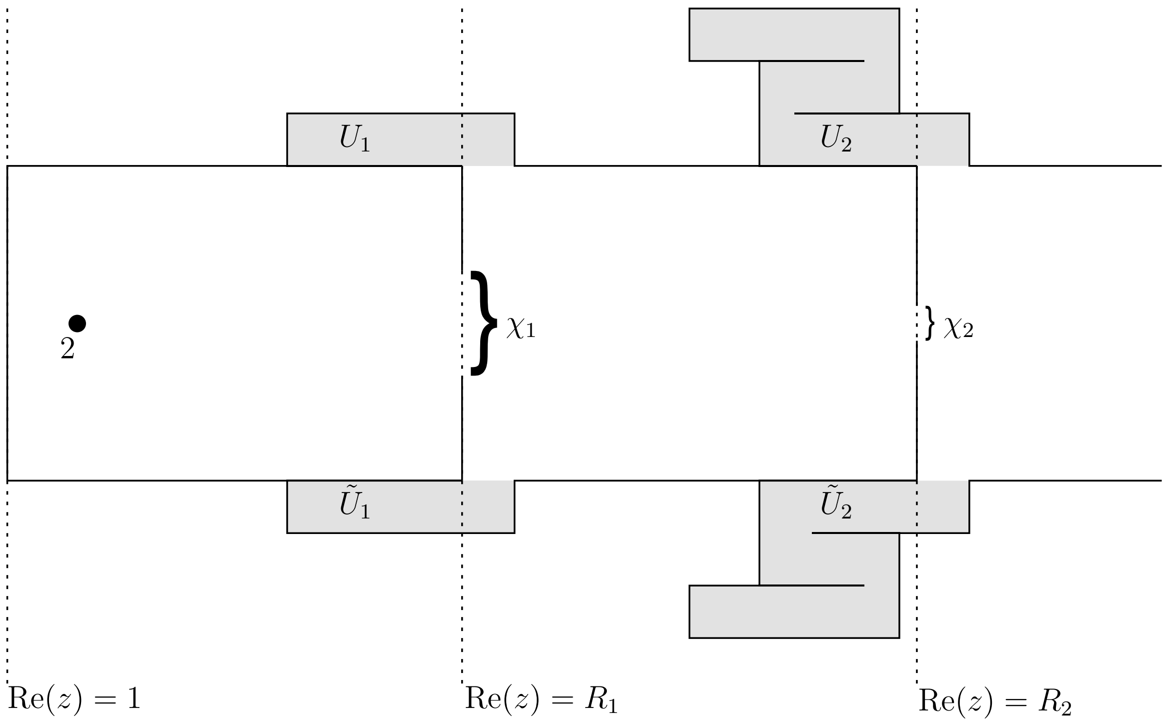


FIGURE 4. The tract T.

The conformal isomorphism $F : T \to \mathbb{H}$ is determined uniquely by requiring that $F(2) = 2$ and $F'(2) > 0$. Since T is symmetric with respect to the real axis, this implies that $F([2, \infty)) = [2, \infty)$. By the expanding property of F, we must have $F(t) > t$ for $t > 2$.

We also define "partial tracts" T_K, with $K \geq 0$, as

$$T_K := \left(S \setminus \bigcup_{k=1}^{K} \{R_k + iy : |y| \geq \chi_k/2\} \right) \cup \bigcup_{k=1}^{K} \left(U_k \cup \overline{U_k} \cup \{x + iy : R_k < x < R_k + 1 \text{ and } |y| = \pi/2\} \right),$$

and let $F_K : T_K \to \mathbb{H}$ be the corresponding conformal isomorphism.

A key fact in our inductive construction is that we can estimate the behavior of the map F on an initial part of the tract by that of the map F_K, independently of further choices.

8.2. Observation (Continuity of the construction)**.** *Let $K, J \in \mathbb{N}$, let $\varepsilon > 0$ and let $A \subset \mathbb{H}$ be compact. Then there is a number $R(T_K, J, \varepsilon, A)$, depending only on the partial tract T_K, on J, on ε and on the set A, with the following property.*

If T is chosen in such a way that $R_{K+1} \geq R(T_k, J, \varepsilon, A)$, then

$$|F^{-j}(z) - F_K^{-j}(z)| \leq \varepsilon \qquad \text{and} \qquad |F_{k'}^{-j}(z) - F_K^{-j}(z)|$$

for all $z \in A$, all $j \leq J$ and all $k' > K$.

Proof. This follows from the fact that, if we let R_{K+1} tend to infinity, then the resulting tracts converge to T_K with respect to Carathéodory kernel convergence. ■

A second observation relates to the expanding properties of our functions:

8.3. Observation (Expansion of the maps F and F_K)**.** *Let $G = F$ or $G = F_K$ for some $K \geq 0$. Then the map G satisfies $|G'(z)| \geq \frac{1}{2}$ whenever $\operatorname{Re} G(z) \geq 1$. Furthermore, $|(G^n)'(z)| \geq 2$ for all $n \geq 5$ and all z with $\operatorname{Re} G^n(z) \geq 1$.*

In addition, if $z \in T$ with $G(z) \in T$ and $\operatorname{Re} z \geq 4$, then $\operatorname{Re} G(z) > \operatorname{Re} z$.

Proof. For each of these claims, we consider the hyperbolic metric on the right half-plane, and let U be the domain of G (i.e., $U = T$ or $U = T_K$). Let $\tilde{S} = \{a + ib : a > 1 \text{ and } |b| < \pi\}$. Since $G : U \to \mathbb{H}$ is a conformal isomorphism, it is an isometry from U to $\mathbb{H}$ with their respective hyperbolic metrics. Furthermore, since U is contained in the strip $\tilde{S} := \{a + ib : a > 1 \text{ and } |b| < \pi\}$, we have $\rho_U(z) \geq \rho_{\tilde{S}}(z) \geq 1/2$ for all $z \in G$, by Pick's theorem. Hence

$$|G'(z)| = \frac{\rho_{\tilde{S}(z)}}{\rho_{\mathbb{H}}(G(z))} \geq \frac{\operatorname{Re} G(z)}{2} \geq \frac{1}{2}.$$

To prove the second claim, we observe that the hyperbolic derivative of G with respect to the hyperbolic metric of $\mathbb{H}$ satisfies

$$\|DG(z)\|_{\mathbb{H}} = \frac{\rho_U(z)}{\rho_{\mathbb{H}}(z)} \geq \frac{\rho_{\tilde{S}}(z)}{\rho_{\mathbb{H}}(z)} =: \lambda(z).$$

We claim that $\lambda(z) \geq 3/2$. Indeed, as above we have $\lambda(z) \geq \operatorname{Re} z/2$ for all z, and hence we are done when $\operatorname{Re} z \geq 3$. On the other hand, $\rho_{\tilde{S}}(z) \geq 1/(\operatorname{Re} z - 1)$ for all $z \in \tilde{S}$, by comparison with a right half plane. Hence, for $\operatorname{Re} z \leq 3$, $\lambda(z) \geq 1 + 1/(\operatorname{Re} z - 1) \geq 3/2$, as claimed.

Hence, $\|DG^{n-1}(z)\|_{\mathbb{H}} \geq (3/2)^{n-1} \geq 4$ for $n \geq 5$ (wherever defined), and therefore

$$|(G^n)'(z)| = \rho_U(z) \cdot \|DG^{n-1}(G(z))\|_{\mathbb{H}} \cdot \frac{1}{\rho_{\mathbb{H}}(G^n(z))} \geq \frac{1}{2} \cdot (3/2)^{n-1} \cdot \operatorname{Re} G^n(z) \geq 2 \operatorname{Re} G^n(z)$$

for all $z \in U$ for which $G^n(z)$ is defined. In the first equality, we again used the fact that $G : U \to \mathbb{H}$ is a conformal isomorphism.

The final claim follows in a similar manner, considering the hyperbolic distances between the fixed point 2 and the points z and $G(z)$. We leave the details to the reader. ■

Inductive construction. The construction of the tract proceeds inductively, along with the construction of a number of additional objects:

(a) Sequences $s(k) \in \mathbb{Z}$ and $N_k \in \mathbb{N}$, for $k \geq 1$, as already mentioned in the theorem. Recall that n_k is determined by the numbers N_k, as $n_k = k + \sum_{j=1}^{k} N_j$.

(b) A sequence δ_k, with $k \geq 0$, for use with Proposition 7.3. These will be chosen to be rational numbers of the form $\delta_k = 1/\Delta_k$, with $\Delta_k \in \mathbb{N}$.

(c) Finite subsets $\Xi_{k-1}, \Omega_k \subset [0, 1]$, for $k \geq 1$. Here

$$\Xi_{k-1} = \left\{0, \frac{1}{\Delta_k}, \frac{2}{\Delta_k}, \frac{3}{\Delta_k}, \dots, \frac{\Delta_k - 1}{\Delta_k}\right\},$$

In the following we abbreviate $\xi^j := \xi_k^j := j/\Delta_k$; so $\Xi_{k-1} = \{\xi^0, \xi^1, \dots, \xi^{\Delta_k - 1}\}$.

(d) A surjective, continuous and non-decreasing map $\varphi_k : [0,\infty] \to [0,1]$, for every $k \geq 0$. This map will have the property that $\varphi_k(x) = 0$ for $x \leq M_k$, and $\varphi_k|_{[M_k,\infty]}$ is a homeomorphism. Furthermore, φ_k is chosen such that $\varphi_k^{-1}(x) = o(-\log(1-x))$ as $x \to 1$.

In a slight abuse of notation, we shall use φ_k^{-1} to denote the inverse of the restriction of φ_k to the interval $[M_k, \infty)$.

The idea of the construction is summarized in Figure 5: the map φ_k provides an identification between the real line in the dynamical plane of F and the domain of the map f_k, or equivalently the range of the map f_{k+1}. The domain U_k is constructed so that a suitable branch of $F^{-(N_k+1)}$ has essentially the same mapping behaviour as the function f_k.

We now begin the inductive construction. To anchor it, let φ_0 be any map with the properties described above. We then proceed as follows, for each $k \geq 1$:

I1. We define Δ_{k-1}, and hence δ_{k-1} and Ξ_{k-1}. We also choose the set Ω_k, depending on Ξ_{k-1}.
I2. We define N_k and R_k.
I3. We define the domain U_k.
I4. We define the opening size χ_k and the entry $s(k)$.
I5. We define the map φ_k.

I1. Begin by choosing Δ_{k-1} sufficiently large such that the following hold. (Recall that Δ_{k-1} determines $\delta_{k-1} = 1/\Delta_{k-1}$ as well as the set Ξ_{k-1}.)

(I1.1) Property (a) of Proposition 7.3 holds (for the sequence δ_j, as far as it has been defined). Since each f_j is uniformly continuous, this will hold whenever δ_{k-1} is sufficiently small, depending on the values of δ_j for $j < k-1$. Note that this condition is empty for $k = 1$.
(I1.2) Let $I = [\xi^j, \xi^{j+1}]$, with $0 \leq j < \Delta_{k-1} - 1$. Then the Euclidean length of $\varphi_{k-1}^{-1}(I)$ is less than $1/6$. (This is possible due to the asymptotic behavior of φ_{k-1}^{-1} at ∞.)
(I1.3) Finally, let $Q_{k-1} \geq M_0 + k$ be minimal such that $F^{n_j}(Q_{k-1}) \geq (2s(j)+1)\pi$ for $j = 1, \dots, k-1$. We assume that Δ_{k-1} is chosen sufficiently large to ensure that $\varphi_{k-1}^{-1}(\xi^{\Delta_{k-1}-1}) \geq F_{k-1}^{n_{k-1}}(Q_{k-1})$. (For $k = 1$, this condition should mean that $\varphi_{k-1}^{-1}(\xi^{\Delta_{k-1}-1}) \geq M_0 + 1$.)

Finally, we let $\Omega_k \subset [0,1)$ be a finite set with the following properties.

(I1.4) The image, under f_k, of any complementary interval of Ω_k has length at most $\delta_{k-1}/2$.
(I1.5) $\Omega_k \cap f_k^{-1}(0) \neq \emptyset$.
(I1.6) No complementary interval of Ω_k has length greater than $1/k$.

Observe that (I1.4), together with the fact that $f_k(1) = 1$, ensures that the largest point of Ω_k is mapped to the interval $[\xi^{\Delta_{k-1}}, 1]$.

Remark. In most situations, it is useful to simply imagine that $\Omega_k := f_k^{-1}(\Xi_{k-1})$. Indeed, if the map f_k is piecewise strictly montone and nowhere locally constant, we can always take Ω_k to be defined in this way. It is well-known that every arc-like continuum can be represented in such a way that all f_k have this property, but for us it will be convenient in some instances e.g. to allow intervals on which the map f_k is constant.

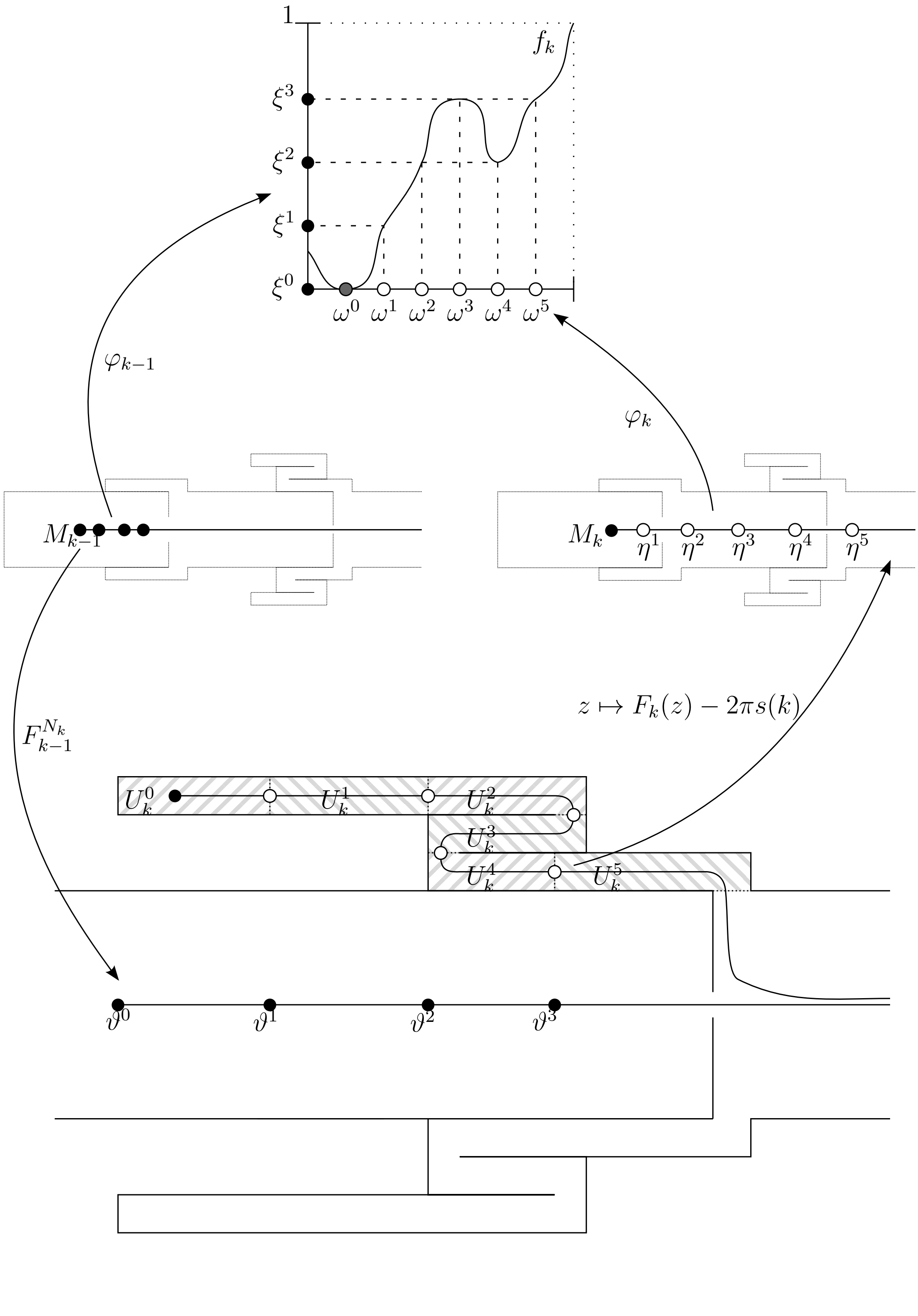


FIGURE 5. Construction of the domain U_k.

For orientation, let us briefly explain these choices, using Figure 5. Property (I1.1) and the conditions on Ω_k are arranged so that the partitions of the domain and range are sufficiently fine to encode the essential mapping properties of the map f_k. The remaining requirements ensure that the preimages of the points ξ^j under the map $\varphi k-1$ – i.e., the black dots in the centre left of the picture – provide a sufficiently fine partition of the interval $[M_k,\infty)$. In particular, (I1.3) ensures that the last of these points lies sufficiently far to the right of the picture.

I2. The next step is to choose the numbers N_k and R_k. Choose $\alpha_{k-1}\leq 1$ such that

$$0<\alpha_{k-1}\leq \min\bigl\{|x-y| : |\varphi_{k-1}(x)-\varphi_{k-1}(y)|\geq \frac{1}{\Delta_k}\bigr\}. \tag{8.1}$$

Now we choose $N_k\geq N$ sufficiently large that

$$F_{k-1}^{N_k}(M_k)>R_{k-1}+1 \qquad \text{and} \tag{8.2}$$

$$N_k>5\cdot\frac{\log(90)-\log(\alpha_{k-1})}{\log 2}. \tag{8.3}$$

(This ensures, by way of Observation 8.3, that the application of $F_{k-1}^{N_k}$ will separate our reference points by a large definite distance, as shown at the bottom of Figure 5.) We next choose

$$R_k>F_{k-1}^{N_k}(\varphi_{k-1}^{-1}(\xi_{k-1}^{\Delta_{k-1}-1})) \tag{8.4}$$

sufficiently large to ensure that, for $G=F$ or $G=F_j$ with $j\geq k$,

$$|G^{-N_k}(x)-F_{k-1}^{-N_k}(x)|\leq\frac{\alpha_{k-1}}{2} \tag{8.5}$$

for $1\leq x\leq F_{k-1}^{N_k}(\varphi_{k-1}^{-1}(\xi_{k-1}^{\Delta_{k-1}-1}))$. This is possible by Observation 8.2. Finally, we claim that we can also choose R_k so large that

$$|F^{-1}(z)-F_{k-1}^{-1}(z)|\leq 1 \qquad\text{or}\qquad \operatorname{Re}F^{-1}(z)\geq R_{k-1}+2 \tag{8.6}$$

for $z=x+2\pi is(k-1)$ with $\operatorname{Re}x\geq 1$. Indeed, let $x_0\geq 2\pi s(k-1)$ be minimal such that $\operatorname{Re}F_{k-1}^{-1}(x_0+2\pi is(k-1))\geq R_{k-1}+7$. For $x\in[1,x_0]$, we can ensure (8.6) using Observation 8.2. On the other hand, for $x\geq x_0$, the hyperbolic distance between z and x in $\mathbb{H}$ is bounded from above by 1, and hence the hyperbolic distance in T between $F^{-1}(z)$ and $F^{-1}(x)$ is also bounded by 1. So $|F^{-1}(z)-F^{-1}(x)|\leq 2$, and thus

$$\begin{aligned}\operatorname{Re}F^{-1}(z)&\geq F^{-1}(x)-2\geq F^{-1}(x_0)-2\geq \operatorname{Re}F^{-1}(x_0+2\pi is(n-1))-4\\ &\geq \operatorname{Re}F_{k-1}^{-1}(x_0+2\pi is(n-1))-5\geq R_{k-1}+2,\end{aligned}$$

as required.

I3. Now we define the domain U_k. For $0\leq j\leq \Delta_{k-1}-1$, let

$$\vartheta^j:=\vartheta^j_{k-1}:=F_{k-1}^{N_k}(\varphi^{-1}(\xi^j_{k-1}));$$

see the bottom of Figure 5. We define U_k so that it follows the same structure as the map f_k; i.e., the domain U_k runs across the real parts ϑ^j in the same order that the graph of f_k runs along the points of Ξ_{k-1}.

More precisely, U_k is defined as the union of a sequence of quadrilaterals as follows. Let $\omega^0<\omega^1<\dots<\omega^m$ be the points of Ω_k. For $j\in\{0,\dots,m\}$, let ℓ_j be maximal such that $f_k(\omega^j)\geq\xi^{\ell_j}$. By choice of Ω, we have $\ell_m=\Delta_k-1$ and $|\ell_k-\ell_{j+1}|\leq 1$ for all j.

Let $(U_k^j)_{j=0}^m$ be a sequence of pairwise disjoint Jordan domains, disjoint from S, together with arcs $(C_k^j)_{j=0}^{m+1}$ in their boundaries, with the following properties. (For simplicity of notation, we suppress the subscript k.)

(I3.1) $C^j \cup C^{j+1} \subset \partial U^j$ for all j, and $C^i \cap \partial U^j \neq \emptyset$ for $i \neq j, j+1$.

(I3.2) $C^{m+1} = \{x + i\pi/2 : R_k < x < R_k + 1\}$.

(I3.3) For all $j \in \{0, \dots, m-1\}$, the set U^j has real parts within distance 1 of the interval between $\vartheta_k^{\ell_j}$ and $\vartheta_k^{\ell_{j+1}}$. That is, if $\ell^- := \min(\ell_j, \ell_{j+1})$ and $\ell^+ := \max(\ell_j, \ell_{j+1})$, then $\vartheta_k^{\ell^-} - 1 < \operatorname{Re} z < \vartheta_k^{\ell^+} + 1$ for all $z \in U^j$.

(I3.4) For all $z \in U^m$, $\vartheta_k^{\ell_m} - 1 < \operatorname{Re} z < R_k + 1$.

(I3.5) For all $j \leq m$ and $z \in C^j$, $\vartheta_k^{\ell_j} - 1 \leq \operatorname{Re} z \leq \vartheta_k^{\ell_j} + 1$.

(I3.6) The modulus of the quadrilateral (U^j, C^j, C^{j+1}) is larger than k. Since $k > 1/2$, this ensures that there is a hyperbolic geodesic of T and T_k that separates C_k^j from C_k^{j+1}; see [Ahl73, Section 4.13]. It also ensures that the hyperbolic distance between C_k^j and C_k^{j+1} (for $j > 0$) tends to infinity as $k \to \infty$.

Remark. There will generally be many different ways to choose the domains U_k^j with these properties. This may lead to different embeddings of the corresponding continua, as discussed at the end of the introduction.

We now define
$$U_k := \bigcup_{j=0}^{m} U_k^j \cup \bigcup_{j=1}^{m} \operatorname{int}(C_k^j).$$
(Here $\operatorname{int}(C_k^j)$ denotes the interior of the arc C_k^j; i.e. the arc without the two end points.)

I4. Next, we choose the number χ_k (and hence complete the choice of the tract T_k and the function F_k). Along with χ_k, we also fix the positive integer $s(k)$. We claim that we can do so in such a way that

(I4.1) the point $F_k^{-1}(x + 2\pi i s(k))$ belongs to the domain U_k^0 for $M_k \leq x \leq M_k + 1$, and

(I4.2) For all $x \geq 0$, the point $F_k^{-1}(x + 2\pi i s(k))$ either belongs to U_k or has real part greater than R_k.

By (I3.6), the condition on the modulus of the quadrilaterals implies that, independently of the choice of χ_k, the domain U_k^0 contains a geodesic γ of T_k separating the arc C_k^0 from the arc C_k^1. Moreover, the hyperbolic distance between γ and the point $R_k + 1$ remains bounded as $\chi_k \to 0$. Now $F_k(\gamma)$ is a geodesic of $\mathbb{H}$; i.e., it is a semi-circle orthogonal to the imaginary axis. As $F_k(R_k + 1) \to \infty$ as $\chi_k \to 0$, the radius of this semi-circle must also tend to infinity. Therefore it eventually surrounds a segment of the form $\{a + is(k) : a \in [M_k, M_k + 1]\}$ for some $s(k) \in 2\pi\mathbb{N}\}$. This establishes (I4.1). Claim (I4.2) follows immediately from Carathéodory convergence.

In Figure 5, the curve $F_k^{-1}(x + 2\pi i s(k))$, $x \geq M_{k+1}$ is indicated in the bottom of the picture. Our choice of χ_k and $s(k)$ ensures precisely that this curve indeed looks essentially as it is sketched there.

I5. Finally, it remains to define the map φ_k, which we do according to Figure 5. More precisely, let $\eta^j > M_k$, for $j = 1, \dots, m$, be minimal such that

$$F_{T_k}^{-1}(\eta^j + 2\pi i s(k)) \in C_{j+1}. \tag{8.7}$$

By construction, we have $M_k < \eta^1 < \dots$, and hence we can define an order-preserving homeomorphism $\varphi_k : [M_k, \infty] \to [0,1]$ such that $\varphi_k(\eta^j) = \omega_j$ for all j. We may extend φ_k to $[0,\infty]$ by setting $\varphi_k(x) := 0$ for $x < M_k$, and such that the inverse φ_k^{-1} satisfies $\varphi_k^{-1}(x) = o(-\log(1-x))$ as $x \to 1$.

This completes the inductive construction.

Analysis of the construction. As indicated in Figure 5, the key property that we are aiming to ensure is that the function

$$x \mapsto \operatorname{Re} F^{-(N_k+1)}(x + 2\pi i s(k))$$

(essentially) maps the interval $[M_k, \infty]$ over the interval $[M_{k-1}, \infty]$ in the same way as the function f_k maps $[0,1]$ to $[0,1]$, using the identification provided by the maps φ_k and φ_{k-1}. The following observation makes this precise.

8.4. Lemma. *Let $k \geq 1$. Suppose that $w \in \mathbb{H}$ with $\operatorname{Re} w \geq M_k - 1$ and*

$$(2s(k) - 1)\pi \leq \operatorname{Im} w \leq (2s(k) + 1)\pi.$$

Set $z := F^{-(N_k+1)}(w)$.

Then $\operatorname{Re} z > M_{k-1} - 1$ and

$$|\varphi_{k-1}(\operatorname{Re} z) - f_k(\varphi_k(\operatorname{Re} w))| \leq 3\delta_{k-1}.$$

Proof. Choose $j \geq 1$ minimal such that $\varphi_k(\operatorname{Re} w) < \omega_k^j$. (Here we use the convention that $\omega_k^{m_k+1} = 1$, so that j is indeed defined.) Equivalently, j is minimal such that $\operatorname{Re} w < \eta^j$, with $\eta^j = \eta_k^j$ as defined in step I5, and using the convention that $\eta^{m_k+1} = \infty$.

Set

$$\zeta^+ := \max\{\xi_k^{\ell_{j'}} : \max(0, j-2) \leq j' \leq j\} \qquad \text{and}$$
$$\zeta^- := \min\{\xi_k^{\ell_{j'}} : \max(0, j-2) \leq j' \leq j-1\}.$$

Then $|\zeta^+ - \zeta^-| \leq 2\delta_k$ by (I1.4). Consider the intervals $I := [\zeta^-, \zeta^+]$ and

$$J := F_{k-1}^{N_k}(\varphi_{k-1}^{-1}(I)).$$

Claim. The Euclidean distance from $F^{-1}(w)$ to J is bounded by 45.

Proof. Let w' denote the point at the same real part as w and with imaginary part equal to $2s(k)\pi$. First suppose that the point $\omega' := F_k^{-1}(w')$ belongs to the domain $U_k^{\tilde{j}}$ for some $\tilde{j} < m_k$. Then $\tilde{j} \leq j$ by choice of j and definition of η^j. Furthermore, by (I3.6), the domain $U_k^{\tilde{j}+1}$ contains a geodesic separating the arc $C_k^{\tilde{j}}$ from $C_k^{\tilde{j}+1}$. Since any two geodesics intersect in at most one point, this implies that $\operatorname{Re} w < \eta^{\tilde{j}+1}$, and hence $\tilde{j} \geq j-1$. If we apply property (I3.3) to $\tilde{j}$, then we notice that ϑ^{ℓ^-} and ϑ^{ℓ^+} belong to J, by choice of J. Hence the Euclidean distance between ω' and the interval J is at most $\pi + 1$.

On the other hand, if $\omega' \in U_k^{m_k}$, or $\omega' \notin U_k^{\tilde{j}}$ for any $\tilde{j}$, then $j = m_k + 1$, and $\operatorname{Re} \omega' > \vartheta_k^{\Delta_k - 1} - 1$ by (I3.4) and (I4.2). Hence we again see that the distance between ω' and J is less than $\pi + 1$.

Recall that either $|\omega' - F^{-1}(w')| \leq 1$ or $\operatorname{Re} F^{-1}(w') \geq R_k + 2$ by (8.6). In the latter case, we have $j = m_k + 1$ (as above), and hence $\operatorname{dist}(F^{-1}(w'), J) \leq \pi$. Hence, using Observation 8.3, we see that

$$\begin{aligned}\operatorname{dist}(F^{-1}(w), J) &\leq |F^{-1}(w) - F^{-1}(w')| + \operatorname{dist}(F^{-1}(w'), J) \\ &\leq |F^{-1}(w) - F^{-1}(w')| + \pi + 2 \leq 4\pi|w - w'| + \pi + 2 \leq 4\pi^2 + \pi + 2 \leq 45,\end{aligned}$$

as claimed. △

So there is $x \in J$ such that $\operatorname{dist}_{\mathbb{H}}(x, F^{-1}(w)) \leq 45$. Then

$$\operatorname{dist}(F^{-N_k}(x), I) \leq \frac{\alpha_{k-1}}{2}. \tag{8.8}$$

Indeed, if $\tilde{x}$ is a finite endpoint of the interval J, then $\tilde{x} \leq \vartheta_{k-1}^{\Delta_{k-1}-1}$, and hence

$$\operatorname{dist}(F^{-N_k}(x), I) \leq \frac{\alpha_{k-1}}{2} \leq |F^{-N_k}(x) - F_{k-1}^{-N_k}(x)| \leq \frac{\alpha_{k-1}}{2}$$

by (8.5). Since I is an interval and F^{N_k} is monotone on $\mathbb{R}$, this implies (8.8).

By (8.3), Observation 8.3 and the Claim, we see that

$$\operatorname{dist}(F^{-N_k}(x), z) \leq 45 \cdot 2^{-\lfloor N_k/N \rfloor} \leq 45 \cdot \frac{\alpha_{k-1}}{90} = \frac{\alpha_{k-1}}{2}.$$

So $\operatorname{dist}(z, I) \leq \alpha_{k-1}$. In particular,

$$\operatorname{Re} z \geq M_{k-1} - \alpha_{k-1} > M_{k-1} - 1,$$

as claimed in the statement of the Lemma. Consider $\zeta := f_k(\omega_k^j)$. By definition of α_{k-1}, and by choice of j, we have

$$\begin{aligned}|\varphi_{k-1}(\operatorname{Re} z) - f_k(\varphi_k(\operatorname{Re} w))| &\leq |\varphi_{k-1}(\operatorname{Re} z) - \zeta| + |\zeta - F_k(\varphi_k(\operatorname{Re} w))| \\ &\leq \operatorname{dist}(\varphi_{k-1}(\operatorname{Re} z), I) + \operatorname{diam}(I) + |\zeta - F_k(\varphi_k(\operatorname{Re} w))| \\ &\leq \frac{1}{\Delta_{k-1}} + \frac{2}{\Delta_{k-1}} + \frac{1}{\Delta_{k-1}} = \frac{4}{\Delta_{k-1}}.\end{aligned}$$

This completes the proof. ■

Let us define

$$X_k := \{x + iy : x \geq M_k - 1 \text{ and } |y| \leq \pi\} \cup \infty$$

for $k \geq 0$, and

$$\Psi_k : X_k \to X_{k-1}; \qquad w \mapsto \begin{cases} \infty & \text{if } w = \infty \\ F^{-N_k+1}(w + 2s(k)\pi i) & \text{otherwise} \end{cases}$$

for $k \geq 1$. Let X be the inverse limit of the maps (Ψ_k); then $\pi_0(X)$ is precisely the Julia continuum of $\tilde{F}$ at the address

$$\underline{s} = 0^{N_1} s(1) 0^{N_2} s(2) 0^{N_3} s(3) \ldots ,$$

where we write $\tilde{F} \in \mathcal{B}_{\log}^{\mathrm{p}}$ for the $2\pi i$-periodic extension of F to avoid confusion. So we have $\pi_0(X) = \hat{J}_{\underline{s}}(\tilde{F})$, and since each Ψ_k is a homeomorphism, we see that $\pi_0 : X \to \hat{J}_{\underline{s}}(\tilde{F})$

is also a homeomorphism. We would like to apply Proposition 7.3 to the functions

$$g_k : X_k \to [0,1]; \qquad z \mapsto \begin{cases} 1 & \text{if } z = \infty \\ \varphi_k(\operatorname{Re} z) & \text{otherwise.} \end{cases}$$

To do so, we shall need to check the following fact.

8.5. Lemma (Points do not stay far to the right)**.** *Let $\hat{z} = z_0 \hookleftarrow z_1 \hookleftarrow \dots \in X$. If $z_0 \neq \infty$, then there is some k_0 such that $\operatorname{Re} z_k < \varphi_k^{-1}(\xi_k^{\Delta_k - 1})$ for all $k \geq k_0$.*

Proof. This is ensured by the choice of Δ_k in (I1.3). Indeed, suppose that k is such that $\operatorname{Re} z_k \geq \varphi_k^{-1}(\xi_k^{\Delta_k - 1})$. Then

$$\operatorname{Re} z_k \geq F_k^{n_k}(Q_k) \qquad \text{and} \qquad 2\pi s(k) + \operatorname{Im} z_k \leq F_k^{n_k}(Q_k),$$

where Q_k is as in (I1.3). In particular, if we set $x_k := |z_k|$, then the hyperbolic distance between x_k and z_k is at most $\pi/4$, and hence $|F^{-1}(x_k) - F^{-1}(z_k)| \leq \pi/2$. By (8.6) and expansion of F, we see that

$$\operatorname{Re} z_{k-1} = \operatorname{Re} F^{-M_k}(F^{-1}(z_k)) \geq \operatorname{Re} F^{-M_k}(F^{-1}(x_k)) - 1 \geq F_k^{n_{k-1}}(Q_k) - 2.$$

Continuing inductively, we see that $\operatorname{Re} z_j \geq F_k^{n_j}(Q_k) - 2$ for $j = 0, \dots, k$. In particular, $\operatorname{Re} z_0 \geq Q_k - 2 \geq k - 2$.

Hence we are done if we take $k_0 > \operatorname{Re} z_0 + 2$. ■

Proof of Theorem 8.1. We apply Proposition 7.3 to the space X and the maps g_k, as defined above, using the sequence $\tilde{\delta}_k := 6\delta_k$. By (I1.1) and Observation 7.4, this sequence satisfies condition (a) of the Proposition. Condition (c) is precisely provided by Lemma 8.4.

By Observation 7.4, condition (b) is implied by the following.

Claim 1. Suppose that $z_0 \mapsto z_1 \mapsto \dots$ and $w_0 \mapsto w_1 \mapsto \dots$ are points of X such that $|g_k(z_k) - g_k(w_k)| \leq \delta_k$ for all k. Then $z_k = w_k$ for all k.

Proof. We may suppose that $z_0 \neq \infty$. By Lemma 8.5, we then have $g_k(z_k) \leq 1 - \delta_k$ for sufficiently large k. In particular, we also have $w_0 \neq \infty$ and $g_k(w_k) \leq 1 - \delta_k$ for sufficiently large k. By (I1.2), this implies that $|z_k - w_k| \leq 1$. The claim follows from the expanding property of F (Observation 8.3). △

By Proposition 7.3, we see that X is homeomorphic to the inverse limit Y. Let $h : X \to Y$ be the homeomorphism be the homeomorphism; then

$$|\pi_k(h(z)) - g_k(z_n)| \leq \tilde{\delta}_k = \delta_k/6$$

for all k and all $z = z_0 \hookleftarrow z_1 \hookleftarrow \dots \in X$. In particular, we see that the point $h(\infty \hookleftarrow \infty \hookleftarrow \dots) = 1 \hookleftarrow 1 \hookleftarrow 1 \hookleftarrow \dots$.

Claim 2. Let $z = z_0 \hookleftarrow z_1 \hookleftarrow \dots \in X$. If $\pi_k(h(z)) = 0$ for some k, then $\operatorname{Re} z_k \leq M_k + 1$.

On the other hand, suppose that there is k_0 such that $\pi_k(h(z)) > \tilde{\delta}_k + \omega_k^2$ for all $k \geq k_0$. Then $\operatorname{Re} z_k \to \infty$.

Proof. In the first case, we have $g_k(z_k) \leq \delta_k/6$, and thus $\operatorname{Re} z_k \leq M_k + 1$ by (I1.2).

If the hypothesis of the second claim is satisfied, then On the other hand, suppose that $\pi_k(h(z)) > \tilde{\delta}_k + \omega_k^2$ for all $k \geq k_0$. Then $\varphi_k(\operatorname{Re} z_k) > \omega_k^2$, and hence $\operatorname{Re} z_k > \eta_k^2$ by definition of φ_k. By (I3.6), we have $\eta_k^2 - \eta_k^1 \to \infty$ as $k \to \infty$, and hence $\operatorname{Re} z_k \to \infty$. △

To deduce the theorem as stated, define $\tilde{h} : \hat{J}_{\underline{s}}(\tilde{F}) \to Y$ by $\tilde{h} \circ \pi_0 := h$. (Recall that $\pi_0 : X \to \hat{J}_{\underline{s}}(\tilde{F})$ is a homeomorphism.)

Then $\tilde{h}(\infty) = 1 \leftarrow\!\shortmid 1 \leftarrow\!\shortmid \dots$, as required. Furthermore, let $z_0 = \pi_0(z) \in \hat{J}_{\underline{s}}(\tilde{F})$, where $z = z_0 \leftarrow\!\shortmid z_1 \leftarrow\!\shortmid \dots$. Then

$$\tilde{F}^{n_k}(z_0) = z_k + 2\pi i s(k)$$

for all $k \geq 0$. In particular,

$$\operatorname{Re} \tilde{F}^{n_k}(z_0) \geq M_k - 1$$

by Lemma 8.4. Furthermore, by Observation 8.3, we have $\operatorname{Re} F^n(z_0) \geq \operatorname{Re} F^{n_k}(z_0)$ for $n = n_k + 1, \dots, n_{k+1} - 1$. This proves the additional claim (a) of Theorem 8.1, while (b) follows from Claim 2. (Recall that $\omega_k^2 \to 0k$ as $k \to \infty$ by choice of Ω_k in step I1.) ■

9. Applications of the construction: point uniformly escaping components and non-escaping points

We now proceed to some applications of Theorem 8.1. The following construction (again together with Theorem 2.4) proves Theorem 1.14, which asserts the existence of a Julia continuum $\hat{C}$ that is homemomorphic to an arc, and such that the finite endpoint of $\hat{C}$ belongs to the escaping set but is not contained in any nondegenerate connected set on which the iterates escape to infinity uniformly. Since Theorem 1.14 implies Theorem 1.3, this also completes the proof of the latter result.

9.1. Proposition (Inverse limit homeomorphic to an arc)**.** *There exists a surjective, continuous function $f : [0,1] \to [0,1]$ with $f(1) = 1$ and the following properties.*

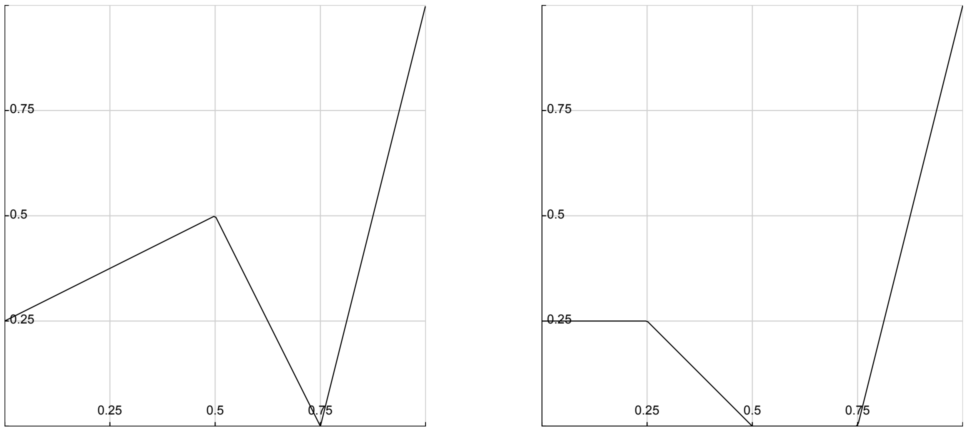
(a) $\varprojlim f$ *is homeomorphic to an arc;*
(b) $\limsup_{n\to\infty} x_n > 0$ *for all inverse orbits* $(x_n) \in \varprojlim f$.

Proof of Theorem 1.14, using Proposition 9.1. Let f be the function from Proposition 9.1, and set $Y := \varprojlim f$. Let F be the function constructed in Theorem 8.1, taking $M_n = 5$ for all n, and let $\hat{C}$ be the corresponding Julia continuum, which is homeomorphic to Y. Then $\hat{C}$ is an arc, and every point of $\hat{C}$ escapes to infinity by (b).

One of the endpoints of Y is at $x^\infty := 1 \mapsto 1 \mapsto 1 \mapsto \dots$; let x be the other endpoint. We claim that $x^k \to x$. Indeed, let $\gamma \subset Y$ be the smallest arc containing both x^∞ and infinitely many x^k. Since $\pi_k(x^k) = 0$ and $\pi_k(x^\infty) = 1$, we see that $\pi_k(\gamma) = [0,1]$ for infinitely many k, and thus $\gamma = Y$. Thus we indeed have $x^k \to x$.

Let $\zeta^k \in \hat{C}$ be the point corresponding to x^k. Then $\operatorname{Re} F^{n_k}(\zeta^k) \leq 6$ for all k, where n_k is defined as in Theorem 8.1, and in particular $n_k \to \infty$. Furthermore, as we just saw, the points ζ^k converge to the finite endpoint ζ of $\hat{C}$. Hence there is no nondegenerate subcontinuum of $\hat{C}$ containing ζ on which the iterates escape to infinity uniformly.

Finally, by Theorem 2.4, there is a disjoint-type entire function having a Julia continuum with the same properties. ■



(a) Proposition 9.1 (b) Proposition 9.3

FIGURE 6. The graph of the function f from the proof of Proposition 9.1 (left), and the function f_a from the proof of Proposition 9.3 (with $a = 1/4$).

Proof of Proposition 9.1. We define f by

$$f : [0,1] \to [0,1]; \qquad x \mapsto \begin{cases} \frac{x}{2} + \frac{1}{4} & \text{if } x \leq \frac{1}{2} \\ \frac{3}{2} - 2x & \text{if } \frac{1}{2} < x \leq \frac{3}{4} \\ 4x - 3 & \text{if } x > \frac{3}{4}. \end{cases}$$

(See Figure 6.)

To see that f has the desired properties, let us define the following subsets of $Y := \varprojlim f$, for all $k \geq 0$:

$$B := \pi_0^{-1}([1/2, 1]); \qquad A_{2k} := \pi_{k+1}^{-1}([3/4, 7/8]); \qquad A_{2k+1} := \pi_{k+2}^{-1}([7/8, 15/16]).$$

Here $\pi_j : Y \to [0,1]$ denotes the projection to the j-th component, as usual. Note that $x_j \leq 1/2$ for $j = 0, \dots, k$ when $x \in A_{2k}$ or $x \in A_{2k+1}$.

Claim. The set B and all A_j are arcs. Furthermore, B shares one endpoint with A_0, and each A_j shares one endpoint with A_{j+1}.

Proof. Since the restriction $f : [3/4, 1] \to [0,1]$ is a homeomorphism, it follows that the projection π_0 maps the set B homeomorphically to $[1/2, 1]$. Similarly, the projections π_{k+1} and π_{k+2} map A_{2k} and A_{2k+1}, respectively, to arcs. The claim about common endpoints follows from the definitions. △

Let $\underline{a} \in Y$ denote the fixed sequence defined by $a_j = 1/2$ for all j. Observe that, on $[0, 1/2]$, $f^n \to 1/2$ uniformly. Hence the sets A_j converge to $\underline{a}$ as $j \to \infty$.

Furthermore, clearly we have

$$Y = B \cup \bigcup_{j=0}^{\infty} A_j \cup \{\underline{a}\}.$$

So it follows that we can define a homeomorphism $\varphi : [0, \infty] \to Y$ that takes $[0, 1]$ to B, $[j + 1, j + 2]$ to A_j and ∞ to $\underline{a}$.

So Y is indeed an arc. Furthermore, for all $(x_n) \in Y \setminus \{\underline{a}\}$, we have $x_n \to 1$ as $n \to \infty$. This completes the proof. ■

We now complete the proof of Theorem 1.8. Recall that the first half of the theorem was established in Theorems 3.8 and 3.10. Also recall that the set of nonescaping points in a given Julia continuum has Hausdorff dimension zero by Proposition 3.9. Hence it remains to prove that there is a disjoint-type entire function f having a Julia continuum $\hat{C}$ such that the set of nonescaping points in $\hat{C}$ is a Cantor set, and an entire function having a Julia continuum containing a dense set of nonescaping points. Both results will be proved using Theorem 8.1.

To prove the second statement, we shall use the following general topological fact.

9.2. Proposition. *Let X be an arc-like continuum. Suppose that x_0 is a terminal point of X, and that E is a finite or countable set of terminal points. such that X is irreducible between each of these points and x_0.*

Then there is a sequence $f_j : [0, 1]$ of continuous and surjective functions such that $Y := \varprojlim(f_j)$ is homeomorphic to X, in such a way that x_0 corresponds to the point $1 \leftarrow\!\shortmid 1 \leftarrow\!\shortmid \dots$ and such that every point of E corresponds to a point $x_0 \leftarrow\!\shortmid x_1 \leftarrow\!\shortmid \dots$ such that $x_j = 0$ for infinitely many j.

Proof. This follows from the proof of [Nad92, Theorem 12.19], similarly as in the proof of Proposition 7.2. We leave the details to the reader. ■

To construct a Cantor set of non-escaping points, we make an explicit inverse limit construction.

9.3. Proposition (Cantor sets in inverse limit spaces)**.** *There exists a sequence (f_j) of surjective continuous maps $f_j : [0, 1] \to [0, 1]$, each fixing 1, such that the inverse limit $Y := \varprojlim(f_j)$ has the following properties.*

- *The set $A := \{(x_n) \in Y : x_j = 0$ for infinitely many $j\}$ is a Cantor set;*
- *every $(x_n) \in Y \setminus A$ satisfies $\liminf_{n\to\infty} x_n > 0$.*

Proof. For $a \in [0, 1)$, let us define

$$f_a : [0,1] \to [0,1]; \qquad x \mapsto \begin{cases} a & \text{if } x \le \frac{1}{4} \\ a \cdot (2 - 4x) & \text{if } \frac{1}{4} < x \le \frac{1}{2} \\ 0 & \text{if } \frac{1}{2} < x \le \frac{3}{4} \\ 4x - 3 & \text{if } \frac{3}{4} < x \le 1. \end{cases}.$$

Each map f_j will be of this form. That is, we set $f_j := f_{a_j}$, where the sequence (a_j) is defined inductively, starting with an arbitrary value $a_1 \in [0, 1)$.

Before we give the definition, we introduce some notation. Let

$$f_{n,k} := f_{k+1} \circ f_{k+2} \circ \cdots \circ f_n$$

for $n \ge k$. (So $f_{n,n-1} = f_n$ for all n; more generally, $f_{n,k}$ takes the n-th component of a point in the inverse limit to the k-th component.) If $n > k$, let us also say that $n \mapsto k$ if $f_{n,k}(0) = 0$ but $f_{n,\tilde{k}}(0) \neq 0$ for $k < \tilde{k} < n$.

Now we can describe the inductive construction. Suppose that $n \geq 1$ is such that $a_1, \ldots, a_n$, and hence $f_1, \ldots, f_n$, have already been chosen. Let $k = k(n+1) \in \{1, \ldots n\}$ be the smallest number such that there are fewer than two values $j \in \{k+1, \ldots, n\}$ with $j \mapsto k$.

Let a_{n+1} be any number such that $f_{n,k}(a) = 0$ and $f_{n,\tilde{k}}(a) \neq 0$ for $k < \tilde{k} < n$. Clearly such a value always exists because 0 has more than one preimage under every f_a.

Setting $f_{n+1} := f_{a_{n+1}}$, we have completed the inductive definition.

Note that $n \mapsto k(n)$ for all $n \geq 2$. Furthermore, for every $k \geq 1$ there are exactly two values of n with $k(n) = k$, and $k(n) \to \infty$ as $n \to \infty$. (Indeed, we have $k(n) = \lfloor k/2 \rfloor$ for all $n \geq 2$.)

It remains to verify that $A := \{(x_n) \in Y : x_j = 0 \text{ for infinitely many } j\}$ has the desired properties. First suppose that $(x_n) \in Y$ and $\liminf_{n \to \infty} x_n = 0$. Then there are infinitely many values of n for which $x_n < 1/2$, and hence $x_{n-1} = f_n(x_n) = a_n = f_n(0)$. Since $n \mapsto k(n)$, we have

$$x_{k(n)} = f_{n,k(n)}(x_n) = f_{n,k(n)}(0) = 0,$$

so $(x_n) \in A$.

Furthermore, if $(x_n) \in A$, then for all $k \geq 0$, there is $n \in \{2^k, \ldots, 2^{k+1} - 1\}$ such that $x_n = 0$. This implies that A is closed, and hence compact.

Finally, since for every $k \geq 1$ there are at least two values of n with $k(n) = n$, we see that A contains no isolated points. So A is a Cantor set, and the proof is complete. ■

9.4. Corollary. *There exists a disjoint-type entire function f having a Julia continuum $\hat{C}$ such that the set of nonescaping points is dense in $\hat{C}$.*

There also exists a disjoint-type entire function f having a Julia continuum $\hat{C}$ such that the set of nonescaping points in $\hat{C}$ is a Cantor set.

Remark. This completes the proof of Theorem 1.8

Proof. Let Y be an arc-like continua containing a terminal point x_0 and a dense countable set E of terminal points such that Y is irreducible between x and e for all $e \in E$. Apply Theorem 8.1 (together with Theorem 2.4) to the representation of Y as an inverse limit guaranteed by Proposition 9.2. Each of the points in E then corresponds to a non-escaping point, and we have proved the first claim.

(An example of a continuum Y with the desired property is given by the pseudo-arc. Indeed, since every point is terminal, hence we can simply select a countably dense subset E of one composant, and choose x_0 in some other composant. We remark that it is also straightforward to construct such an inverse limit Y with the desired properties directly, without using Proposition 9.2.)

The second claim follows from Proposition 9.3. ■

Remark. A continuum containing a dense set of terminal points must necessarily be either indecomposable or the union of two indecomposable continua [CM87].

10. Bounded Julia continua

We now turn to proving that any arc-like continuum Y can be realized as a bounded-address Julia continuum, provided that Y has two terminal points between which Y is irreducible. (Again, the fact that the same function can give all such continua will be deferred until Section 13.)

The construction is very similar to the proof of Theorem 8.1. In order to construct a bounded address, we cannot, however, use “side channels” of a single tract as in Section 8. Hence we will instead construct a function with two logarithmic tracts (the results of Section 6 show that this is indeed necessary).

10.1. Theorem (Bounded-address Julia continua). *Let $(f_k)_{k\geq 1}$ be a sequence of continuous functions $f_k : [0,1] \to [0,1]$ with $f_k(0) = 0$ and $f_k(1) = 1$ for all k. Let $Y := \varprojlim(f_k)$.*

Then there exista a disjoint-type function $F \in \mathcal{B}^{\mathrm{p}}_{\log}$ and two tracts S and T of f, and an external address $\underline{s}$ involving only the tracts S and T, such that $\hat{J}_{\underline{s}}(F)$ is homeomorphic to Y. Furthermore, the point $0 \leftarrow 0 \leftarrow \cdots \in Y$ corresponds to the unique point in $\hat{J}_{\underline{s}}(F)$ that has a bounded orbit, while $1 \leftarrow 1 \leftarrow \dots$ corresponds to ∞.

The first of the two tracts we consider is the half-strip

$$S := \{x + iy : x > 1 \text{ and } |y| < \pi/2\},$$

together with the function $F_S : S \to \mathbb{H}$ with $F_S(2) = 2$ and $F_S'(2) > 0$. This tract will play essentially the same role as the central strip of the tract from Section 8. The second tract,

$$T \subset \{x + iy : x > 1 \text{ and } \pi/2 < y < 3\pi/2\}$$

is again constructed in a recursive fashion. It is determined by numbers R_k^-, R_k^+ with $1 < R_k^- < R_k^+ < R_{k+1}^-$, domains

$$U_k \subset \{x + iy : R_k^- < x < R_k^+ \text{ and } \pi/2 < y < 3\pi/2\}$$

and arcs C_k^- and C_k^+, with

$$C_k^\pm \subset \partial U_k \cap \{x + iy : x = R_k^\pm\}.$$

(See Figure 7.) The tract T is defined as

$$T := \bigcup_{k\geq 1} \{x + iy : R_{k-1}^+ < x < R_k^- \text{ and } \pi/2 < y < 3\pi/2\} \cup C_k^- \cup U_k \cup C_k^+.$$

(Here we use the convention that $R_0^+ = 1$.) The function $F_T : T \to \mathbb{H}$ is chosen such that $F_T(2 + \pi i) = 2$ and $F_T(\infty) = \infty$.

We remark that, again, the tracts are not necessarily Jordan domains, and they do not have pairwise disjoint closures, but that this can easily be achieved by restricted the functions F_S and F_T to the preimages of a slightly smaller half-plane, once the construction is complete.

The construction proceeds inductively, and for this purpose we define partial tracts

$$T_m := \left(\bigcup_{k=1}^{m} \{x + iy : R_{k-1}^+ < x < R_k^- \text{ and } \pi/2 < y < 3\pi/2\} \cup C_k^- \cup U_k \cup C_k^+ \right) \cup \{x + iy : x > R_m^+ \text{ and } \pi/2 < y < 3\pi/2\},$$

and associated conformal isomorphisms $F_m : T_m \to \mathbb{H}$. Observe that, by choosing R_{m+1}^- sufficiently large, we can always ensure that F_m is close to the final function F on T_m.

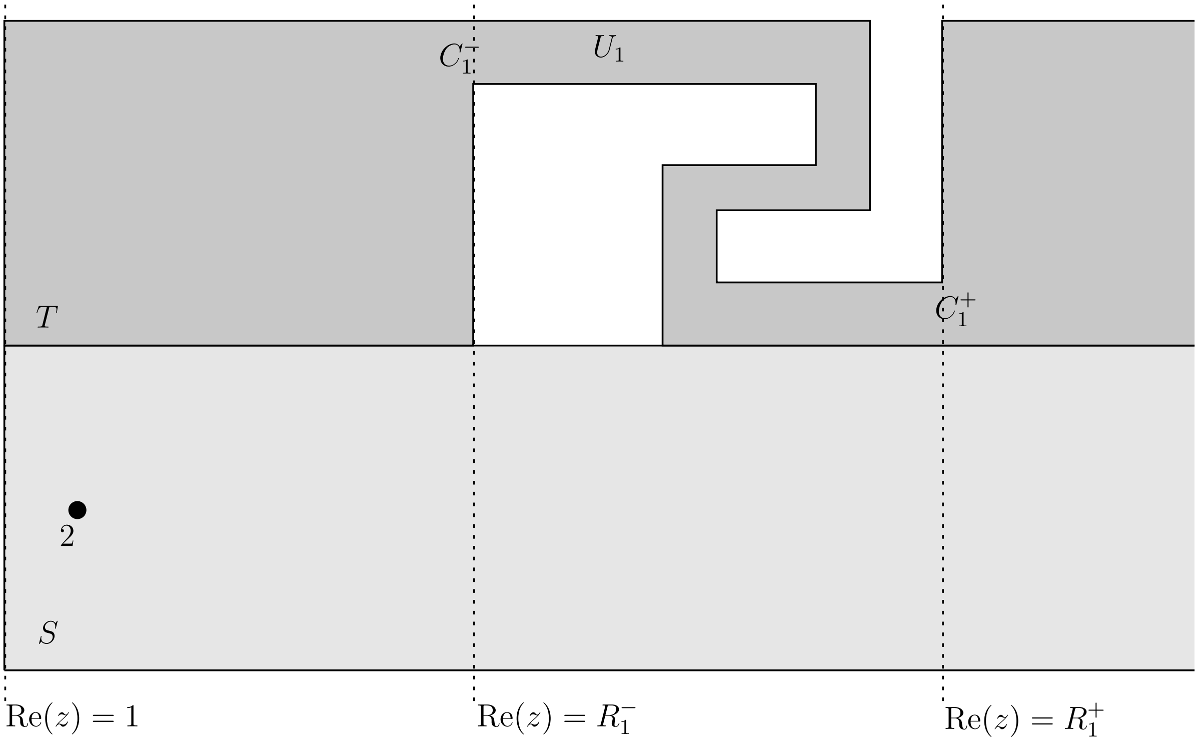


FIGURE 7. The tracts S and T.

The external address in question will be of the form

$$\underline{s} = S^{N_1} T S^{N_2} T S^{N_3} T \ldots.$$

The construction is very similar to the proof of Theorem 8.1, and in fact perhaps slightly simpler, since the function F_S remains the same throughout the construction. The domain U_k is chosen precisely so that the behaviour of the map $F_S^{-N_k} F_T^{-1}$ mirrors that of the function f_k. (See Figure 8.) We shall leave the details to the reader.

11. Periodic Julia continua

We now turn to proving Theorem 1.11, about the construction of periodic Julia continua.

11.1. Theorem (Constructing invariant Julia continua)**.** *Let $f : [0,1] \to [0,1]$ be a continuous function fixing 0 and 1, and satisfying $f(x) < x$ for all $x \in (0,1)$.*

There exists a logarithmic tract T of bounded slope and with bounded decorations, and satisfying $\overline{T} \subset \mathbb{H}$, and a conformal isomorphism $F : T \to \mathbb{H}$ with $F(\infty)$, such that the following holds. Set

$$C := \{z \in T : F^n(z) \in T \text{ for all } n \geq 0\}$$

and $\hat{C} := C \cup \{\infty\}$. Also let z_0 be the unique fixed point of F in C. Then there is a homeomorphism $\Theta\colon \hat{C} \to \varprojlim f$ such that $\Theta(z_0) = 0 \leftarrow\!\shortmid 0 \leftarrow\!\shortmid \ldots$ and $\Theta(\infty) = 1 \leftarrow\!\shortmid 1 \leftarrow\!\shortmid$ dots.

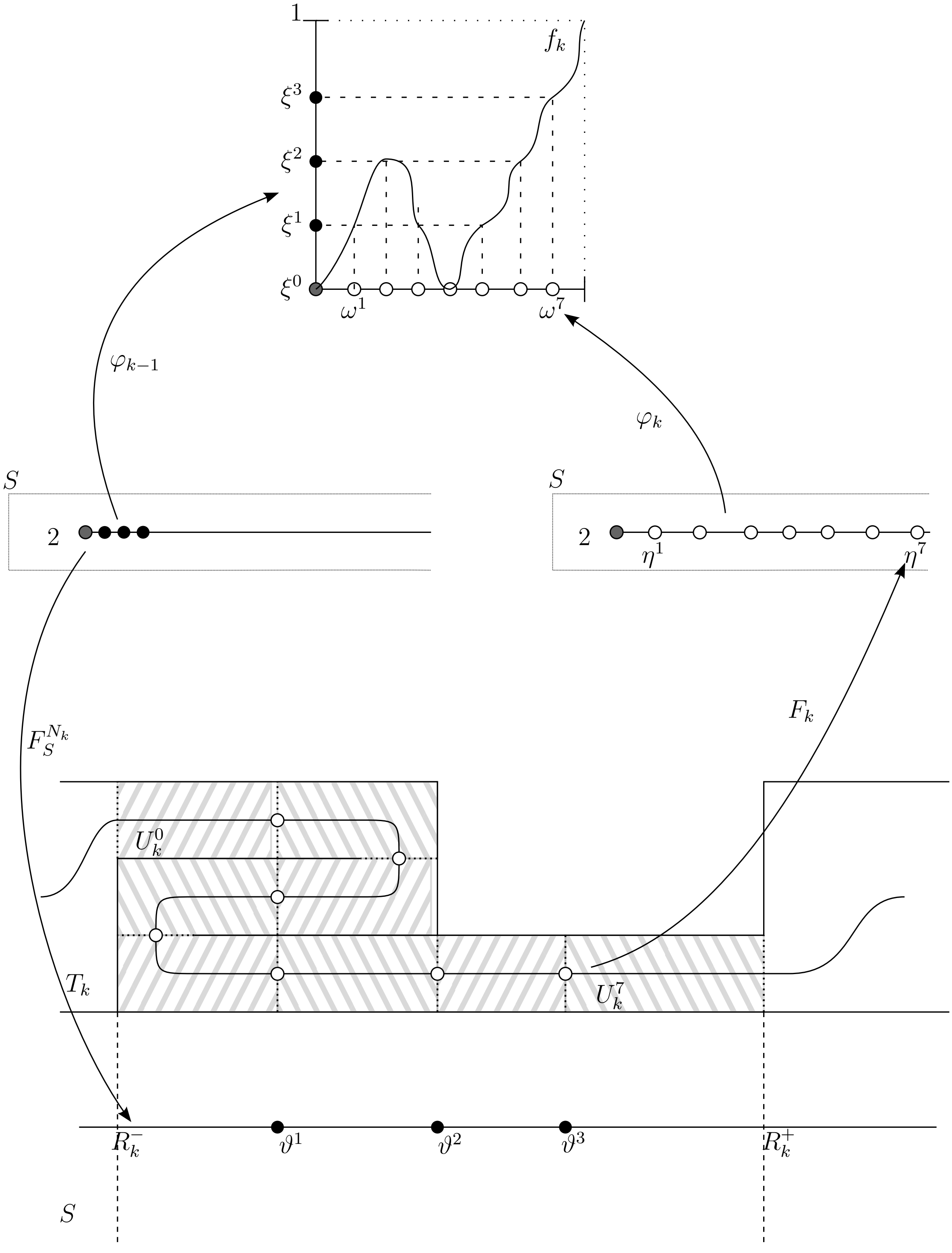


FIGURE 8. Construction of the tracts for Theorem 1.10.

Let us fix a function f as in the statement for the rest of the section, and set $Y := \varprojlim f$. The function F will be normalized such that $F(2) = 2$, and the idea is to construct the tract in such a way that the graph of function $[2, \infty] \to [2, \infty]; x \mapsto \operatorname{Re} F^{-1}(x)$ has the same shape as the graph of f (in a way that becomes more and more precise as $x \to \infty$). That is, there is a function $h\colon [0, 1] \to [2, \infty]$ such that $\operatorname{Re} F^{-1}(h(x)))$ is close to $h(f(x))$ for all x (in a way to be made precise). The assumption that $f(x) < x$ for all $x \in (0, 1)$ allows us to define both the tract and the map h in stages, without ever running into issues.

More precisely, let $(\vartheta_k)_{k\geq 0}$ in $(0, 1)$ be a strictly increasing sequence such that

(1) $f(x) < f(\vartheta_k)$ for all $x < \vartheta_k$ and
(2) $f(\vartheta_{k+2}) = f(\vartheta_k)$

for all $k \geq 0$. Clearly such a sequence exists. Indeed, if $a \in (0, 1)$, then there is a unique number $\vartheta(a) > a$ satisfying (1) and such that $f(\vartheta(a)) = a$. We can choose e.g. $\vartheta_0 := \vartheta(1/2)$ and $\vartheta_1 := \vartheta(a)$ for some a between $1/2$ and ϑ_0; the remainder of the sequence is then uniquely determined via (2). Observe that $\vartheta_k \to 1$, since 1 is the only positive fixed point of f^2.

The tract T will be constructed piece-by-piece, with the k-th piece representing the behaviour of the map f on the interval $I_k := [\vartheta_{k+1}, \vartheta_{k+2}]$. More precisely,

$$T \subset \{x + iy : x > 1, |y| < \pi\}$$

and

$$T = \bigcup_{j\geq 0} U_j \cup C_j.$$

Here $(C_j)_{j\geq 0}$ is a sequence of cross-cuts of T, with C_{j-1} and C_j bounding the subdomain U_j in T for $j \geq 1$. Furthermore,

$$\begin{aligned} &U_0 = \{x + iy : 1 < x < R_0, |y| < \pi/2\}, \\ &C_j \subset \{R_j + iy : |y| \leq \pi\}, \qquad \text{and} \\ &U_j \subset \{x + iy : 1 < x < R_j, |y| < \pi \cdot (1 - 1/k + 1)\} \qquad (j \geq 1), \end{aligned}$$

where (R_k) is a strictly (and rapidly) increasing sequence of numbers $R_k > 2$. The conformal isomorphism $F : T \to \mathbb{H}$ is chosen such that $F(2) = 2$ and $F(\infty) = \infty$.

Once more, we define partial tracts

$$T_k := \left(\bigcup_{j=0}^{k} U_j \cup C_j \right) \cup \{x + iy : x > R_k, |y| < \pi\}$$

and associated functions $F_k : T_k \to \mathbb{H}$.

The inductive construction will determine:

(A) The numbers R_k, as well as the sets U_k and C_k.
(B) Order-preserving homeomorphisms $h_k\colon [\vartheta_{k-1}, \vartheta_k] \to [R_{k-1}, R_k]$ for each $k \geq 0$. (Here, and in the following, $\vartheta_{-1} = 0$ by convention.) The maps constructed, up to h_k, combine to form an order-preserving homeomorphism $h_{0,\dots,k} : [0, \vartheta_{k+1}] \to [2, R_{k+1}]$.

(C) A finite set $\Omega_k \subset (\vartheta_{k-1}, \vartheta_k]$ for each $k \geq 0$. We shall write this set in ordered form as

$$\Omega_k = \{\omega_k^0 < \cdots < \omega_k^{m_k}\}.$$

with $\omega_k^0 = \vartheta_{k-1}$ and $\omega_k^{m_k} = \vartheta_k$.

(D) A finite set $\Xi_k \subset (\vartheta_{k-1}, \vartheta_k]$ with $\Xi_k \supset \Omega_k$. We write this set as

$$\Xi_k = \{\xi_k^0 < \xi_k^1 < \dots M\xi_k^{\Delta_k}$$

and will also consider the union

$$\Xi_{0,\dots,k} := \bigcup_{j=0}^{k} \Xi_j.$$

Our only additional requirements on the sets Ξ_k will be that

$$|h_k(\xi_k^j) - h_k(\xi_k^{j+1})| \leq 1 \tag{11.1}$$

for all j, and that every complementary interval of Ω_k contains at least three additional points of Ξ_k.

(E) For $k \geq 2$ and $1 \leq j \leq m_k$, a crosscut C_{k-2}^j of U_{k-2}, with $C_{k-2}^{m_k} = C_{k-2}$.

We begin by choosing $R_0 > 2$ sufficiently large (see below) and set $R_1 := 2R_0$. The cross-cut C_0 is defined to be the entire right-hand side of the rectangle U_0.

We also define $h_0 \colon [0, \vartheta_0] \to [2, R_0]$ and $h_1 \colon [\vartheta_0, \vartheta_1] \to [R_0, R_1]$ to be order-preserving linear homeomorphisms. Also set $\Omega_j := \{\vartheta_{j-1}, \vartheta_j\}$ for $j = 0, 1, 2$. (Note that this also uniquely determines the crosscut $C_0^1 = C_0$.)

Finally, choose $\Xi_0 \supset \Omega_0$ and $\Xi_1 \supset \Omega_1$ so that (11.1) is satisfied.

To begin the inductive step of the construction, let $k \geq 1$ be such that the domains U_j have been constructed for $j < k$, and such that R_k, $h = h_{0,\dots,k}$, as well as $\Xi_{0,\dots,k}$ and Ω_{k+1} have been defined.

I1. We begin by defining the set Ω_{k+2}. This set should be chosen with the following properties.

I1.1 For all $n \geq 1$, and all $0 \leq j < m_{k+2} - 1$, the image of the interval $[\omega_j, \omega_{j+1}]$ under the map f^n has diameter at most $1/k$.

I1.2 For all $n \geq 1$, and all $0 \leq j < m_{k+2} - 1$, the image of the interval $[\omega_j, \omega_{j+1}]$ under f intersects at most two complementary intervals of $\Xi_{0,\dots,k}$.

I2. We now construct the domain U_k. For each $j \in \{0, \dots, m_{k+2}\}$, let $\xi_{k_j}^{\ell_j}$ be a closest point of $\Xi_{0,\dots,k}$ to $f(\omega_{k+2}^j)$. We define U_k similarly as in step I3 of the proof of Theorem 8.1, as a union of domains U_k^j, for $j = 0, \dots, m_{k+2} - 1$, each bounded by two cross-cuts C_k^j and C_k^{j+1} such that the corresponding quadrilateral has modulus greater than k. The domains should be chosen such that the real parts of U_k^j are within distance 1 of the interval $[h(\xi_{k_j}^{\ell_j}), h(\xi_{k_{j+1}}^{\ell_{j+1}})]$.

Observe that, since $\xi_{k_0}^{\ell_0} = f(\vartheta_{k+1}) = \vartheta_{k-1}$, we can make the construction in such a way that $C_k^0 \subset C_{k-1}$. Likewise, we can assure that all points of $C_k^{m_{k+2}}$ have real part R_k. This completes the construction of U_k, and of $C_k := C_k^{m_{k+2}}$.

I3. We now define R_{k+1} and the map $h_{k+1} : [\vartheta_k, \vartheta_{k+1}] \to [R_k, R_{k+1}]$. To do so, for every $j \in \{1, \dots, m_{k+1}\}$, define $h_{k+1}(\omega_{k+1}^j)$ to be the minimal value of x with

$$F_k^{-1}(x) \in C_{k-1}^j.$$

For $j = m_{k+1}$, this also defines the value R_{k+1}. We then define h_{k+1} by linear interpolation.

We remark that it follows from the assumption on the modulus of the domains U_{k-1}^j that

$$\text{dist}_{\mathbb{H}}(h(\omega_{k+1}^j), h(\omega_{k+1}^{j+1}) \to \infty \tag{11.2}$$

as $k \to \infty$.

Finally, we choose $\Xi_{k+1} \supset \Omega_{k+1}$ such that (11.1) is satisfied. This completes the inductive construction.

11.2. Lemma (F is close to F_k)**.** *Suppose that the number R_0 was chosen sufficiently large. Then*

$$|F_k^{-1}(z) - F^{-1}(z)| \leq 1$$

for all $z = x + iy$ with $1 \leq x \leq R_{k+1}$ and $|y| \leq \pi$.

Proof. If R_0 was chosen sufficiently large, then the modulus of each of the domains U_k is very large (and, in fact, tend to infinity rapidly). The claim follows from hyperbolic geometry. ∎

As mentioned, the key property of the construction is that the behaviour of F reflects that of the function f. The following is a consequence of our construction.

11.3. Lemma (Connection between F and f)**.** *There is a constant K with the following property. For all $x \in [0, 1)$, we have*

$$\text{dist}_{\mathbb{H}}(F^{-1}(h(x)), h(f(x))) \leq K.$$

Proof of Theorem 11.1. Let T be the tract we have just constructed. Then T has bounded slope, and we can clearly ensure in step I2 that the tract also has bounded decorations, using Proposition 5.3. (Indeed, we only need need to make sure that every point on the "upper" side of each U_k^j can be connected to some point on the "lower" side by a curve of bounded length, and vice versa.)

Set $Y := \varprojlim f$, and define maps $\Theta_k : Y \to F^{-k}(\mathbb{H})$ by $\Theta_k(1 \leftarrow 1 \leftarrow \dots) = \infty$ and

$$\Theta_k(x) := F^{-k}(h(\pi_k(x))) \qquad \text{for } x < 1.$$

It follows from Lemma 11.2 and the expanding property of F that

$$\text{dist}_{\mathbb{H}}(\Theta_{k+1}(x), \Theta_k(x)) \leq K \cdot \Lambda^{-k},$$

where Λ is the expansion factor of F. In particular, the maps Θ_k form a Cauchy sequence, and converge uniformly to a continuous function $\Theta : Y \to \hat{C}$ (where $C = \{\infty\} \cup \bigcap_{k \geq 0} F^{-k}(\overline{T})$).

Let σ denote the shift map on Y. Then $\Theta \circ \sigma = F \circ \Theta$ by construction. In particular, Θ maps $0 \leftarrow 0 \leftarrow \dots$ to the unique finite fixed point z_0 of F. Since $\hat{C}$ is irreducible between z_0 and ∞ by Theorem 3.8, it follows that Θ is surjective.

It remains to prove that Θ is injective. Indeed, let $x^0, x^1 \in Y$ with $\Theta(x^0) = \Theta(x^1)$. Then $h(\pi_j(x^0))$ and $h(\pi_j(x^1))$ are within a bounded hyperbolic distance from each other, for every j. By (11.2), this implies that $\pi_j(x^0)$ and $\pi_j(x^1)$ are contained in a union of two adjacent complementary intervals of $\bigcup \Omega_k$. By choice of Ω_k, this implies that $x^0 = x^1$, as required. ■

Proof of Theorem 1.11. The fact that every invariant Julia continuum of a function with bounded-slope tracts is of the required form was already established in Theorem 5.6. Conversely, by Theorem 2.4, we can realize the invariant Julia continuum from Theorem 11.1 by a disjoint-type entire function. ■

12. A function where all Julia continua are pseudo-arcs

We have already shown that the pseudo-arc can arise as an invariant Julia continuum. In this section, we shall prove Theorem 1.2, which states that there is a function for which *every* Julia continuum is a pseudo-arc. This will follow from Theorem 11.1 and the following fact.

12.1. Proposition (Pseudo-arcs). *Suppose that $F \in \mathcal{B}_{\log}^{\mathrm{p}}$ is a disjoint-type function having a unique tract up to translation by $2\pi i$, and such that this tract T has bounded slope and bounded decorations. Suppose that the invariant Julia continuum contained in T is a pseudo-arc.*

Then every Julia continuum of F is a pseudo-arc.

Proof. Let C be a constant such that

$$\operatorname{diam}_H(\{z \in T : \operatorname{Re} z = x\}) \leq C \tag{12.1}$$

for all $x \geq 0$, where H is the range of F. This is possible because T has bounded slope.) Since T has bounded decorations, we can also choose C sufficiently large to ensure that

$$\operatorname{diam}_H(F^{-1}(\{z + 2\pi i s : z \in T \text{ and } \operatorname{Re} z \leq 2\pi|s|\})) \leq C \tag{12.2}$$

for all $s \in \mathbb{Z}$. Indeed, by the bounded-slope condition this set is within a bounded hyperbolic distance of a vertical geodesic of T, and this geodesic has bounded hyperbolic diameter by assumption. Let $\Lambda > 1$ be the hyperbolic expansion factor of F.

For $s \in \mathbb{Z}$, set $T_s := T + 2\pi i s$. Since F is $2\pi i$-periodic by assumption, the map $F_s := F|_{T_s} : T_s \to H$ is given by $F_s(z) = F(z - 2\pi i s)$. As before, we shall identify an external address $T_{s_0} T_{s_1} \ldots$ with the sequence $\underline{s} = s_0 s_1 \ldots$, for convenience. Let $\underline{s}$ be an allowable address; we shall show that $\hat{C} := \hat{J}_{\underline{s}}(F)$ is a pseudo-arc. Since $\hat{C}$ is arc-like by Proposition 5.5, it is sufficient to show that $\hat{C}$ is hereditarily indecomposable.

Let K be a non-degenerate subcontinuum of $\hat{C}$, and set

$$\vartheta := C \cdot \max\left(\frac{1}{\Lambda - 1}, 1\right).$$

By hyperbolic expansion of F, we can assume without loss of generality that $\operatorname{diam}_H(K)$ has hyperbolic diameter greater than $C + \vartheta$. (Otherwise, we replace K by a suitable forward iterate.) By expansion, $f^n(K)$ then has hyperbolic diameter greater than $C + \vartheta$ for all $n \geq 0$. (If $\infty \in K$, then in the following we use the convention that $F(\infty) = \infty$, and that $\operatorname{Re} \infty = +\infty$.)

For every $n, j \geq 0$, we inductively define a continuum $K_n^j \subset F^n(K)$ as follows. We set $K_n^0 := F^n(K)$. If K_{n+1}^j has been defined, choose a point $\zeta_{n+1}^j \in K_{n+1}^j$ with maximal real part, and let $\tilde{K}_{n+1}^j$ be the connected component of

$$\{z \in K_{n+1}^j : \operatorname{Re} z \geq 2\pi|s_{n+1}|\}$$

that contains ζ_{n+1} (provided that $\operatorname{Re} \zeta_{n+1}^j \geq 2\pi|s_{n+1}|)$). We then define

$$K_n^{j+1} := F_{s_n}^{-1}(\tilde{K}_{n+1}^j).$$

Claim. Let $n \geq 0$. Then the above construction defines a continuum K_n^j for every j, and furthermore every point of $f^n(K)$ has hyperbolic distance at most ϑ from K_n^j.

Proof. We prove the claim by induction on j. The claim is trivial for $j = 0$.

Suppose that the claim is true for j (and all n). Since $F^n(K)$ has diameter greater than $\vartheta + C$, for all k, it follows from (12.2) that K_n^j contains a point at real part greater than $2\pi|s_{n+1}|$.

Hence we see that $\tilde{K}_{n+1}^{j+1}$ is indeed defined for all n. By the boundary bumping theorem (Theorem 2.9), we see that either $\tilde{K}_{n+1}^{j+1} = \tilde{K}_{n+1}^j$, in which case we are done by the expansion property of F, or $\tilde{K}_{n+1}^j$ contains a point at real part $2\pi|s_{n+1}|$.

Let $z \in f^n(K)$. If $\operatorname{Re} F(z) \leq 2\pi|s_{n+1}|$, then $\operatorname{dist}_H(z, K_n^{j+1}) \leq C \leq \vartheta$ by (12.2). Otherwise, the inductive hypothesis and (12.1) imply that

$$\operatorname{dist}_H(F(z), \tilde{K}_{n+1}^j) \leq C + \vartheta.$$

So

$$\operatorname{dist}_H(z, K_n^{j+1}) \leq \frac{C + \vartheta}{\Lambda} \leq \frac{\vartheta \cdot (\Lambda - 1) + \vartheta}{\Lambda} = \vartheta.$$

This completes the proof. △

Note that the inductive construction ensures that $K_n^{j+1} \subset K_n^j$ and $F(K_n^{j+1}) \subset K_{n+1}^j$ for all n and j. We define

$$K_n := \bigcap_{j \geq 0} K_n^j.$$

Then, for all n,

- K_n is a non-degenerate subcontinuum of $f^n(K)$;
- every point of $f^n(K)$ has distance at most ϑ from K_n;
- $f(K_n) \subset K_{n+1}$;
- $\operatorname{Re} F(z) \geq 2\pi|s_{n+1}|$ for all $z \in K_n$.

Let $\underline{t}$ be the fixed address $\underline{t} = 000\ldots$. By assumption, the Julia continuum $\hat{J}_{\underline{t}}(F)$ is a pseudo-arc. By Proposition 6.1, there is a continuum $A_n \subset \hat{J}_{\underline{t}}(F)$ that is homeomorphic to K_n. It follows that K_n is hereditarily indecomposable.

Now let A and B be subcontinua of K such that $A \cup B = K$, and assume that $B \neq K$. To show that K is indecomposable, we must show that $A = K$. Since K does not contain any subset that separates the plane, we see that $f^n(A) \cap K_n$ and $f^n(B) \cap K_n$ are continua for all n.

By assumption, there is some point $a \in A \setminus B$. For sufficiently large n, we have $\operatorname{dist}_H(f^n(a), f^n(B)) > \vartheta$. It follows that $K_n \not\subset f^n(B)$. Since K_n is indecomposable, we see that $K_n \subset f^n(A)$ for sufficiently large n.

Let $z \in K$. We have $\operatorname{dist}_H(z, f^n(A)) \leq \vartheta$ for all sufficiently large n, and hence $\operatorname{dist}_H(z, A) \leq \vartheta/\Lambda^n$. Hence $z \in A$, as required. ■

Proof of Theorem 1.2. By a classical result of Henderson [Hen64], the pseudo-arc can be written as an inverse limit of a single function $f : [0,1] \to [0,1]$ with $f(x) < x$ for all $x \in (0,1)$. By Theorem 11.1, there is a disjoint-type function $F \in \mathcal{B}_{\log}^{\mathrm{p}}$, having only one tract T, such that the invariant Julia continuum in T is a pseudo-arc. Furthermore, this tract has bounded slope and bounded decorations, and hence the preceding theorem implies that every Julia continuum is a pseudo-arc.

Finally, by Theorem 2.4, there is a disjoint-type function $f \in \mathcal{B}$ with the same property. ■

13. Realizing all arc-like continua by a single function

In this section, we complete the proof of Theorems 1.6 and 1.10, by showing that all the arc-like continua in question can be realized by a single function. Recall from Proposition 7.5 that there is a countable set $\mathcal{F}$ of functions $f : [0,1]$ with $f(1) = 1$ such that every arc-like continuum with a terminal point can be written as an inverse limit with bonding maps in $\mathcal{F}$.

13.1. Theorem (A universal Julia set for arc-like continua)**.** *Let $(M_k)_{k\geq 5}$ be any sequence with $M_k \geq 5$. Then there exists a function $F \in \mathcal{B}_{\log}^{\mathrm{p}}$, having a single tract T up to translation by $2\pi i\mathbb{Z}$, such that F satisfies the conclusion of Theorem 8.1 for all continua Y whose sequence (f_k) of bonding maps is chosen from $\mathcal{F}$.*

Proof. Let $(S_m)_{m\geq 1}$ be an enumeration of all finite sequences of maps in $\mathcal{F}$. That is, $S_m = (f_{m,0}, f_{m,1}, \dots, f_{m,\ell_m})$ with $f_{m,j} \in \mathcal{F}$, and every finite sequence S of maps in $\mathcal{F}$ appears at exactly one position $m(S)$. Furthermore, we may assume that no such sequence S appears before any of its prefixes. That is, if we denote by $\sigma(S)$ the sequence obtained from S by forgetting the final entry, then $m(\sigma(S)) < \sigma(S)$.

We can now carry out exactly the same construction as in the proof of Theorem 8.1, but at the k-th stage of the construction, we insert a tract that mimics the behaviour of the map f_{m,ℓ_m}. For each sequence S of length ℓ, we shall keep track of different numbers $N(S)$ and $s(S)$. If S is an infinite sequence in $\mathcal{F}$, then the external address that gives rise to the corresponding Julia continuum will be

$$0^{N(S^1)}s(S^1)0^{N(S^2)}s(S^2)0^{N(S^3)}s(S^3)\dots,$$

where S^ℓ denotes the prefix of S of length ℓ. Since the construction involves a considerable amount of bookkeeping, but does not require new ideas otherwise, we shall omit the details. ■

Proof of Theorem 1.6. The first part of the theorem was established in Theorems 3.5, 3.7 and 5.5. The second part of the theorem follows from Theorem 13.1, again combined with Theorem 2.4 to realize the example by a disjoint-type entire function. ■

In exactly the same manner, we can prove the analogous result for bounded-address Julia continua, and hence establish Theorem 1.10.

14. Making examples with a finite number of tracts and singular values

We now turn to Theorem 1.15. So far, we have shown that each of the examples in question can be constructed in the class $\mathcal{B}$, with the desired number of tracts (one or two). With the exception of Theorem 1.2, in which we require control over *all* Julia continua, Theorem 2.4 also shows that they can be constructed in the class $\mathcal{S}$, but with a potentially infinite number of tracts.

In order to show that we can also realize our examples in the class $\mathcal{S}$, without having to add addional tracts, we need to use a more precise version of Bishop's resultfrom [Bis13]. Let G be an infinite locally bounded tree in the plane. Following Bishop, we say that G has *bounded geometry* if the following hold.

- The edges of G are C^2, with uniform bounds.
- The angles between adjacent edges are bounded uniformly away from zero.
- Adjacent edges have uniformly comparable lengths.
- For non-adjacent edges e and f, we have $\operatorname{diam}(e) \leq C \cdot \operatorname{dist}(e, f)$ for some constant C depending only on G.

Bishop's theorem [Bis13, Theorem 1.1] is as follows:

14.1. Theorem (Construction of entire functions with two singular values)**.** *Suppose that G has bounded geometry. Suppose furthermore that $\tau : \mathbb{C} \setminus G \to \mathbb{H}$ is holomorphic with the following properties.*

(a) For every component Ω_j of $\mathbb{C} \setminus G$, $\tau : \mathbb{C} \setminus \mathbb{H}$ is a conformal isomorphism whose inverse σ_j extends continuously to the closure of $\mathbb{H}$ with $\sigma_j(\infty) = \infty$.

(b) For every (open) edge $e \in G$, and each j, every component of $\sigma_j^{-1}(e) \subset \partial\mathbb{H}$ has length at least π.

Then there is an entire function $f \in \mathcal{S}$ and a K-quasiconformal map φ such that $f \circ \varphi = \cosh \circ \tau$ on the complement of the set

$$G(r_0) := \bigcup_{e \in G} \{z \in \mathbb{C} : \operatorname{dist}(z, e) < r \operatorname{diam}(e)\}.$$

(Here r_0 is a universal constant, and the union is over all edges of G.)

The only critical points of f are ± 1 and f has no asymptotic values. If $d \geq 4$ is such that G has no vertices of valence greater than d, then f has no critical points of degree greater than d.

In order to use this result to obtain the desired examples, there are two steps that we need to take.

- Firstly, we must check that the logarithmic tracts $\mathcal{T}$ of our examples can be constructed in such a way that $\mathbb{C} \setminus \exp(\mathcal{T})$ is a bounded geometry tree. In particular, we should make sure that $\overline{\mathcal{T}}$ fills out the entire plane, which will mean that we make the individual tracts fill out a complete horizontal strip.
- Once this is achieved, let f_0 be the function from Theorem 14.1, and set $f := \lambda f$, where λ is chosen sufficiently small to ensure that f is of disjoint type. The functions f and $g := \cosh \circ \tau$ are no longer necessarily "quasiconformally equivalent" near infinity in the sense of [Rem09], and hence we cannot conclude that they are quasiconformally equivalent near their Julia sets. However, if the set

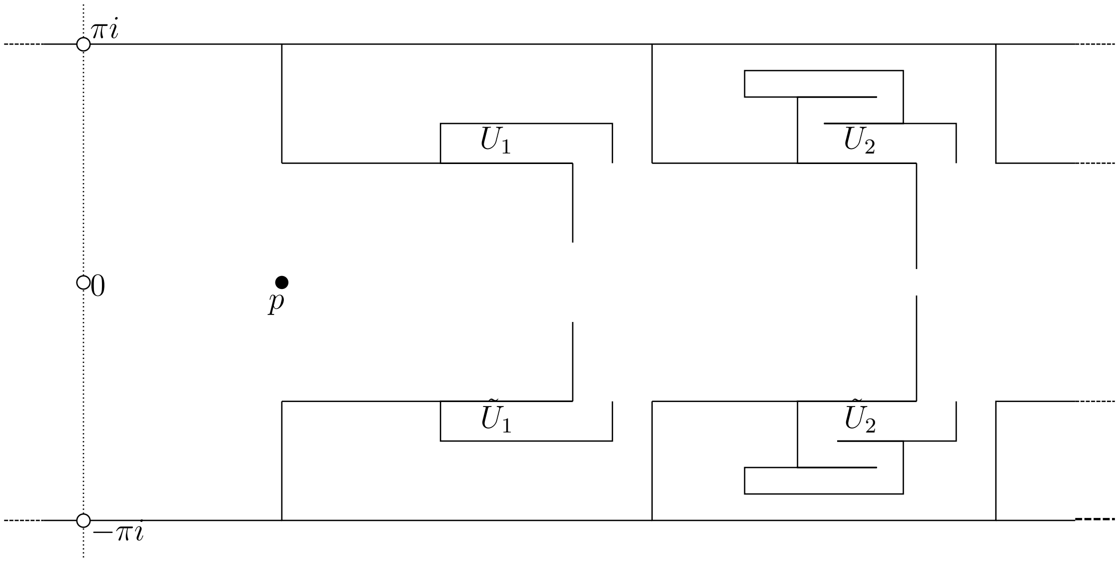


FIGURE 9. A modification of the tract T from Figure 4, which allows the function from Theorem 1.6 to be constructed in the class $\mathcal{S}$, without additional tracts.

$T(r_0)$ is disjoint from the orbit of the Julia continuum $\hat{C}$ of g under consideration, then the arguments from [Rem09] still apply to show that there is a corresponding Julia continuum of f homeomorphic to $\hat{C}$.

(We remark that the set $T(r_0)$ can in fact be replaced by a smaller set $V_{\mathcal{I}}$ (compare [Bis13, Lemma 1.2]), and this set will essentially be automatically disjoint of the *bounded-address* continua that we construct in Theorems 1.10 and 1.11. However, some care is still required in the case of Theorem 1.6.)

We shall now discuss in some detail how to ensure these properties in the case of Theorem 8.1.

14.2. Proposition. *Let the assumptions of Theorem 8.1 be satisfied, with the additional requirement that $M_k \geq P$ for all k, where $P > 3$ is a universal constant.*

Then there is a simply-connected domain $\tilde{T} \subset \{a + ib : |b| \leq \pi\}$ and a conformal isomorphism $\tilde{F} : \tilde{T} \to \mathbb{H}$ with the following properties.

- *(a) The domain $T := \tilde{F}^{-1}(\{z \in \mathbb{H} : \operatorname{Re} z > 1\})$ and the map $F : T \to \mathbb{H}; F(z) = \tilde{F}(z) - 1$ satisfy the conclusion of Theorem 8.1.*
- *(b) $G := \mathbb{C} \setminus \exp(T)$ is a bounded-geometry tree T, and the function τ defined by $\tau(\exp(z)) := \tilde{F}(z)$ satisfies the hypotheses of Theorem 14.1.*
- *(c) The hyperbolic distance (in T) between the orbit of the constructed Julia continuum and the set $\exp^{-1}(G(r_0))$ is bounded from below.*

Proof. We slightly modify the construction of the tract T from the proof of Theorem 8.1 as shown in Figure 9, where the function $\tilde{F}$ is chosen such that $\tilde{F}(p) = p$. Here p is a sufficiently large universal constant, and $P = p + 3$. This ensures that $\mathbb{C} \setminus \exp(T)$ is a tree. Also, clearly we can make the construction in such a way that edges only meet at angles of $\pi/2$.

It remains to ensure that we can subdivide the edges of the tree in such a way that adjacent edges have comparable length. We can use edges of a fixed length along the horizontal edges of the central strip, and edges whose length decreases geometrically along the "iris" that closes off the gap up to a small opening at real part R_k. When constructing the domain U_k, we may choose the domains U_k^j as rectangles, each with a width that is bounded from below (or indeed tends to infinity with k), and with a height that is bounded from below for the final domain $U_k^{m_k}$, and decreases geometrically as j changes from m_k to 0. Hence the length of edges used in ∂U_k^j decrease geometrically of j, and are only ever adjacent to edges that are comparable in size. If the initial size of edges is chosen sufficiently small, then we can easily ensure (c).

Finally, having chosen p sufficiently large, it is enough to let ξ_k be sufficiently small in step I4 to ensure that the image of an edge under τ has length at least π, as required. ■

The construction for Theorem 10.1 is similar. Here we should choose a partition of the boundary of the strip S into edges first, such that these edges have length at least π when mapped forward under the conformal map S, and choose R_k^- is chosen sufficiently large so that the boundary of the domain U_k can be subdivided in such a way to ensure bounded geometry; this is clearly possible because the set Ω_k, and hence the number of rectangles in U_k, is known before the value R_k^- is chosen. We should also make sure that the domain U_k fills out the piece of the strip between real parts R_k^- and R_k^+ (unlike in Figure 7), and, after U_k is chosen, reduce the size of the cross-cut C_k^- in order to ensure τ-length π for each of the edges.

Finally, the construction of Theorem 11.1 is a little more delicate, because the domain U_k, for large k, can potentially reach very far back to the left. More precisely, for fixed k, there may be some large values of $\tilde{k}$ such that the interval $[\vartheta_{\tilde{k}-1}, \vartheta_{\tilde{k}}]$ contains points whose image is below ϑ_k. If the set $\Omega_{\tilde{k}}$ (which is not known at the time that R_k is chosen) is very large, then potentially there may be many pieces intersecting the line $\{\operatorname{Re} z = R_k\}$, and hence it may be difficult to control the size of the corresponding edges of the tree here.

To resolve this problem, we observe that we may assume that the function f is piecewise linear on the interval $[0, 1)$ (with countably many points of nonlinearity, which accumulate only at 1), and everywhere locally non-constant. This means that we can choose the domain U_k to consist (essentially) of finitely many rectangles, one for each interval of monotonicity of f in the interval $[\vartheta_{k+1}, \vartheta_{k+2}]$. This means that the shape of the tract depends only on the function f; only the way that the tract is stretched along the real axis depends on the initial construction. This observation allows us to carry out the desired construction (once more, once the piece U_k is chosen, we should shrink the opening size of the cross-cut U_{k-1} so that edges have τ-length at least π. This also allows us to ensure that the sequence (R_k) grows sufficiently rapidly. We shall leave the details to the reader.

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